Quantized Vortex Rings in Superfluid Helium, A Phenomenological Theory*

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Rayfield and Reif (RR) and Careri, Cunsolo, and Mazzoldi (CCM) have done some experiments involving ion complexes moving through liquid helium in a constant uniform electric field. The results of RR give direct and compelling evidence that the ion complexes become attached to quantized vortex rings, the existence of which was conjectured by Feynman and Onsager. Their experiments, however, do not yield details concerning the interactions between ion complex and vortex ring. The results of CCM, on the other hand, exhibit a wealth of gross and fine structures, but they do not seem to be immediately understandable in simple terms. This paper attempts to understand all the results of RR and CCM in terms of a phenomenological theory of quantized vortex rings. Guided by experiments, we hypothesize that an ion complex can create a vortex ring only if its velocity v and effective radius R satisfy the condition $6\pi vR=nh/m$, ($n=1, 2, 3, \cdots$), where h is Planck's constant and m the helium mass. The number 6π is semiempirical. An experiment is suggested to test this hypothesis directly. After creation, the interaction of the vortex ring and the ion complex is described phenomenologically, with all parameters determined directly by experiments. This leads to a definite model that describes in detail the history of an ion complex in liquid helium. The determination of parameters makes extensive use of experimental data, but only one number from CCM. The model then reproduces quantitatively all the data of CCM.

1. INTRODUCTION

IT has been conjectured by Feynman[1] and Onsager[2] that the superfluid component of liquid helium may possess modes of hydrodynamic flow analogous to vortex motion in a classical liquid. The assumption that the superfluid is represented by a coherent quantum-mechanical wave function leads to the requirement that the circulation of the superfluid velocity along any closed path in the liquid be nh/m, where h is Planck's constant, m is the helium mass, and $n=0, 1, 2, \cdots$. The core of a quantized vortex is a line (or tube) in the liquid, about which the circulation is nh/m, with a fixed $n\geqslant 1$. A vortex ring is a vortex whose core is circular. Using very plausible arguments one can show that a vortex ring of radius R, and circulation nh/m about the core, has a definite energy E, and a translational velocity v in the direction of the superfluid velocity at the center of the circle, normal to the plane of the circle. They are given by

$$E=\tfrac{1}{2}(nh/m)^2\rho R(\eta-7/4)\,, \qquad (1)$$

$$v=(nh/m)(4\pi R)^{-1}(\eta-\tfrac{1}{4})\,, \qquad (2)$$

with

$$\eta\equiv\ln(8R/a)\,,$$
$$\rho=0.145\ \text{g/cm}^3,\quad \text{(density of liquid helium)} \qquad (3)$$
$$h/m=0.997\times10^{-3}\ \text{cm}^2/\text{sec}.$$

The density of the superfluid should approach zero as the center of the vortex core is approached. One approximates this behavior by assuming that the core has a finite radius a, and that the energy of the vortex ring comes exclusively from the superfluid motion outside of this radius. One expects a to be of the order of angstroms, and a vortex ring has meaning only if $R\gg a$. When R is eliminated between (1) and (2), one obtains v as a decreasing function of E as shown in Fig. 1 for $a=1$ Å. Table I contains a table of values for R, v, and E for $a=1$ Å. It should be emphasized that (1) and (2) are in no sense classical, even though they have the same forms as those for a vortex ring in a classical liquid. The derivation of the formulas in the present case is purely quantum-mechanical. Approxi-

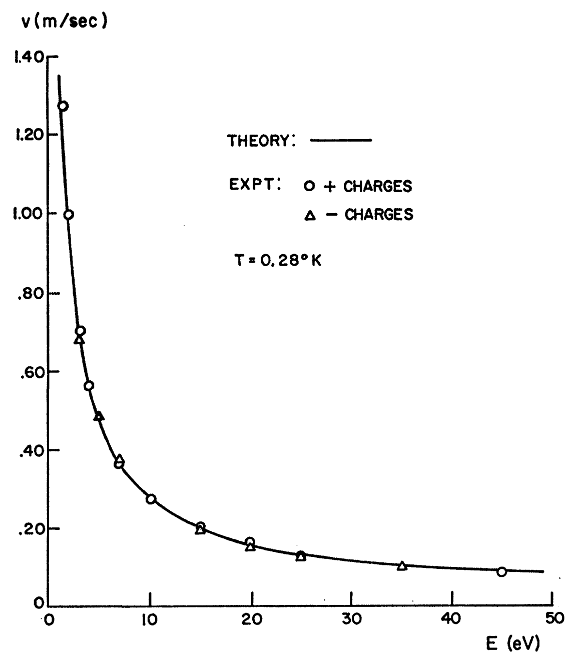

FIG. 1. Data of RR superimposed on theoretical v-E curve of a vortex ring of circulation h/m and core radius $a=1$ Å.

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[1] R. P. Feynman, in *Progress in Low Temperature Physics*, edited by C. J. Gorter (Interscience Publishers, Inc., New York, 1955), Vol. 1, pp. 34–53.
[2] L. Onsager, Nuovo Cimento **6**, Suppl. 2, 249 (1949).

TABLE I. Radius R, translational velocity v, and energy E of a vortex ring of circulation h/m and core radius $a=1$ Å.

