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Model for Dynamical Calculation of Inelasticity*

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We propose a model in which inelasticity can be calculated dynamically. By this we mean, given the left-hand cut contribution (or force), our model gives a prescription for calculating inelasticity $\eta_l(=e^{-2\delta_l^I}$ where δ_l^I is the imaginary part of the phase shift). The basic assumption of the model is that there is one inelastic ν_i above which a large number of reaction channels open, so that the partial-wave amplitude is essentially imaginary in the inelastic region. Our amplitude satisfies elastic unitarity below the inelastic threshold and inelastic unitarity above it. We illustrate the use of the model by applying it to the π-π p-wave system, where we approximate the left-hand-cut contribution by one pole and by two poles.

I. INTRODUCTION

A PROBLEM which theorists face at present is that there is no simple method for calculating inelasticity at high energy. In the Chew-Mandelstam[1,2] N/D method, inelasticity occurs through an unknown function $R_l(=\sigma_l^{\text{tot}}/\sigma_l^{\text{el}})$. In the N/D method of Froissart[3] and that of Frye and Warnock,[4] *a priori* knowledge of the inelasticity is necessary for the determination of the scattering amplitude. A useful method for calculating inelasticity dynamically is the ND^{-1} matrix formulation of Bjorken,[5] which is suitable when a few inelastic channels are open. However, at high energy, the method becomes not only prohibitive, because of the opening of a large number of inelastic channels, but also cannot be applied, since inelastic channels involving large numbers of particles become important. Thus, at high energy, in any phenomenological investigation, one not only has to find the force (or the left-hand-cut contribution), but also the inelasticity.[6] It is, therefore, physically interesting to see whether the two problems can be reduced to one, say, that of finding the force, while the inelasticity becomes a calculable function.[7] This has been the basic motivation of our model. An approach, similar to ours in spirit, but with a very different scheme for calculating inelasticity, has been outlined by Olesen.[8]

In Sec. II, we present the mathematical formulation of our model. The N/D method with inelastic cut by Froissart[3,9] is used. A practical difficulty, which may arise because of a simple approximation for the driving force, is discussed in Sec. III. In Sec. IV, we present the results of applying our model to the π-π p-wave system. Finally, in Sec. V, a few concluding remarks are made.

II. FORMULATION OF THE MODEL

We consider the scattering of two equal-mass spinless particles. The partial-wave scattering amplitude is given by

$$A_l(\nu)=(e^{2i\delta_l(\nu)}-1)/2i\rho(\nu)\,, \tag{2.1}$$

where

$$\rho(\nu)=[\nu/(\nu+1)]^{1/2}.$$

Here ν is the square of the c.m. momentum[10] and $\delta_l(\nu)$ is the phase shift. If ν_i is the inelastic threshold, then δ_l is real for $\nu<\nu_i$ and δ_l is complex for $\nu>\nu_i$ $(\delta_l=\delta_l^R+i\delta_l^I)$. For $\nu>\nu_i$, we can write

$$A_l(\nu)=(\eta_l e^{2i\delta_l^R}-1)/2i\rho(\nu)\,, \tag{2.2}$$

where

$$\eta_l=e^{-2\delta_l^I}.$$

We call η_l the inelasticity.[11]

Let us now introduce the following function[3,9]:

$$F_l(\nu)=\exp\left[\frac{2i\nu^{l+1/2}}{\pi}\int_{\nu_i}^{\infty}\frac{\delta_l^I(\nu')d\nu'}{\nu'^{l+1/2}(\nu'-\nu)}\right]\equiv\exp[2i\theta_l]. \tag{2.3}$$

The function

$$\theta_l(\nu)=\frac{\nu^{l+1/2}}{\pi}\int_{\nu_i}^{\infty}\frac{\delta_l^I(\nu')d\nu'}{\nu'^{l+1/2}(\nu'-\nu)} \tag{2.4}$$

is an analytic function which is real for $0<\nu<\nu_i$ and becomes complex for $\nu>\nu_i$:

$$\theta_l(\nu)=\Delta_l(\nu)+i\delta_l^I(\nu)\,,\quad(\nu>\nu_i). \tag{2.5}$$

Δ_l is the principal value of the integral in (2.4). Here, it is noted that the factor $\nu^{l+1/2}$ is used rather than $\nu^{1/2}$, as has been done by Froissart.[3] The reason for this is given later on.

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[1] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).
[2] G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin and Company, Inc., New York, 1961).
[3] M. Froissart, Nuovo Cimento **22**, 191 (1961).
[4] G. Frye and R. L. Warnock, Phys. Rev. **130**, 478 (1963).
[5] J. D. Bjorken, Phys. Rev. Letters **4**, 473 (1960).
[6] Usually the left-hand cut and the inelasticity are treated as given separately in the S-matrix approaches. See for example, G. F. Chew and S. Frautschi, Phys. Rev. **124**, 264 (1961).
[7] Frye and Warnock (Ref. 4) have found that the left-hand cut contribution and the inelasticity cannot be chosen independently, in general, and they observed that the asymptotic behaviors of the left-hand cut contribution and the inelastic effect should be precisely matched.
[8] P. Olesen, Phys. Letters **10**, 352 (1964).
[9] The method was independently developed by one of us (K. K.) and M. Ross in an unpublished work.
[10] The particle mass is taken as unity.
[11] The function $\eta_l(\nu)$ is usually called the inelasticity factor, the absorption coefficient or the transmission coefficient.

Let us further introduce a new partial-wave amplitude $a_l(\nu)$ by the following relation:

$$[1+2i\rho(\nu)A_l(\nu)]=F_l(\nu)[1+2i\rho(\nu)a_l(\nu)]. \quad (2.6)$$

The new amplitude then takes the form

$$a_l(\nu)=(e^{2i\alpha_l(\nu)}-1)/2i\rho(\nu), \quad (2.7)$$

where

$$\alpha_l(\nu)=\delta_l(\nu)-\theta_l(\nu). \quad (2.8)$$

Since δ_l and θ_l are both real for $0<\nu\leqslant\nu_i$ and have the same imaginary part for $\infty>\nu>\nu_i$, therefore α_l is real throughout the physical region $0<\nu<\infty$. Equation (2.7) then implies that $a_l(\nu)$ always obeys elastic unitarity, i.e.,

$$\mathrm{Im}a_l(\nu)=\rho(\nu)|a_l(\nu)|^2 \quad \text{for} \quad \infty>\nu>0. \quad (2.9)$$

Let us now consider the threshold behavior of α_l. When $\nu\rightarrow 0$, we have $\delta_l\propto\nu^{l+1/2}$. Further, from Eq. (2.4), we have $\theta_l\propto\nu^{l+1/2}$ as $\nu\rightarrow 0$. Thus, from (2.8), $\alpha_l\propto\nu^{l+1/2}$ as $\nu\rightarrow 0$; that is, the new amplitude $a_l(\nu)$ has the same threshold behavior as the physical amplitude $A_l(\nu)$. This is essentially the reason for our using the factor $\nu^{l+1/2}$ rather than $\nu^{1/2}$ in $F_l(\nu)$. The discontinuity of $a_l(\nu)$ on the left-hand cut is given by

$$\mathrm{Im}a_l(\nu)=\mathrm{Im}A_l(\nu)/F_l(\nu) +[F_l(\nu)-1]/2\rho(\nu)F_l(\nu), \quad (\nu<-1). \quad (2.10)$$

In our model, we shall assume that $\mathrm{Im}a_l(\nu)$ for $\nu<-1$ or equivalently, the left-hand cut contribution of $a_l(\nu)$ is known. Then, the N/D method of Chew and Mandelstam[1,2] or the inverse method[12] allows us to calculate the amplitude $a_l(\nu)$, which has the correct threshold behavior, obeys elastic unitarity, and has the given left-hand cut contribution. This, in turn, gives the phase shift $\alpha_l(\nu)$ of $a_l(\nu)$. Therefore, the function $\alpha_l(\nu)$ will be considered by us to be a known function of ν.

We shall now present arguments that δ_l^R is an approximately known quantity. First we note that if ν_i is large, then for $\nu>\nu_i$ a large number of inelastic channels open. In that case, the elastic scattering can be considered as the shadow scattering of inelastic processes and $A_l(\nu)$ is, therefore, essentially imaginary. This corresponds to $\delta_l^R\simeq n\pi$ where n is an integer. Again, from a generalized Levinson's theorem,[13] we know that δ_l^R goes asymptotically to $n\pi$ where n is related to the number of bound states and the number of Castillejo-Dalitz-Dyson (CDD) poles.[14] For a given force, we shall assume that these numbers are known from physical considerations.

Thus, in our model, α_l and δ_l^R are both assumed as known quantities. Writing Eq. (2.8) as $\alpha_l=\delta_l^R-\Delta_l$, with $\nu>\nu_i$, we get,

$$\Delta_l(\nu)=\delta_l^R(\nu)-\alpha_l(\nu)\simeq n\pi-\alpha_l(\nu), \quad (\nu>\nu_i) \quad (2.11)$$

which is, therefore, a known quantity.

From Eq. (2.4), we get,

$$\Delta_l(\nu)=\frac{\nu^{l+1/2}}{\pi}P\int_{\nu_i}^{\infty}\frac{\delta_l^I(\nu')d\nu'}{\nu'^{l+1/2}(\nu'-\nu)}, \quad (\nu>\nu_i). \quad (2.12)$$

The left-hand side of the above equation is known. If we can now invert Eq. (2.12) so that $\delta_l^I(\nu)/\nu^{l+1/2}$ is expressed as an integral over $\Delta_l(\nu)$, then $\delta_l^I(\nu)$ will be known and the problem of finding inelasticity will be solved. To solve the corresponding mathematical problem, we proceed in the following way:

Let us write

$$\theta_l(\nu)/\nu^{l+1/2}=\phi(\nu),$$
$$\Delta_l(\nu)/\nu^{l+1/2}=g(\nu),$$

and

$$\delta_l^I(\nu)/\nu^{l+1/2}=h(\nu). \quad (2.13)$$

Then, from Eq. (2.4), we have, for $\nu>\nu_i$,

$$\phi(\nu+)+\phi(\nu-)=2g(\nu), \quad (2.14)$$

and

$$\phi(\nu+)-\phi(\nu-)=2ih(\nu). \quad (2.15)$$

In our model, $g(\nu)$ is a known function and $h(\nu)$ is unknown. Finding $\phi(\nu)$ from Eq. (2.14) is a standard Hilbert arc problem.[15,16] The solution, in our case, corresponding to some physical restrictions, is given by

$$\phi(z)=\frac{(z-\nu_i)^{1/2}}{2\pi i}\int_{\nu_i}^{\infty}\frac{2g(\nu')d\nu'}{(\nu'-\nu_i)^{1/2}(\nu'-z)}. \quad (2.16)$$

The detailed derivation of Eq. (2.16) and the physical restrictions imposed on the solution are considered in the Appendix.

From (2.16), we now get,

$$\delta_l^I(\nu)/\nu^{l+1/2}=-\frac{(\nu-\nu_i)^{1/2}}{\pi} \times P\int_{\nu_i}^{\infty}\frac{\Delta_l(\nu')d\nu'}{\nu'^{l+1/2}(\nu'-\nu_i)^{1/2}(\nu'-\nu)}, \quad (\nu>\nu_i) \quad (2.17)$$

and

$$\theta_l(\nu)/\nu^{l+1/2}=\frac{(\nu_i-\nu)^{1/2}}{\pi}\int_{\nu_i}^{\infty}\frac{\Delta_l(\nu')d\nu'}{\nu'^{l+1/2}(\nu'-\nu_i)^{1/2}(\nu'-\nu)}. \quad (2.18)$$

In the above equations, $\Delta_l(\nu)$ is given by Eq. (2.11). Equation (2.17) gives the inelasticity in our model. The amplitude $A_l(\nu)$, for $\nu>\nu_i$, can be calculated from the inelasticity and the relation $\delta_l^R\simeq n\pi$. For $\nu<\nu_i$, the

[12] J. W. Moffat, Phys. Rev. **121**, 926 (1961); P. T. Mathews and A. Salam, Nuovo Cimento **13**, 381 (1959); also see Ref. 23.

[13] R. L. Warnock, Phys. Rev. **131**, 1320 (1963).

[14] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956).

[15] N. I. Muskhelishvili, *Singular Integral Equations* (P. Noordhoff Ltd., Groningen, The Netherlands, 1953).

[16] J. D. Jackson, in *Scottish Summer School Notes, 1960*, edited by G. R. Screaton (Interscience Publishers, Inc., New York, 1961).

amplitude $A_l(\nu)$ can be calculated by using the relation $\delta_l=\alpha_l+\theta_l$, where θ_l is given by the Eq. (2.18). Finally, we would like to point out that $\delta_l{}^I/\nu^{l+1/2}$, as given by Eq. (2.17), always satisfies Eq. (2.12). This can be shown by inserting (2.17) in (2.12) and using the Poincaré-Bertrand formula.[17] The details are also given in the Appendix.

III. DIFFICULTY ASSOCIATED WITH APPROXIMATE FORCE

In presenting our model in the previous section, we assumed that the force was correctly given. However, in an actual situation, the force itself is not generally known, and we have to make some phenomenological approximation for it. In Eq. (2.17) the ν' integration runs over the values of $\alpha_l(\nu')$ from $\nu'=\nu_i$ up to $\nu'=\infty$ (note, $\Delta_l=n\pi-\alpha_l$). Now, an approximate phenomenological force can only be realistic in a limited energy region and therefore, the corresponding function $\alpha_l(\nu)$ cannot be considered seriously beyond that range of ν. This, in turn, implies that $\delta_l{}^I(\nu)$ as given by (2.17) can get an appreciable contribution from values of $\alpha_l(\nu)$ which are not physical and thus can yield a $\delta_l{}^I(\nu)$ which is unphysical, say, $\delta_l{}^I(\nu)<0$. However, from unitarity, we know $\delta_l{}^I(\nu)>0$ $(0<\eta_l<1)$. Therefore, in such a case, further investigation of the function $\Delta_l(\nu)$ or $\alpha_l(\nu)$ is necessary, so that Eq. (2.17) does not give an unphysical result. We shall show that, in such a situation, our model can still be used to give $\delta_l{}^I(\nu)$ in the nearby inelastic region, while the far-off inelastic cuts have to be taken into account in a phenomenological fashion.