R (Å)	v (m/sec)	E (eV)	$\eta \equiv \ln(8R/a)$
5	53.2	0.0042	3.59
6	48.0	0.0058	3.87
7	43.0	0.0072	4.02
8	38.9	0.0087	4.16
9	35.6	0.0103	4.28
10	32.9	0.0129	4.38
11	30.6	0.0136	4.48
12	28.6	0.0153	4.56
15	24.1	0.0206	4.78
20	19.2	0.0301	5.07
25	16.1	0.0403	5.30
30	13.9	0.0508	5.48
40	11.0	0.0728	5.76
50	9.14	0.0962	5.99
60	7.87	0.120	6.17
70	6.90	0.145	6.33
80	6.18	0.171	6.46
90	5.60	0.197	6.58
100	5.12	0.224	6.68
200	2.84	0.511	7.38
300	2.00	0.821	7.78
400	1.56	1.15	8.07
500	1.28	1.48	8.29
10^3	0.695	3.29	8.99
2×10^3	0.375	7.20	9.68
10^4	0.0879	43.3	11.29
10^5	0.0108	549	13.84
10^6	0.0011	5.6×10^3	14.07

mations have been used, to be sure, but not classical approximations.[3]

Whenever it can be well defined (i.e., $R\gg a$) a vortex ring is a macroscopic flow pattern of the superfluid. It does not contribute to the entropy of the whole liquid, which is still solely due to the normal-fluid component consisting of elementary excitations, i.e., phonons and rotons. The collection of these elementary excitations are not represented by a single wave function, but by an incoherent superposition of wave functions (an ensemble characterized by a density matrix), which multiplies the single wave function of the flowing superfluid to give the effective wave function of the entire liquid.

We expect physically that a vortex ring has a finite lifetime, and cannot exist in absolute thermodynamic equilibrium. Owing to interactions with phonons and rotons, it should in the course of time break up into smaller rings, and still smaller rings. As the radii of these small rings approach the core radius, there comes a point when these rings can no longer be regarded as macroscopic flow patterns. Their degradation from that point would be so complicated dynamically that it can only be described as a thermodynamically irreversible process, involving an entropy increase of the entire liquid. We assume, however, that the lifetime is macroscopically long. To the extent that it is longer than the duration of a physical experiment, we regard the vortex ring as a stable entity.

[3] Some theoretical considerations concerning vortex rings are contained in Refs. 1 and 4.

Rayfield and Reif (RR),[4] and Careri, Cunsolo, and Mazzoldi[5] (CCM) have carried out some interesting experiments studying the motion of ion complexes through liquid helium in an external electric field. For brevity we refer to an ion complex as an ion, although it is in reality a relatively large group of atoms bound to a charge by electric polarization. We summarize the experimental results later, and give only the most cursory description at this point. RR observed the velocity of positive ions at 0.28°K, after they have traveled through a certain distance in liquid helium under a constant uniform electric field, and found compelling evidence that these ions coupled themselves to quantized vortex rings of $n=1$. CCM measured an average drift velocity of the ions at about 1°K over a very wide range of field strengths. Their results exhibit some striking phenomena suggesting violence encountered by the ions of their journey through liquid helium. However, the phenomena do not appear to be similar to those of RR, and are not immediately understandable in terms of quantized vortex rings.

This paper attempts to understand these experiments on the basis of a single phenomenological model of the interaction between ions and quantized vortex rings. The model is concerned with the journey of an ion through liquid helium. It describes how it may create a vortex ring, and either get trapped by it, or drift apart from it. Starting from a hypothesis motivated by the experiments, we construct a well-defined model, in which all parameters are determined directly by experiments. The model is then capable of giving a detailed account of the history of the ion. Fitting one number to CCM, we reproduce quantitatively the data of CCM at a particular temperature over their entire range of field strengths. The model oversimplifies many physical effects, and its region of validity is restricted to narrow temperature regions about $T=0$°K and $T=1$°K. Although we do not understand why the agreement between theory and experiment should be so good, we feel that the agreement justifies the basic picture of the model. We therefore conclude that quantized vortex rings do exist in superfluid helium, and that Rayfield and Reif, and Careri, Cunsolo, and Mazzoldi, did observe their effect.

2. EXPERIMENTAL RESULTS AND SOME IMMEDIATE INTERPRETATIONS

A. Rayfield and Reif

In the experiments of RR,[4] ions, either positive or negative, are drawn through liquid helium at temperature T, by a constant uniform electric field $\mathcal{E}$, over a

[4] G. W. Rayfield and F. Reif, Phys. Rev. Letters **11**, 305 (1963); Phys. Rev. **136**, A1194 (1964); hereafter referred to as RR.

[5] G. Careri, S. Cunsolo, and P. Mazzoldi, Phys. Rev. Letters **7**, 151 (1961); Phys. Rev. **136**, A303 (1964); hereafter referred to as CCM.

distance L, with

$$L=0.3 \text{ cm}, \quad T=0.28°\text{K}, \quad 10<\mathcal{E}<150 \text{ V/cm}. \tag{4}$$

At this low temperature, L is much smaller than the mean free path for collisions between the charge carrier and elementary excitations—a fact which RR show by observing that charge carriers suffer no appreciable energy loss when they traverse a field-free region of several cm. The energy E gained by a charged carrier in the electric field is much greater than its initial kinetic energy, which may be neglected. Hence its final energy is essentially $E=e\mathcal{E}L$. The final velocity v of the charged carrier is measured as a function of $\mathcal{E}$ by a kind of velocity selector, which makes use of the fact that there is no energy loss. The experimental points are shown in Fig. 1, in which the solid curve is a theoretical curve obtained by eliminating R between (1) and (2) with $n=1$. The core radius a, chosen to fit the experimental points, turns out to be

$$a=(1.28\pm0.13)\times10^{-8} \text{ cm}. \tag{5}$$

The reasonable value for a, and the remarkable agreement between theoretical and experimental results seem to leave no doubt that the ions somehow became coupled to quantized vortex rings of $n=1$, forming what RR called "charged vortex rings". The density of charges vortex rings at the end of L is estimated to be 10^4-10^5 cm^{-3},[6] which is low enough for us to assume that they do not interact with one another.