We shall first show an example where $\delta_l{}^I$ as calculated from (2.17) will always be negative. Let us consider $l=1$, $n=0$ (no bound state or CDD pole). In this case, we can write Eq. (2.17) as

$$\delta_1{}^I(\nu)/\nu^{3/2}=\frac{(\nu-\nu_i)^{1/2}}{\pi}P\int_{\nu_i}^{\infty}\frac{\alpha_1(\nu')d\nu'}{\nu'^{3/2}(\nu'-\nu_i)^{1/2}(\nu'-\nu)}$$
$$=\frac{(\nu-\nu_i)^{1/2}}{\pi}\int_{\nu_i}^{\infty}\frac{[\alpha_1(\nu')/\nu'-\alpha_1(\nu)/\nu]}{\nu'^{1/2}(\nu'-\nu_i)^{1/2}(\nu'-\nu)}d\nu'$$
$$+\frac{(\nu-\nu_i)^{1/2}\alpha_1(\nu)}{\pi\nu}P\int_{\nu_i}^{\infty}\frac{d\nu'}{\nu'^{1/2}(\nu'-\nu_i)(\nu'-\nu)}. \quad (3.1)$$

If, now, $\alpha_1(\nu)>0$ for $\nu>\nu_i$ and $\alpha_1(\nu)/\nu$ is a monotonically decreasing function, then the integrand of the first term is always negative for $\infty>\nu>\nu_i$. The second term in Eq. (3.1) is given by

$$-\frac{\alpha_1(\nu)}{\pi\nu^{3/2}}\ln\left(\frac{1+[(\nu-\nu_i)/\nu]^{1/2}}{1-[(\nu-\nu_i)/\nu]^{1/2}}\right),$$

and is also negative. Thus $\delta_l{}^I(\nu)$ in this example will always be negative.

[17] See Ref. 15, p. 57.

Let us now consider an approximate phenomenological force which is realistic in the low-energy region and for $\nu\sim\nu_i$. We shall see how, in this case, our model yields $\delta_l{}^I(\nu)$ in the nearby inelastic region. Let us define

$$\Phi(z)=\phi(z)/(z-\nu_i)^{1/2}$$
$$=-\frac{1}{2\pi i}\int_{\nu_i}^{\infty}\frac{2\alpha_l(\nu')d\nu'}{\nu'^{l+1/2}(\nu'-\nu_i)^{1/2}(\nu'-z)}. \quad (3.2)$$

[See Eq. (2.16). For simplicity we take $n=0$.] From (3.2) we have,

$$\Phi(\nu+)-\Phi(\nu-)=-2\alpha_l(\nu)/[\nu^{l+1/2}(\nu_+-\nu_i)^{1/2}]. \quad (3.3)$$

Let us now consider the function

$$\psi(z)=-\alpha_l(z)/z^{l+1/2}(z-\nu_i)^{1/2}. \quad (3.4)$$

The discontinuity of the function $\psi(z)$ for $\infty>\nu>\nu_i$ is[18]

$$\psi(\nu_+)-\psi(\nu_-)=-\alpha_l(\nu)/\nu^{l+1/2}(\nu_+-\nu_i)^{1/2}$$
$$+\alpha_l(\nu)/\nu^{l+1/2}(\nu_--\nu_i)^{1/2}$$
$$=-2\alpha_l(\nu)/\nu^{l+1/2}(\nu_+-\nu_i)^{1/2}. \quad (3.5)$$

Therefore,

$$\Phi(\nu_+)-\Phi(\nu_-)=\psi(\nu_+)-\psi(\nu_-). \quad (3.6)$$

This means that $\Phi(z)-\psi(z)$ is an analytic function which does not have the branch cut $\nu=\nu_i$ to $\nu=\infty$.

Equation (3.2) shows that $\Phi(z)$ is regular in the whole complex plane except for the branch cut $\nu=\nu_i$ to $\nu=\infty$. The function $\psi(z)$ of Eq. (3.4) is analytic in the whole complex plane except for a right-hand cut from $\nu=\nu_i$ to $\nu=\infty$ and a left-hand cut from $\nu=-1$ to $\nu=-\infty$. This left-hand cut arises because the phase shift $\alpha_l(\nu)$ has this cut. The function $\Phi(z)-\psi(z)$ is, therefore, regular inside a circle of radius (ν_i+1) with the center $\nu=\nu_i$. So we can expand it in a Taylor series and obtain[19]

$$\Phi(z)=-\alpha_l(z)/z^{l+1/2}(z-\nu_i)^{1/2}$$
$$+a_0+a_1(z-\nu_i)+a_2(z-\nu_i)^2+\cdots. \quad (3.7)$$

The series in (3.7) will be uniformly converging for $|z-\nu_i|<\nu_i+1$. From Eq. (3.7), we get

$$\tfrac{1}{2}[\Phi(\nu_+)+\Phi(\nu_-)]$$
$$=a_0+a_1(\nu-\nu_i)+a_2(\nu-\nu_i)^2+\cdots. \quad (3.8)$$

Again,

$$\tfrac{1}{2}[\Phi(\nu_+)+\Phi(\nu_-)]=\tfrac{1}{2}[\phi(\nu_+)-\phi(\nu_-)]/(\nu_+-\nu_i)^{1/2}$$
$$=i\delta_l{}^I(\nu)/[\nu^{l+1/2}(\nu_+-\nu_i)^{1/2}]. \quad (3.9)$$

From Eqs. (3.8) and (3.9) we, therefore, obtain

$$\delta_l{}^I(\nu)/\nu^{l+1/2}=-i(\nu_+-\nu_i)^{1/2}$$
$$\times[a_0+a_1(\nu-\nu_i)+a_2(\nu-\nu_i)^2+\cdots]$$
$$=(\nu_+-\nu_i)^{1/2}[C_0+C_1(\nu-\nu_i)+C_2(\nu-\nu_i)^2+\cdots], \quad (3.10)$$

[18] The function $\alpha_l(\nu)/\nu^{l+\frac{1}{2}}$ is analytic in the complex ν plane except for the cut $\nu=-1$ to $\nu=-\infty$. Since $\alpha_l(\nu)$ is the purely elastic scattering phase shift, so the right-hand cut of $\alpha_l(\nu)$ is removed by dividing by the factor $\nu^{l+\frac{1}{2}}$.

[19] See Ref. 15, p. 75.

where the coefficients C_0, C_1, C_2, etc. have to be real, since $\delta_l^I(\nu)$ is real. We now have, from Eqs. (2.4) and (3.10),

$$\begin{aligned}\theta_l(\nu)/\nu^{l+1/2} &= \phi(\nu) = \frac{1}{\pi}\int_{\nu_i}^{\infty} \frac{\delta_l^I(\nu')d\nu'}{\nu'^{l+1/2}(\nu'-\nu)} \\ &= \frac{1}{\pi}\int_{\nu_i}^{2\nu_i+1} \frac{\delta_l^I(\nu')d\nu'}{\nu'^{l+1/2}(\nu'-\nu)} + \frac{1}{\pi}\int_{2\nu_i+1}^{\infty} \frac{\delta_l^I(\nu')d\nu'}{\nu'^{l+1/2}(\nu'-\nu)} \\ &= [C_0I_0+C_1I_1+C_2I_2+\cdots] + \frac{1}{\pi}\int_{2\nu_i+1}^{\infty} \frac{\delta_l^I(\nu')d\nu'}{\nu'^{l+1/2}(\nu'-\nu)}, \quad (3.11)\end{aligned}$$

where

$$I_j(\nu) = \frac{1}{\pi}\int_{\nu_i}^{2\nu_i+1} d\nu'(\nu'-\nu_i)^{j+1/2}/[\nu'^{l+1/2}(\nu'-\nu)], \quad (j=0, 1, 2, \cdots).$$

The last term in (3.11) can be interpreted as the contribution of the distant inelastic cuts, while the terms in the square bracket can be interpreted as the contribution of the nearby inelastic cuts.

To proceed further, let us assume that we are interested in values of ν in the low energy region and in the nearby inelastic region (i.e., $\nu\sim\nu_i$). In that case, we can consider that the effect of distant inelastic cuts will be reasonably taken into account by a phenomenological constant, i.e., we replace the last term in Eq. (3.11) by a constant. Also, we can expect the first few terms inside the square bracket in (3.11) to take into account the effect of nearby inelastic cuts, because the series in Eq. (3.10) is uniformly converging. Specifically, we have made the following 3-parameter approximation:

$$\theta_l(\nu)/\nu^{l+1/2} \simeq C_0I_0+C_1I_1+d_0$$

i.e.,

$$\theta_l(\nu)/\nu^{l+1/2} \simeq [C_0+C_1(\nu-\nu_i)]I_0(\nu) + 2C_1(\nu_i+1)^{3/2}/3\pi + d_0, \quad (3.12)$$

where, for $l=1$,

$$\begin{aligned}I_0(\nu) &= \frac{1}{\pi}\left[2(\nu_i+1)^{1/2} - (\nu-\nu_i)^{1/2} \times \ln\left(\frac{(\nu_i+1)^{1/2}+(\nu-\nu_i)^{1/2}}{(\nu_i+1)^{1/2}-(\nu-\nu_i)^{1/2}}\right) + i\pi(\nu-\nu_i)^{1/2}\right], \quad (\nu>\nu_i) \\ &= \frac{1}{\pi}\left[2(\nu_i+1)^{1/2} + 2(\nu_i-\nu)^{1/2} \times \tan^{-1}\left[\frac{(\nu_i-\nu)}{(\nu_i+1)}\right]^{1/2} - \pi(\nu_i-\nu)^{1/2}\right], \quad (\nu<\nu_i) \quad (3.13)\end{aligned}$$

and

$$I_1(\nu) = 2(\nu_i+1)^{3/2}/3\pi + (\nu-\nu_i)I_0(\nu). \quad (3.14)$$

To determine the parameters in Eq. (3.12), we note that $\mathrm{Re}\theta_l=\Delta_l=-\alpha_l$ and is, therefore, known. Thus, by comparing the real parts of the left-hand and the right-hand sides of Eq. (3.12), we can determine C_0, C_1, and d_0. The inelasticity can then be calculated from the equation

$$\delta_l^I(\nu)/\nu^{l+1/2} \approx [C_0+C_1(\nu-\nu_i)](\nu-\nu_i)^{1/2}. \quad (3.15)$$

Further, the amplitude $A_l(\nu)$ can be calculated in the elastic region using Eq. (3.12) and the relation $\delta_l=\theta_l+\alpha_l$. One point worth emphasizing here is that in determining the parameters C_0, C_1, d_0 we have used only the values of α_l in the nearby inelastic region. On the other hand, in the application of Eqs. (2.17) and (2.18), one needs values of α_l throughout the inelastic region.

IV. APPLICATION TO π-π p-WAVE SYSTEM

We have applied our model to the π-π p-wave state. The left-hand cut contribution is approximated in two ways: (i) by one pole and (ii) by two poles. The pole positions in the two-pole approximation are chosen by the Balázs prescription[20] and the residues are adjusted so that a resonance occurs around 550 MeV in the two-pion p-wave state. In the one-pole approximation, the pole position is taken the same as the second pole position of the two-pole approximation. Again, the residue is adjusted to give a resonance at about the same energy. Our resonance position is appreciably below the physical ρ mass (760 MeV). The value 550 MeV corresponds approximately to the number that has been calculated by using crossing symmetry.[20,21]

For each input force, we have considered two inelastic thresholds $\nu_i=12.5$ and $\nu_i=17.5$. The phase shift $\alpha_1(\nu)$ of amplitude $a_1(\nu)$ is calculated using the N/D method. We have

$$a_1/\nu = N/D, \quad (4.1)$$

where (i) for one pole

$$N=\frac{b}{\nu+\omega},$$

$$D=1-\frac{\nu+\omega}{\pi}\int_0^{\infty}\frac{d\nu'}{(\nu'+\omega)(\nu'-\nu)}\left(\frac{\nu'^3}{\nu'+1}\right)^{1/2}N(\nu'), \quad (4.2)$$

and (ii) for two poles

$$N=\frac{b_1}{\nu+\omega_1}+\frac{b_2}{\nu+\omega_2},$$

$$D=1-\frac{\nu}{\pi}\int_0^{\infty}\frac{d\nu'}{\nu'(\nu'-\nu)}\left(\frac{\nu'^3}{\nu'+1}\right)^{1/2}N(\nu'). \quad (4.3)$$

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[20] L. A. P. Balázs, Phys. Rev. **128**, 1939 (1962).
[21] T. Kanki and A. Tubis, Phys. Rev. **136**, B723 (1964).

In (4.2) the D function is normalized at the pole position and in (4.3) at the threshold. The subtraction-point independence of the N/D formalism has been shown by several authors[22,23] and it can be shown explicitly when the N function is given by pole terms.[24]

The parameters C_0, C_1, and d_0 are determined from the phase shift $\alpha_1(\nu)$, as described in Sec. III. We cannot apply Eqs. (2.17) and (2.18) directly for the inelasticity calculation, since $\alpha_1(\nu)$ obtained from either (4.2) or (4.3) decreases monotonically for $\nu>\nu_i$, thus giving unphysical $\delta_l^I(\nu)(<0)$.[25] The results of our calculation are shown in Figs. 1(a)–1(d) for the one-pole input force and in Figs. 2(a)–2(d) for the two-pole input force. In Fig. 1(a), $\rho_1 \cot\alpha_1$ and $\rho_1 \cot\delta_1$ are plotted (ρ_1 being $\nu\rho$); in Fig. 2 the corresponding cross sections are given. It has been shown that when inelasticity is included, the width of the cross section becomes narrower than that of the elastic cross section.[26] This is expected in the actual physical situation, when the force is correctly given. Figure 1(a) shows that for a one pole approximation when inelasticity is taken into account, the width is narrowed. However, the cotangent of the phase shift $\cot\delta_1$, does not go through zero, i.e., we do not have a resonance of the Breit-Wigner type. This is clearly unphysical and we feel that it essentially indicates that our one pole input force is crude in the inelastic region, so that the inelasticity parameters determined from it are bad. In Fig. 1(c) η_1 is plotted and shows a sharp fall in the nearby inelastic region. In Fig. 1(d), $\alpha_1(\nu)/\nu^{3/2}$ is plotted and compared with the expression

$$-\{[C_0+C_1(\nu-\nu_i)]\mathrm{Re}I_0(\nu)+2C_1(\nu_i+1)^{3/2}/3\pi+d_0\}; \quad (4.4)$$

it indicates how well the inelasticity parameters C_0, C_1, d_0 are determined.

The values of the parameters for the one-pole case are $b=2.7$, $\omega=50.0$, and

(i) $C_0=0$, $C_1=0.00201$, $d_0=-0.0607$ for $\nu_i=12.5$,

(ii) $C_0=0$, $C_1=0.00073$, $d_0=-0.03563$ for $\nu_i=17.5$.