Theoretical arguments suggested by RR show that a charged vortex ring should experience a viscous force $-F_1(v,T)$ of the form

$$F_1(v,T)=n(\eta-\tfrac{1}{4})\alpha(T). \tag{6}$$

where η is given by (3), and depends on v through R. Independent experiments by RR verify this form for $n=1$, in the temperature range $0.4<T<0.7$°K, and the same range of $\mathcal{E}$ as in (4). The measured values of $\alpha(T)$ are shown in Fig. 2. RR also suggest that near 1°K, where roton scattering is the dominant contribution to $F_1(v,T)$ the quantity $\alpha(T)$ should be proportional to $\exp(-\Delta/kT)$, where Δ is the "roton energy gap," available from other experiments. Accepting this argument, we can find the proportionality constant by fitting an asymptote to the data, shown as the straight line in Fig. 2. We obtain in this manner the semiempirical formula

$$\alpha(T)=\exp(13.9-\Delta/kT) \text{ eV/cm}, \quad (T\approx1°\text{K}), \qquad \Delta/k=8.65°\text{K (roton energy gap)}. \tag{7}$$

The theoretical considerations leading to (6), however, ignore the presence of the bound ion in the charged vortex ring. The vindication of (6) by experiments must mean that in the region of $\mathcal{E}$ and T investigated, the additional viscous drag due to the presence of the ion is negligible. We later suggest a correction to (6) that includes the effect of the bound ion.

[6] F. Reif (private communication).

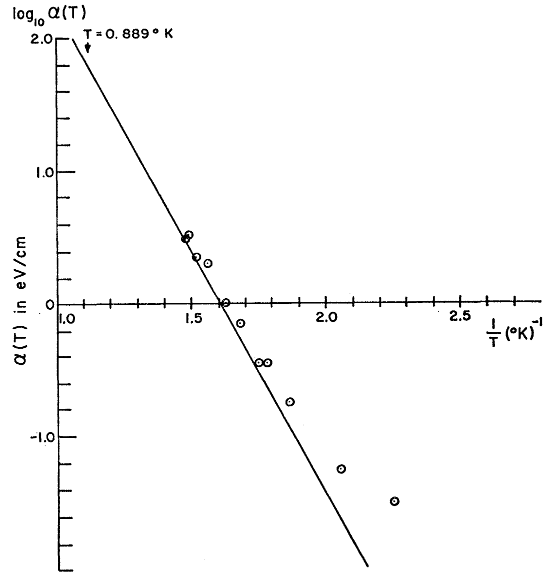


FIG. 2. Straight line is the asymptote fitted to the experimental data of RR for the function $\alpha(T)$ defined in Eq. (6).

B. Careri, Cunsolo, and Mazzoldi

The experiments of CCM[5] are similar in arrangement to those of RR, with

$$L=0.6 \text{ cm}, \quad T\approx1°\text{K}, \quad 0<\mathcal{E}<350\text{V/cm}. \tag{8}$$

What they measure, however, is not the final velocity of the ions, but an average drift velocity $\langle v\rangle$, defined by

$$\langle v\rangle\equiv L/\tau, \tag{9}$$

where τ is the total time required for the ions to cover the distance L, measured by introducing the ions into the field $\mathcal{E}$ in regularly spaced pulses. For $T\approx1$°K, the normal fluid is almost entirely composed of rotons, whose density is about 9×10^{18} cm^{-3}, greater than that at 0.28°K by roughly a factor of 10^{10}. The mean free path of an ion may be crudely estimated to be 10^{-7} cm. Thus, in these experiments the energy of a charge carrier is in general not $e\mathcal{E}L$. Despite the similarity in experimental arrangements, the experiments of CCM probe a very different physical domain from that of RR.

We describe an experimental run of CCM having the largest range in $\mathcal{E}$, that for positive ions at $T=0.889$°K. The experimental points for $\langle v\rangle$ as a function of the field strength $\mathcal{E}$ are shown in Fig. 3. The most striking feature is the existence of a critical field strength

$$\mathcal{E}_c\approx300 \text{ V/cm} \quad (T=0.889°\text{K}), \tag{10}$$

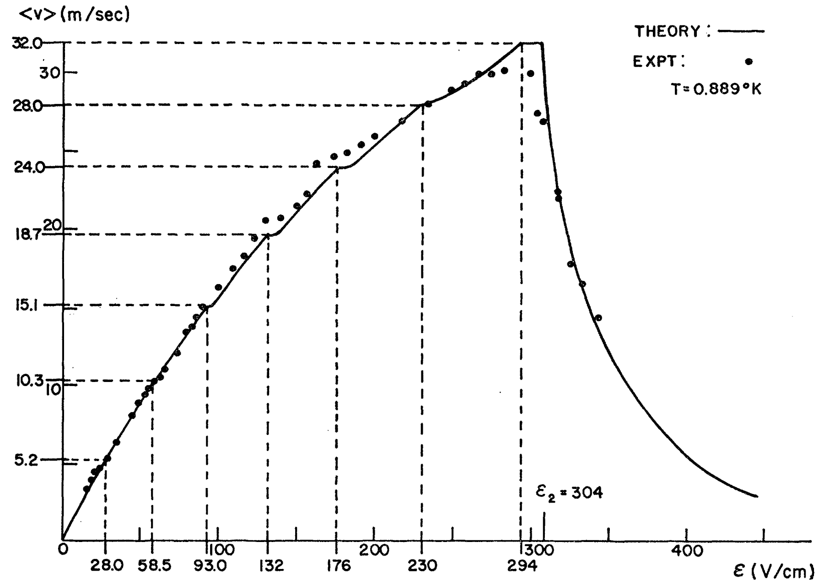


FIG. 3. Average drift velocity as function of electric field. Data of CCM superimposed on theoretical curve calculated from the present model. For $\mathcal{E}<\mathcal{E}_c$, the theoretical curve is a graph of Eq. (35). For $\mathcal{E}>\mathcal{E}_c$ it is a graph of Eq. (37). The only number fitted to the data shown is $v_c=5.2$ m/sec.

which is the dividing line between two drastically different regimes. The value of $\mathcal{E}_c$ is temperature-dependent, as shown by other runs.

For $\mathcal{E}<\mathcal{E}_c$, the average drift velocity $\langle v\rangle$ goes through a series of small discontinuities in slope, of which 5 may be discernible. The first one occurs at the critical velocity

$$v_c=5.2\times10^2\ \text{cm/sec}. \tag{11}$$

The successive ones occur roughly at multiples of v_c, but there is a tendency for them to become more closely spaced. These discontinuities show up in more detail in a plot of the average mobility

$$\langle\mu\rangle\equiv\langle v\rangle/\mathcal{E} \tag{12}$$

as a function of $\langle v\rangle$, as shown in Fig. 4. Other runs of CCM, carried out nearer $T=1$°K, exhibit these discontinuities more convincingly. This particular run is chosen for discussion because it covers both $\mathcal{E}<\mathcal{E}_c$ and $\mathcal{E}>\mathcal{E}_c$. Comparison between different runs at different temperatures shows that v_c is independent of the temperature.

At $\mathcal{E}=\mathcal{E}_c$, the slope of $\langle v\rangle$ undergoes a giant downward discontinuity. For $\mathcal{E}>\mathcal{E}_c$, $\langle v\rangle$ decreases montonically. Observations in this regime has been extended to an electric field of 2000 V/cm,[7] some features of which will be discussed later. One is tempted to make a correspondence between the behavior of $\langle v\rangle$ in this regime and the behavior in Fig. 1 observed by RR. A direct experimental comparison, however, is impossible, because the relation between $\mathcal{E}$ and the energy of the charge carrier is not directly known here. For the same reason, it is not possible to make an immediate connection between these results and quantized vortex rings.

For negative ions, qualitatively the same results are found for $\langle v\rangle$. The parameters are $v_c=2.42\times10^2$ cm/sec, which is independent of temperature, and $\mathcal{E}_c\approx400$ V/cm at $T=0.895$°K.

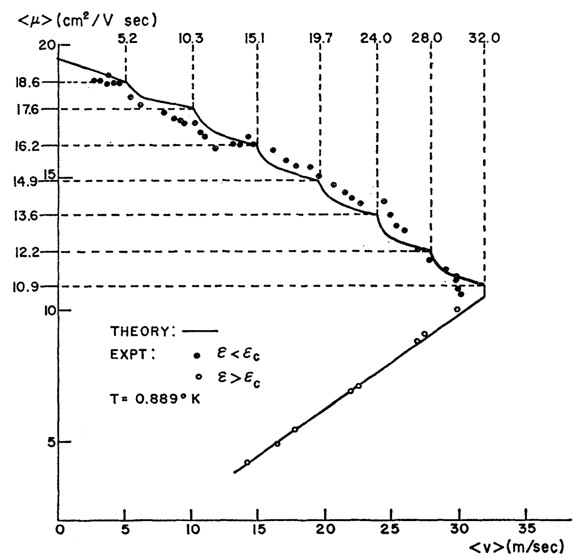


FIG. 4. Average mobility as function of average drift velocity.

C. Reif and Meyer

Reif and Meyer (RM)[8] had earlier carried out experiments of identical type to CCM covering part of the

[7] Report by G. Careri *et al.* at the International Conference on Low Temperature Physics, Columbus, Ohio, 1964 (unpublished).

[8] F. Reif and L. Meyer, Phys. Rev. **119**, 1164 (1960), hereafter referred to as RM.

region

$$0.5<T<0.7°\mathrm{K}\,,\quad 0<\mathcal{E}<100\ \mathrm{V/cm}\,, \tag{13}$$

which lies between the regions of RR and CCM. The measurements were not sufficiently accurate to show any discontinuities of the type seen by CCM, even if they exist. It is, however, interesting to compare the gross features of RM and CCM, for they reveal a general pattern. RM discovered that when $\langle v\rangle$ is plotted against some reduced field quantity, their experimental points fall approximately on a universal curve independent of T. To understand and to interpret such a plot, we first review some theoretical arguments.