Figures 2(a)–(d) correspond to Figs. 1(a)–(d) when the N function is approximated by two poles. The values of the parameters for the two-pole case are $b_1=-2.75$, $b_2=23.75$, $\omega_1=6.25$, $\omega_2=50.0$ and

(i) $C_0=0$, $C_1=0.00248$, $d_0=-0.07307$ for $\nu_i=12.5$,

(ii) $C_0=0$, $C_1=0.00090$, $d_0=-0.04236$ for $\nu_i=17.5$.

Here it will be noticed that the cross sections with inelasticities are wider than the purely elastic cross section. This is presumably unphysical. In this case, $\rho_1 \cot\delta_1$ for $\nu_i=17.5$ has a zero indicating a Breit-Wigner resonance. However, $\rho_1 \cot\delta_1$ for $\nu_i=12.5$ not only does not go through zero, but also develops a pole around $\nu=0.5$. This occurs because θ_1 is negative and becomes equal to α_1 in magnitude near threshold and $\delta_1(\delta_1=\theta_1+\alpha_1)$ develops a zero. As before, these features indicate that the input force is very bad in the inelastic region, so that the corresponding inelastic effect is unrealistic. Obviously, the simple criterion which we have used to determine the input force, namely, that it will produce a low-energy resonance of the type obtained in self-consistent calculations, is not enough to give a physical force for the high-energy region.[27]

V. CONCLUDING REMARKS

The basic assumption of our model is very similar to that of the optical model,[28] viz., at high energy the elastic scattering is essentially the diffraction scattering associated with inelastic processes and is purely imaginary. However, to calculate inelasticity in the optical model, one has either to assume some kind of absorptive potential,[29] or to use some type of phenomenological description.[30] On the other hand, in our case, inelasticity is obtained from the left-hand cut contribution and this, in principle, can be dynamically calculated by considering exchange of particles or systems. Besides, our model also shows how the effect of inelasticity can be taken into account in the elastic region.

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[22] A. W. Martin, Phys. Rev. **135**, B967 (1964).

[23] G. Q. Hassoun and K. Kang, Phys. Rev. **137**, B955 (1965).

[24] For example, see M. L. Mehta and P. K. Srivastava, Phys. Rev. **137**, B423 (1965).

[25] We have considered $n=0$ for the π-π p-wave amplitude. This choice is based on the work of Warnock (Ref. 12). He has proved that $\delta_l^R(\infty)=n\pi=(-n_b+n_c)\pi$, where n_b=number of bound states and n_c=number of CDD poles. The number n_c for his B class amplitudes is given by $n_c=n_u-n_d-\epsilon+n_b+n_\infty$, where n_u=number of times the phase shift δ_l^R goes up through an integral multiple of π, n_d=number of times the phase shift δ_l^R goes down through an integral multiple of π, $\epsilon=0$ or 1 if the phase shift is positive or negative near the threshold, $n_\infty=0$ or 1 if $\delta_l^R(\nu)$ approaches its limit from above or below. For the class B amplitudes $n_d+\epsilon\geqslant n_b$, while for the class C amplitudes (which involve CDD poles of second kind) $n_d+\epsilon<n_b$. Since there is no experimental evidence of π-π p-wave bound state, we have $n_b=0$. This shows that we are not dealing with the class C amplitude. If our phase shift $\alpha_1(\nu)$ for the purely elastic-scattering case is examined, it will be seen that it is positive near threshold, rises to a maximum value $(<\pi)$ and then falls montonically. This happens for both the one-pole and the two-pole input forces. Such behavior corresponds to $n_u=n_d=\epsilon=n_\infty=0$, i.e., no CDD pole.

[26] J. R. Fulco, G. L. Shaw, and D. Y. Wong, Phys. Rev. **137**, B1242 (1965); P. Coulter and G. L. Shaw, Phys. Rev. **138**, B1273 (1965).

[27] A question that arises is whether we bring in any CDD pole when we introduce inelasticity. This can be checked by examining the behavior of the phase shift $\delta_1(\nu)$. For the single-pole input force, and $\nu_i=12.5$ and 17.5, $\delta_1(\nu)$ is positive near threshold, rises to a maximum $(<\pi)$ and then falls to zero at $\nu=\nu_i$. This behavior corresponds $n=0$, as pointed out before. The same behavior is also exhibited by $\delta_1(\nu)$ for the two-pole input force and $\nu_i=17.5$. However, $\delta_1(\nu)$ for the two-pole input force and $\nu_i=12.5$ shows that it is negative near threshold, goes up through zero around $\nu=0.5$, reaches a maximum $(<\pi)$ and then falls to zero at $\nu=\nu_i$. This behavior corresponds to $\epsilon=1$, $n_u=1$ and $n_d=n_\infty=0$. Therefore, $n_c=n_u-n_d-\epsilon+n_b+n_\infty=0$ again, i.e., no CDD pole.

[28] S. Fernbach, R. Serber, and T. B. Taylor, Phys. Rev. **75**, 1352 (1949); H. Feshbach, C. Porter, and V. F. Weisskopf, Phys. Rev. **96**, 448 (1954); K. K. Greider and Glassgold, Ann. Phys. (N.Y.) **10**, 100 (1960).

[29] R. Serber, Rev. Mod. Phys. **36**, 649 (1964).

[30] A. Baiquini, Phys. Rev. **137**, B1009 (1965).