Suppose the effect of the normal fluid on an ion of velocity v can be described in terms of a viscous force $-F_2(v,T)$. In an electric field $\mathcal{E}$, the ion would eventually reach a terminal velocity $v_\infty(\mathcal{E},T)$, which is the root of the equation

$$e\mathcal{E}=F_2(v_\infty,T)\,. \tag{14}$$

One expects that

$$v_\infty(\mathcal{E},T)\xrightarrow[\mathcal{E}\to\infty]{} v_0\,,\qquad v_\infty(\mathcal{E},T)\xrightarrow[\mathcal{E}\to 0]{}\mu_0(T)\mathcal{E}\,, \tag{15}$$

where v_0 is a limit independent of T, the critical velocity at which roton creation can occur, and $\mu_0(T)$ is the zero-field mobility. From known roton kinematics, one finds that $v_0=58$ m/sec for an infinitely heavy ion, and is higher for an ion of finite mass; but for an ion mass greater than $10m$, the velocity v_0 is essentially the same as the infinite-mass case. The dominant temperature dependence of $\mu_0(T)$ is expected to be of the form $\exp(\Delta/kT)$ for $T\approx 1°$K, where roton scattering is the main cause of energy loss. The data of RM are consistent with this expectation, although Δ deviates slightly from the value in (7). We take Δ as given in (7) and fit the data of RM by choosing

$$\mu_0(T)=\exp(\Delta/kT-6.76)\ \mathrm{cm^2/V\,sec}\,.\quad (T\approx 1°\mathrm{K})\,. \tag{16}$$

Qualitatively speaking, then, $v_\infty(\mathcal{E},T)$ increases linearly from 0 at $\mathcal{E}=0$, and approaches v_0 as $\mathcal{E}\to\infty$. We may define a characteristic field parameter

$$\mathcal{E}_0(T)\equiv v_0/\mu_0(T)=\exp(15.43-\Delta/kT)\quad \mathrm{V/cm}\,, \tag{17}$$

such that $v_\infty\propto\mathcal{E}$ for $\mathcal{E}\ll\mathcal{E}_0$, and $v_\infty\approx v_0$ for $\mathcal{E}\gg\mathcal{E}_0$. If we plot $v_\infty(\mathcal{E},T)$ against $\mathcal{E}/\mathcal{E}_0(T)$, then the asymptotes of v_∞ for both $\mathcal{E}\to 0$ and $\mathcal{E}\to\infty$ are independent of T.

The measurements of RM and CCM yield $\langle v\rangle$ instead of v_∞. Nevertheless, it might still be interesting to plot their data for $\langle v\rangle$ against $\mathcal{E}/\mathcal{E}_0$. Such a plot is shown in Fig. 5, from which we see that all the points of RM lie approximately on a universal curve. Those of CCM lie approximately on the same universal curve for $\mathcal{E}<\mathcal{E}_c(T)$, but deviate sharply from it for $\mathcal{E}>\mathcal{E}_c(T)$. Thus, a truly universal curve does not exist, but for a given temperature, all data treated in this manner lie approximately on a T-independent curve for $\mathcal{E}<\mathcal{E}_c(T)$. To explain the mechanism for the existence of $\mathcal{E}_c(T)$ is part of the task of our model described later.

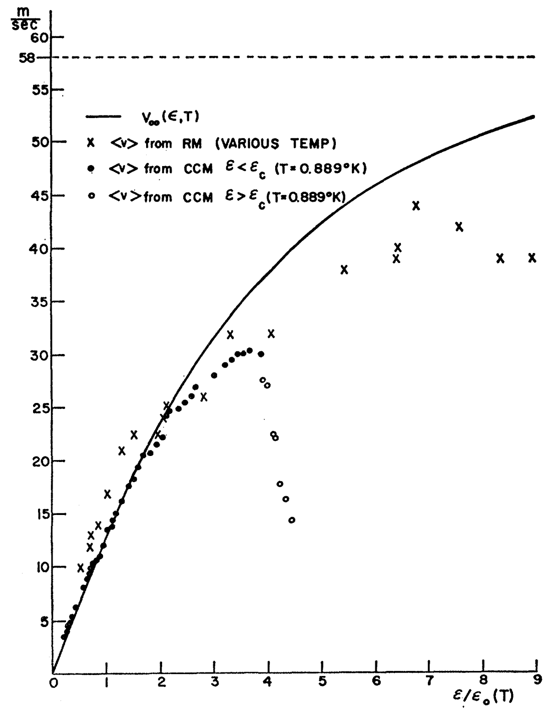


FIG. 5. Comparison between terminal velocity $v_\infty(\mathcal{E},T)$ [Eq. (18)] of ion and its average drift velocity $\langle v\rangle$. The asymptote at 58 m/sec is the minimum velocity required for an ion to create rotons.

It can also be seen from Fig. 5 that as $\mathcal{E}\to\infty$ the data seem to approach an asymptote lying appreciably below v_0. That $\langle v\rangle$ differs slightly from v_∞ is to be expected, for it takes an ion a finite length of time to accelerate from rest to a neighborhood of v_∞; but this difference is completely negligible, because $L\approx 1$ cm, whereas the acceleration distance is of the order of angstroms. One interpretation of the appreciable discrepancy between $\langle v\rangle$ and v_∞ is that there exists another mechanism of energy loss not accounted for by the viscosity due to the normal fluid. We later show that the extra mechanism is the creation of vortex rings, which, when taken into account in a model, enables us to obtain $\langle v\rangle$ in agreement with the data.

In the model we shall describe, we need to know $v_\infty(\mathcal{E},T)$. In the absence of independent experimental data, we crudely interpolate (15) by the simple formula

$$v_\infty(\mathcal{E},T)=v_0[1-\exp(-\mathcal{E}/\mathcal{E}_0)]\,, \tag{18}$$

where $v_0=58$ m/sec and $\mathcal{E}_0(T)$ is given by (17). Substituting (18) into (14), we deduce that to the same approximation the viscous drag due to the normal fluid is

$$F_2(v,T)=-e\mathcal{E}_0(T)\ln(1-v/v_0)\,. \tag{19}$$

3. A MODEL

A. The Idealized Picture

We consider an ion of charge e, effective mass M_{ion}, and effective radius R_{ion}, moving in a constant uniform electric field $\mathcal{E}$, through a macroscopic distance L in liquid helium at temperature T. We take as rough estimates $M_{\text{ion}} \gtrsim 10m$ and $R_{\text{ion}} \gtrsim 10$ Å.[9] Only two temperature ranges will be considered, corresponding to the regions of RR and CCM, and are characterized by vastly different magnitudes of the ion mean free path l:

$$\text{RR region:} \quad T \ll 1^\circ\text{K}, \quad l \gg L.$$
$$\text{CCM region:} \quad T \approx 1^\circ\text{K}, \quad l \lesssim R_{\text{ion}}.$$

We neglect the mutual interactions between ions and consider them one at a time. In the RR region, the only force on the ion is $e\mathcal{E}$; but in the CCM region it experiences additional forces due to collisions with rotons. The motion of ion in the CCM region may be approximately separated into two parts: an average motion and random fluctuations. The average motion may be described by an equation of motion with a viscous force term arising from the scattering and creation of rotons, plus forces due to the superfluid, to be postulated later. The fluctuations have small amplitudes because $l \lesssim R_{\text{ion}}$. We ignore the fluctuations altogether, and in doing so should perhaps replace R_{ion} by a larger effective value. Whenever we refer to the velocity of the ion from now on, we shall mean the velocity of the average smooth motion.

As far as the superfluid is concerned, the ion is regarded as a time-dependent boundary condition on the superfluid wave function Ψ_0, e.g., $\Psi_0 = 0$ on the surface of a moving sphere. The creation of rotons from the superfluid, which has an important effect on the motion of the ion in the real problem, is taken into account only in so far as it contributes to the viscous force acting on our idealized ion.

A model based on the picture adopted can be valid only if T lies in some small neighborhood of $T=0$ or $T=1^\circ$K. Otherwise the basis for the picture fails, either because $R \ll l \ll L$, in which case we cannot neglect the random fluctuations in the motion of the ion, or because the normal fluid becomes so dense that the quasiparticle model of liquid helium breaks down.

B. Creation of Vortex Ring by Ion

How does a moving boundary condition on the superfluid wave function affect the flow of the superfluid? It seems reasonable to assume that normally the superfluid flows in streamlines around the ion; but when certain conditions on the velocity of the ion are met, the ion is capable of exciting distinctive flow patterns such as vortex flow. We make some phenomenological postulates guided by the experiments described previously.

In CCM, the near regularity with which the slope of $\langle v \rangle$ changes discontinuously points to the interpretation that an ion can create a vortex only if $v_{\text{ion}} = nv_c$, $(n=1, 2, 3, \cdots)$. (It must be rembered that $v_{\text{ion}} \neq \langle v \rangle$.) It is reasonable to assume that at creation the vortex ring has a radius solely determined by the ion radius, and that it moves with the same velocity as the ion. Since for fixed radius the translational velocity of a vortex ring is quantized, we are forced by the previous assumptions to conclude that an ion moving at nv_c can only create a vortex ring of circulation nh/m, which for brevity shall be called an n ring.

To put these ideas in a suggestive form, we hypothesize that an ion can create vorticity only if the circulation about its circumference has the right value. More precisely we assume that an ion of velocity v_{ion} and radius R_{ion} can create an n ring only if

$$\theta v_{\text{ion}} R_{\text{ion}} = nh/m\,, \quad (n=1, 2, 3, \cdots) \tag{20}$$

and that at creation the vortex ring moves with the ion. Here θ is a dimensionless number independent of the temperature. Assuming that (20) is satisfied for $v_{\text{ion}} = v_c$ and $n=1$, we obtain $v_c = (h/m\theta R_{\text{ion}})$, or $\theta R_{\text{ion}} = 190$ Å for positive ions, 410 Å for negative ions. Hence $R_{\text{ion}}^+/R_{\text{ion}}^- = 0.46$. If we put $R_{\text{ion}}^+ \approx 10$ Å, we find $\theta \approx 6\pi$. We later suggest a direct experimental test of (20). (See Sec. 4A.)

The assumption that at creation an n ring moves with velocity nv_c enables us to calculate its radius and energy. For creation by a positive ion the results are

$$\begin{aligned} v &= nv_c = 5.2n \text{ m/sec}\,, \\ R &= 98 \text{ Å} \quad (\text{independent of } n)\,, \\ E &= 0.22n^2 \text{ eV}\,. \end{aligned} \tag{21}$$

As we can see the energy is so large that the only possible source seems to be the external electric field. This suggests the following picture of the creation process. To create an n ring, an ion must have a terminal velocity $v_\infty > nv_c$, and the creation process must take place before the velocity reaches v_∞, and after it passes $(n-1)v_c$. As the ion accelerates towards v_∞, it reaches nv_c at some time, when it begins to create "turbulence" in the superfluid. The ion remains at the constant speed nv_c to supply energy to the growing turbulence at the rate $(e\mathcal{E} - F_2)nv_c$, where F_2 is the viscous force due to the normal fluid. When the total energy imparted to the superfluid equals $0.22n^2$, the turbulence in some way becomes an n ring.

For a typical field strength $\mathcal{E} = 100$ V/cm, the creation process lasts $4.2 \times 10^{-6}n$ sec, during which the ion traverses a distance of $2{,}200n$ Å. These numbers seem reasonable, for an ion should not be able to bring about

[9] The precise definitions and values for M_{ion} and R_{ion} are not important to our development. For theoretical results on the structure of ions in liquid helium and for references see K. R. Atkins, in C. Careri, *Liquid Helium, Proceedings of the International School of Physics "Enrico Fermi"* (Academic Press Inc., New York, 1963), p. 403 ff.

an orderly flow pattern without pushing a large number of atoms around.