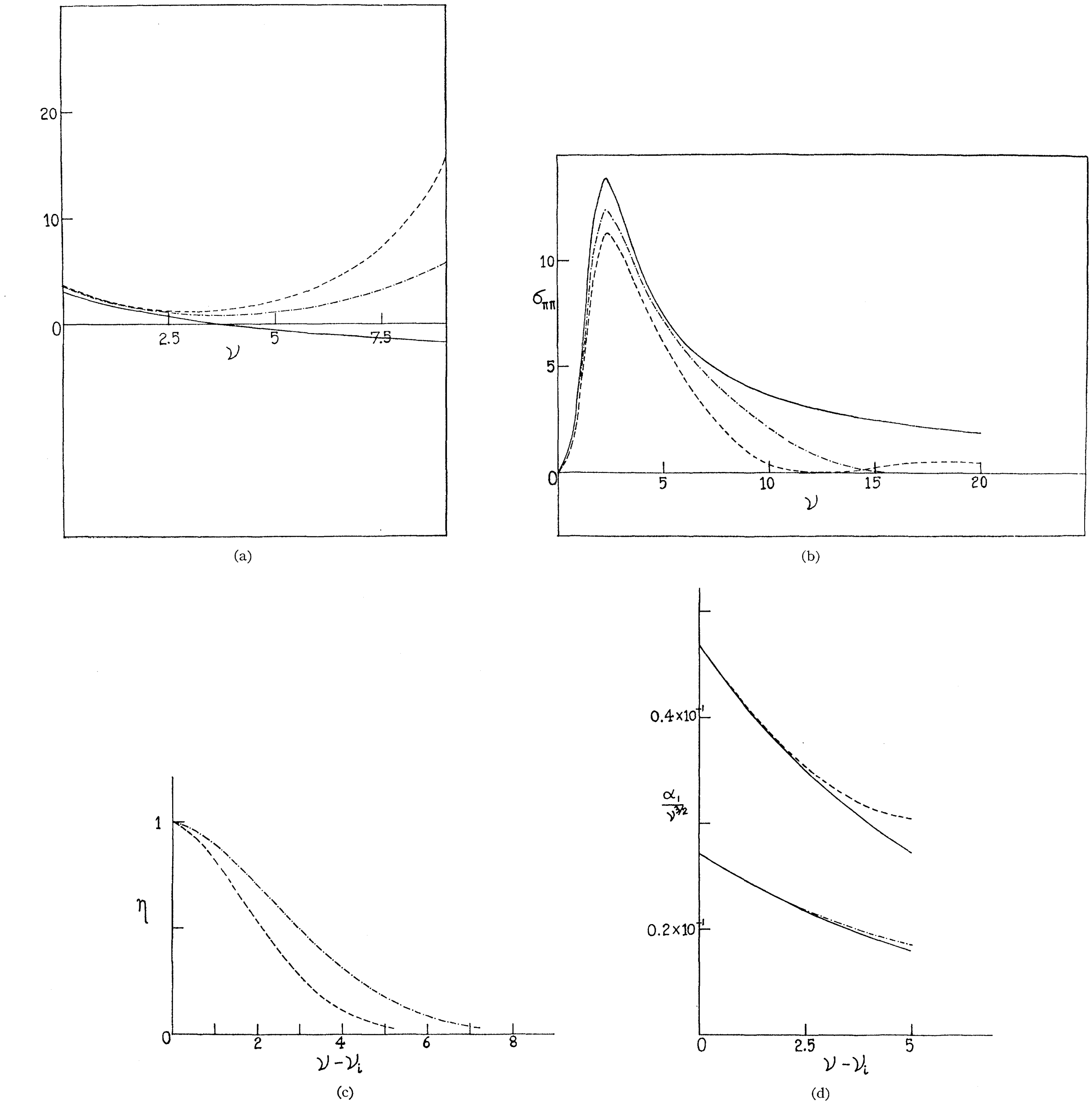

FIG. 1. (a) $\rho_1 \cot\alpha_1$ and $\rho_1 \cot\delta_1$ are plotted for the one-pole input force $\rho_1=[\nu^3/(\nu+1)]^{\frac{1}{2}}$. The solid curve corresponds to $\rho_1 \cot\alpha_1$; the dashed curve corresponds to $\rho_1 \cot\delta_1$ for $\nu_i=12.5$ and the dash-dot curve to $\rho_1 \cot\delta_1$ for $\nu_i=17.5$. (b) The π-π p-wave cross section is plotted for the one-pole input force in units of pion Compton wavelength. The solid curve corresponds to $\sigma_{\pi\pi}$ for the purely elastic scattering; the other two curves correspond to $\sigma_{\pi\pi}$ when inelasticity is taken into account. The dash curve represents $\sigma_{\pi\pi}$ for $\nu_i=12.5$ and the dash-dot curve represents $\sigma_{\pi\pi}$ or $\nu_i=17.5$. (c) The inelasticity η calculated from Eq. (3.15) is plotted against $(\nu-\nu_i)$ and compared with the expression (4.4) for the one-pole input force. The dash curve is for $\nu_i=12.5$ and the dash-dot curve for $\nu_i=17.5$. (d) $\alpha_1(\nu)/\nu^{3/2}$ is plotted against $(\nu-\nu_i)$ and compared with the expression (4.4) for the one-pole input force. The dash curve is for $\nu_i=12.5$ and the dash-dot curve for $\nu_i=17.5$.

In Sec. II, we assumed that the left-hand cut contribution of $a_l(\nu)$ [say, $a_l^L(\nu)$] rather than that of $A_l(\nu)$ [say, $A_l^L(\nu)$] is given. However, if the left-hand cut is due to some known particle exchange, then $A_l^L(\nu)$ is known. In this case, we can consider as a first approximation $a_l^L(\nu)\simeq A_l^L(\nu)$. If a better approximation is desired, we can calculate $F_l(\nu)$ with this $a_l^L(\nu)$, analytically continue it to the left and then obtain a better $a_l^L(\nu)$.

In Sec. III, we pointed out the possibility that for an approximate force which is unrealistic in the high-energy region, δ_l^I as calculated from Eq. (2.17) can be negative.

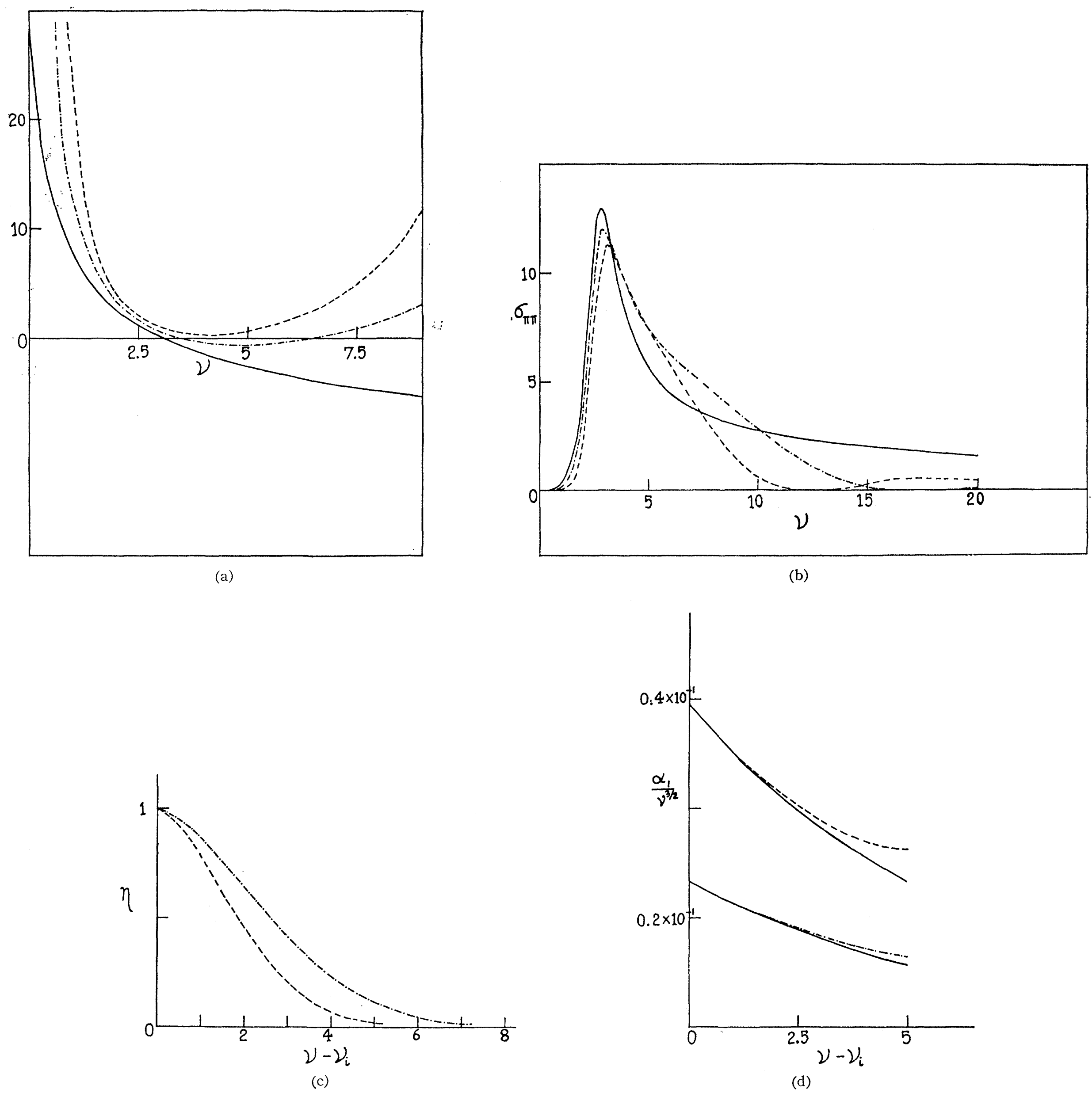

FIG. 2. (a) Same as Fig. 1(a) for the two-pole input force. (b) Same as Fig. 1(b) for the two-pole input force. (c) Same as Fig. 1(c) for the two-pole input force. (d) Same as Fig. 1(d) for the two-pole input force.

In such a case, the model cannot be directly applied. However, if the approximate force is such that $\alpha_l(\nu)$ is fairly realistic near the inelastic threshold, then the equations of our model yield δ_l^I for the nearby inelastic region. We have used this method in Sec. IV for the π-π p-wave system. The results obtained indicate that inelasticity can have an appreciable effect on the elastic scattering and this effect is sensitive to the input force. However, we feel that our results should be taken as illustrating the use of the model, rather than as physical, because the input forces are not good approximations even in the nearby inelastic region. One can, of course, invert the problem and use our model to find out what force, with inelastic effect included, describes the π-π p-wave state.

A physical situation that may be conceived is where the input force is known to be realistic not only in the low-energy region but also in the high-energy region. For this force, if application of Eq. (2.17) with $n=0$ gives δ_l^I unphysical (<0), then it may imply that n is different from zero. In this case, we can search for a value of n that will give δ_l^I physical (>0) and thus get an idea about the CDD ambiguity occurring in the problem.

It has been recently pointed out that the single-channel calculation may not be equivalent to the multi-channel calculation.[31,32] Such a circumstance may imply zeros of the S matrix and these can be taken into account by multiplying the function $F_l(\nu)$ occurring in Eq. (2.6) with a suitable rational function.[31] However, in methods where a dispersion relation for the phase shift is used,[33] such zeros of the S matrix bring in extra branch cuts and cannot be easily incorporated.

ACKNOWLEDGMENTS

The authors would like to thank Professor D. Feldman and Professor H. Fried for their interest in this work.

APPENDIX

We have to find the function $\phi(\nu)$ which is analytic in the complex ν plane except for a right-hand cut from $\nu=\nu_i$ to $\nu=\infty$ and satisfies the boundary condition

$$\phi(\nu_+)+\phi(\nu_-)=2g(\nu), \quad (\nu_i<\nu<\infty). \qquad \text{(A1)}$$

The function $g(\nu)$ is known. Next we want to obtain the imaginary part $h(\nu)$ of $\phi(\nu)$ from the equation

$$\phi(\nu_+)-\phi(\nu_-)=2ih(\nu), \quad (\nu_i<\nu<\infty). \qquad \text{(A2)}$$

We introduce a new variable $\zeta=-(1/\nu)$ so that the cut $\nu=\nu_i$ to $\nu=\infty$ in the ν plane is transformed into a finite cut $\zeta=\zeta_i$ to $\zeta=0$ in the ζ plane ($\zeta_i=-1/\nu_i$). Let us denote the values of ζ on this cut by t and denote the functions $\phi(\nu)$, $g(\nu)$, and $h(\nu)$ by primed ones in the ζ plane; i.e., $\phi(\nu)=\phi'(\zeta)$ etc. Equations (A1) and (A2) now become

$$\phi'(t_+)+\phi'(t_-)=2g'(t) \qquad \text{(A3)}$$

and

$$\phi'(t_+)-\phi'(t_-)=2ih'(t). \qquad \text{(A4)}$$

The present problem is then to find the sectionally holomorphic function $\phi'(\zeta)$, as given by Eqs. (2.4) and (2.12), and satisfying the boundary condition

$$\phi'(t_+)=G(t)\phi'(t_-)+2g'(t) \qquad \text{(A4a)}$$

on the cut, where $G(t)=-1$.

Before we solve the problem, some definitions of the solutions are in order. We follow closely those of Ref. 15. The ends are called special (or nonspecial) if $G(t)$ as defined in (A4a), is (or is not) a real positive quantity. If solutions are bounded at nonspecial ends $c_1, c_2, \cdots, c_n$ they are called solutions of the class $h(c_1,c_2,\cdots,c_n)$. The solution of the homogeneous equation vanishes at these nonspecial ends. Each solution is characterized by an index which is the negative of the highest power of the expansion at infinity.

[31] M. Bander, P. W. Coulter, and G. L. Shaw, Phys. Rev. Letters **14**, 270 (1965).
[32] E. J. Squires, Nuovo Cimento **34**, 1751 (1964).
[33] J. S. Ball and W. R. Frazer, Phys. Rev. Letters **7**, 204 (1961).

Now returning to the present problem, the ends ζ_i and 0 are nonspecial. Let us first consider the corresponding homogeneous problem

$$\chi(t_+)+\chi(t_-)=0. \qquad \text{(A5)}$$

The fundamental solution of the class $h(c_1)$ of the homogeneous problem (A5) is

$$\chi_1(\zeta)=C_1[R_1(\zeta)]^{1/2}/[R_2(\zeta)]^{1/2}=C_1[R_1(\zeta)/R_2(\zeta)]^{1/2}, \qquad \text{(A6a)}$$

where

$$R_1(\zeta)=\zeta-\zeta_i, \quad R_2(\zeta)=\zeta$$

and C_1 is an arbitrary nonzero constant. The index of this class is 0. The fundamental solution of the class $h(C_2)$ is

$$\chi_2(\zeta)=C_2[R_2(\zeta)]^{1/2}/[R_1(\zeta)]^{1/2} \qquad \text{(A6b)}$$

and the index of this class is also zero. The fundamental solution of the largest class h_0 is

$$\chi_3(\zeta)=C_3[R_1(\zeta)R_2(\zeta)]^{-1/2} \qquad \text{(A6c)}$$

and the index of this class is 1. The fundamental solution of the smallest class $h(c_1,c_2)$ is

$$\chi_4(\zeta)=C_4[R_1(\zeta)R_2(\zeta)]^{1/2} \qquad \text{(A6d)}$$

and its index is -1.

It can be easily seen that each of the above solutions has a cut from ζ_i to 0 and satisfies Eq. (A5). Using Eq. (A5), Eq. (A3) can be written as

$$\phi'(t_+)=[\chi(t_+)/\chi(t_-)]\phi'(t_-)+2g'(t)$$

or

$$\phi'(t_+)/\chi(t_+)-\phi'(t_-)/\chi(t_-)=2g'(t)/\chi(t_+). \qquad \text{(A7)}$$

The function $\phi(\zeta)/\chi(\zeta)$ is regular in the complex plane except for the cut from $\zeta=\zeta_i$ to $\zeta=0$ and the discontinuity across the cut is given by (A7). From the results obtained in Chap. 10 of Ref. 15, the general solution of Eq. (A7) for a given class is given by

$$\phi'(\zeta)/\chi(\zeta)=(2\pi i)^{-1}\int_{\zeta_i}^{0} 2g'(t)[\chi(t_+)(t-\zeta)]^{-1}dt+P_j(\zeta)$$

or

$$\phi'(\zeta)=[\chi(\zeta)/2\pi i] \times\int_{\zeta_i}^{0} 2g'(t)[\chi(t_+)(t-\zeta)]^{-1}dt+\chi(\zeta)P_j(\zeta), \qquad \text{(A8)}$$

where $P_j(\zeta)$ is an arbitrary polynomial of degree j and $P_{-1}(\zeta)=0$. The second term of (A8) is the general solution for the given class of the corresponding homogeneous problem (A5), while the first term is a particular solution of the nonhomogeneous problem (A3).