C. Capture of Vortex Ring by Ion

The experiments of RR strongly suggest that ions are captured by vortex rings to form a bound state, referred to as a charged vortex ring. The capture probability is not precisely known, but estimated to be near unity.[6] We take this to mean that an ion is almost always captured by the vortex ring it created. RR suggest that the mechanism for the capture might be the trapping of the ion in the vortex core, which should appear as an attractive potential to the ion, by virtue of the high superfluid velocity and consequently low pressure near the core. If we adopt this picture, then after creation the vortex ring captures the ion by a kind of hydrodynamic suction. The time between creation and capture, on dimensional grounds, should be R/c, where $R\approx 100$ Å and c is the sound velocity, or about 10^{-11} sec.

In the RR region, where one may neglect any energy loss, the charge vortex ring moves in the electric field essentially undisturbed by the normal fluid. When there is energy dissipation, as in the CCM region, it is strongly affected by the normal fluid. In the latter case we have to investigate whether or not the charged vortex ring can be stable.

As mentioned earlier, the semiempirical formula (6) for the viscous force acting on a charged vortex ring ignores the presence of the ion, which should contribute an extra term due to its scattering and creation of rotons. We take this additional term to have the same form as that for an isolated ion, but reduced in magnitude, so that the total viscous drag on a charged vortex ring is

$$F(v,T)=F_1(v,T)+\xi F_2(v,T)\,, \tag{22}$$

where ξ is an undetermined parameter, and F_1, F_2 are given, respectively, by (6) and (19). It would be helpful if experiments can be carried out to test (22) directly, and to determine the temperature dependence of ξ. For the present, we can only guess that $\xi\approx\frac{1}{3}$,[10] which leads to the neat form

$$F(v,T)=n\alpha(T)[\eta-\tfrac{1}{4}-n^{-1}\ln(1-v/v_0)]\,, \tag{23}$$

where η, as a function of v, was tabulated earlier in Table I. Equation (23) does not contradict the experiments of RR verifying (6), for in their experiments $v\ll v_0$ and $\eta\gg 1$, and hence the last term is negligible.

Steady-state motion for a charged vortex ring is possible only if $e\mathcal{E}=F$, or

$$e\mathcal{E}=n\alpha(T)[\eta-\tfrac{1}{4}-n^{-1}\ln(1-v/v_0)]\,. \tag{24}$$

The right-hand side, as a function of v, has a minimum at v_1, which is the root of the equation

$$\frac{v_0}{v_1}=1+\frac{1}{n}\frac{\eta_1-1/4}{\eta_1-5/4}\,,\quad \eta_1\equiv\eta(v_1)\,, \tag{25}$$

and may be found by graphical solution. Thus (24) cannot be satisfied unless $\mathcal{E}>\mathcal{E}_c(T)$, where

$$\mathcal{E}_c(T)=\eta\alpha(T)[\eta_1-\tfrac{1}{4}-n^{-1}\ln(1-v_1/v_0)]\,. \tag{26}$$

If $\mathcal{E}>\mathcal{E}_c(T)$, the charged vortex ring can (and we assume will) adjust its velocity to a value required by (24), whatever its initial velocity. It is assumed that a stable charged vortex ring does not create turbulence in the superfluid, whatever its velocity. If $\mathcal{E}<\mathcal{E}_c(T)$, the charge vortex ring always suffers energy loss, which tends to accelerate a vortex ring but decelerate an ion. Thus, after a short time, the bound state would be pulled apart and the freed ion quickly accelerates or decelerates to the terminal velocity $v_\infty(\mathcal{E},T)$ for an isolated ion.

The critical field $\mathcal{E}_c(T)$ is roughly proportional to n. For later use we need to consider only $n=1$. In Fig. 6 is shown a graph of $(\eta-\frac{1}{4})-\frac{1}{2}\ln(1-v/v_0)$ against v, from which we obtain, for $n=1$,

$$\mathcal{E}_c(T)=304\exp[-(\Delta/kT)+(\Delta/.889k)]\ \text{V/cm}\,,\quad v_1=33\ \text{m/sec}\,. \tag{27}$$

The velocity v_1 is independent of the temperature.

For $\mathcal{E}>\mathcal{E}_c(T)$ there are two values of v satisfying (24), of which only the lower one can lead to a stable charged vortex ring, (except if $\mathcal{E}\approx\mathcal{E}_c$, in which case it matters little which root is chosen). The higher root generally corresponds to $R<10$ Å, and should be rejected on the ground that the radius is too small either for the vortex ring to bind the ion, or for the vortex ring to exist at all. In either case, the ion will be released and quickly accelerate or decelerate to the terminal velocity $v_\infty(\mathcal{E},T)$, and will not be detected in the experiment. The possibility of a choice of roots, however, reduces the effective capture probability of an ion by a vortex ring in the CCM region.