Using (A8) and (A6), we obtain

$$h'(t_0) = -[\chi(t_0)/2\pi] \times P\int_{\zeta_i}^{0} 2g'(t)[\chi(t)(t-t_0)]^{-1}dt - i\chi(t_0)P_j(t_0) \quad \text{(A9)}$$

for $\zeta_i < t_0 < 0$, and we write $\chi(t_+) = \chi(t)$.

To find out which of the four fundamental solutions in (A6) is appropriate to our physical problem, let us first examine the behavior of $h'(t_0)$ near the end ζ_i. For $\chi_1(\zeta) = C_1[(\zeta-\zeta_i)/\zeta]^{1/2}$ and $\chi_4(\zeta) = C_4[\zeta(\zeta-\zeta_i)]^{1/2}$, the integrand in Eq. (A9), $[2g'(t)]/[\chi(t)]$, behaves as $1/(t-\zeta_i)^{1/2}$ near the end ζ_i. Application of the results obtained in Chap. 4 of Ref. 15 gives, in this case, $h'(t_0) \propto (t_0-\zeta_i)^{1/2-\alpha_0}$ where $\alpha_0 < \frac{1}{2}$, that is, $h'(t_0)$ vanishes at the end ζ_i. For $\chi_2(\zeta) = C_2[\zeta/(\zeta-\zeta_i)]^{1/2}$ and $\chi_3(\zeta) = C_3[\zeta(\zeta-\zeta_i)]^{-1/2}$, the integrand $[2g'(t)]/[\chi(t)]$ vanishes as $(t-\zeta_i)^{1/2}$ near the end ζ_i, so that in this case, $h'(t_0)$ behaves as $(t_0-\zeta_i)^{-1/2}$ i.e., $h'(t_0)$ becomes infinite at the end ζ_i.

Now, $h'(t_0) = h(\nu)$ $(\infty > \nu > \nu_i)$

$$= \delta_l^I(\nu)/\nu^{l+1/2} \quad \text{(A10)}$$

and $\delta_l^I(\nu) = 0$ at $\nu = \nu_i$. Therefore, $h'(t_0)$ should vanish at the end ζ_i. Thus, only the solutions $\chi_1(\zeta)$ and $\chi_4(\zeta)$ in (A6) are relevant to our problem.

Let us next examine the behavior of $h'(t_0)$ near the end $t=0$. For $\chi_4(\zeta) = C_4[\zeta(\zeta-\zeta_i)]^{1/2}$ the integrand in (A9), $2g'(t)/\chi(t)$, behaves $\propto 1/t^{1/2}$, so that, as before, $h'(t_0) \propto t^{1/2-\alpha_0}$ where $\alpha_0 < \frac{1}{2}$; that is, $h'(t_0)$ vanishes at the end $t=0$. On the other hand, for $\chi_1(\zeta) = C_1[(\zeta-\zeta_i)/\zeta]^{1/2}$, we have $h'(t_0) \propto t_0^{-1/2}$, i.e., it becomes infinite at the end $t=0$. Now recalling

$$\phi(\nu) = \frac{1}{\pi}\int_{\nu_i}^{\infty} \frac{\delta_l^I(\nu')d\nu'}{\nu'^{l+1/2}(\nu'-\nu)}$$

we notice that in writing this equation, we have considered $\delta_l^I/\nu^{l+1/2}$ to vanish as $\nu \to \infty$. Therefore, from (A10), $h'(t_0)$ should vanish at $t=0$. Thus, the fundamental solution appropriate to our problem is

$$\chi_4 = C_4[\zeta(\zeta-\zeta_i)]^{1/2}. \quad \text{(A11)}$$

The solution (A11) behaves as $0(\zeta)$ for $\zeta \to \infty$. Inserting (A11) in (A8), we find that the first term on the right-hand side in (A8) behaves as constant for $\zeta \to \infty$, while the second term behaves as ζ^{1+j} for $j \geq 0$. Now,

$$\phi'(\zeta) = \phi(\nu) = \theta_l(\nu)/\nu^{l+1/2} \quad \text{(A12)}$$

and $\theta_l/\nu^{l+1/2}$ behaves as a constant when $\nu \to 0$. Therefore, $\phi'(\zeta)$ should behave as constant when $\zeta \to \infty$. This means that in Eq. (A8) the polynominal $P_j(z)$ should not occur. Thus, we arrive at the following results:

$$\phi'(\zeta) = (2\pi i)^{-1}[\zeta(\zeta-\zeta_i)]^{1/2} \times \int_{\zeta_i}^{0} 2g'(t)[t(t-\zeta_i)]^{-1/2}(t-\zeta)^{-1}dt \quad \text{(A13)}$$

and

$$h'(t_0) = -(2\pi)^{-1}[t_0(t_0-\zeta_i)]^{1/2} \times P\int_{\zeta_i}^{0} 2g'(t)[t(t-\zeta_i)]^{-1/2}(t-t_0)^{-1}dt. \quad \text{(A14)}$$

If we change from ζ to the original variable ν, we get from the above equations

$$\theta_l(\nu)/\nu^{l+1/2} = \phi(\nu) = (2\pi i)^{-1}(\nu_i-\nu)^{1/2} \times \int_{\nu_i}^{\infty} 2g(\nu')(\nu_i-\nu')^{-1/2}(\nu'-\nu)d\nu' \quad \text{(A15)}$$

and

$$\delta_l^I(\nu)/\nu^{l+1/2} = h(\nu) = -(2\pi)^{-1}(\nu-\nu_i)^{1/2} \times P\int_{\nu_i}^{\infty} 2g(\nu')(\nu'-\nu_i)^{-1/2}(\nu'-\nu)d\nu', \quad \text{(A16)}$$

where $g(\nu') = \Delta_l(\nu')/\nu'^{l+1/2}$.

We now want to verify that $\delta_l^I(\nu)/\nu^{l+1/2}$ given by (A16) satisfies the equation:

$$\Delta_l(\nu)/\nu^{l+1/2} = (\pi)^{-1}P\int_{\nu_i}^{\infty} [\delta_l^I(\nu')/\nu'^{l+1/2}](\nu'-\nu)^{-1}d\nu'. \quad \text{(A17)}$$

Inserting (A16) in the above equation, we get

$$\Delta_l(\nu)/\nu^{l+1/2} = g(\nu) - \frac{1}{\pi^2}\int_{\nu_i}^{\infty} d\nu'' \frac{g(\nu'')}{(\nu''-\nu_i)^{1/2}} \times P\int_{\nu_i}^{\infty} d\nu' \frac{(\nu'-\nu_i)^{1/2}}{(\nu'-\nu)(\nu''-\nu')}, \quad \text{(A18)}$$

where we have used the Bertrand-Poincaré formula for the repeated principal value integrals. The last integral in (A18) vanishes, so that the solution (A16) always satisfies Eq. (A17).