In summary we adopt the following picture. Shortly after an ion creates a vortex ring, the later captures it to form a charged vortex ring with a capture probability essentially unity. If $\mathcal{E}>\mathcal{E}_c(T)$, and $F\neq 0$, the velocity of the charged vortex ring quickly adjusts to the smaller root of (24), which then gives the $v-\mathcal{E}$ relation of a stable charged vortex ring. For $n=1$, the portion of the

[10] As a guide to an estimate of ξ, we consider the limit of small velocities, where $F_2\approx ev/\mu_0$. The reduction of F_2 may be attributed to the increase in the mobility μ_0. Assuming $\mu_0\propto(M_{\text{ion}}\sigma)^{-1}$, where M_{ion} is the effective mass and σ the scattering cross section for rotons, we see that ξ is the factor by which $M_{\text{ion}}\sigma$ is reduced. An ion trapped in the vortex core loses that part of M_{ion} due to displacement of the superfluid, which amounts to roughly $\frac{1}{2}$ the "mechanical mass." Hence M_{ion} is reduced by roughly $\frac{2}{3}$. The reduction of σ may be estimated from L. Meyer and F. Reif, Phys. Rev. **123**, 727 (1961), which gives measurements of μ_0 at different pressures and temperatures. The reduction factor of σ is the ratio of μ_0 at zero pressure to that at the helium vapor pressure, at $T\approx 1$°K. The result is uncertain because we have to extrapolate the data to zero pressure, and μ_0 falls off extremely rapidly with decreasing pressure. As an order-of-magnitude estimate we take $\frac{1}{2}$ as the reduction factor for σ. This leads to $\xi\approx\frac{1}{3}$.

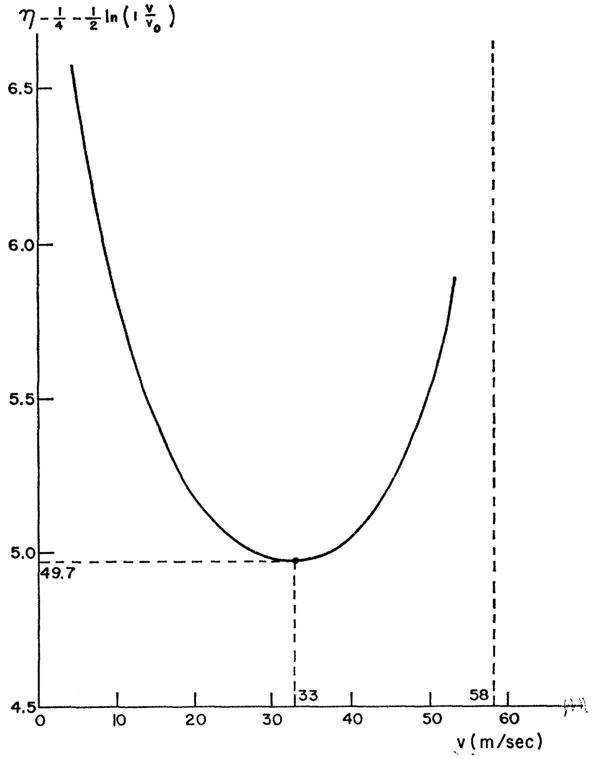


FIG. 6. The function plotted is equal to $e\mathcal{E}/\alpha(T)$, where $\mathcal{E}$ is the electric field in which a charged vortex ring of $n=1$ can be stable, with steady state velocity v. The branch to the right of the minimum should be discarded because it yields too small a radius for the charged vortex ring to be well-defined.

curve in Fig. 6 for $v<v_1$ is a graph of $e\mathcal{E}/\alpha(T)$ versus v. If $\mathcal{E}<\mathcal{E}_c(T)$ and $F\neq 0$, then the charged vortex breaks up, freeing the ion to settle to the terminal velocity $v_\infty(\mathcal{E},T)$, while the ring disappears owing to energy loss.

The adjustment times involved in the processes mentioned above are all negligible. The time constant for an ion moving under the force $e\mathcal{E}-F_2$ is $M_{\text{ion}}\mu_0(T)/e \approx 10^{-10}$ sec at about 1°K. The time between the creation of a vortex ring by an ion and its capture by the former was estimated earlier to be $R/c\approx 10^{-11}$ sec. With an energy loss of $F(v,T)$ per unit distance traveled, the equation of motion for a neutral vortex ring is

$$\frac{dv}{dt}=\frac{8\pi v^3(\eta-5/4)F(v,T)}{\rho(nh/m)^3(\eta-3/4)(\eta-1/4)^2}. \tag{28}$$

At $T\approx 1$°K, the time it takes for a neutral vortex ring of 100 Å ($v=v_c$) to shrink to 1 Å, at which it disappears, is of the order of $\rho(h/m)^3[8\pi v_c^2\alpha(T)]^{-1}\approx 10^{-8}$ sec. The time it takes for an unstable charged vortex ring to break up would be much smaller, because the breakup process is hastened by the opposite tendency of the ion to decelerate instead of accelerate. We shall neglect these adjustment times in comparison to the total transit time of an ion through $L\approx 1$ cm, about 10^{-3} sec; or to the time required for the creation of a vortex ring, about 10^{-6} sec.

D. Summary of the Model

To summarize the considerations presented so far we state a series of assertions, in the nature of a recipe, that serves to describe phenomenologically the behavior of an ion in liquid helium. We deal only with steady state situations and neglect all transient times.

(a) In its average motion an isolated ion can be described by an equation of motion

$$M_{\text{ion}}\dot{v}_{\text{ion}}=e\mathcal{E}-F_2(v_{\text{ion}},T), \tag{29}$$

except whenever $v_{\text{ion}}=nv_c$, at which the ion experiences additional drag due to the creation of "turbulence" in the superfluid.

(b) Whenever $v_{\text{ion}}=nv_c$, the ion remains at this constant speed, converting electrical energy to the energy of "turbulence" at the rate $(e\mathcal{E}-F_2)nv_c$, until such time that the total energy acquired by the turbulence becomes equal to $0.22n^2$ eV, whereupon an n ring is created.

(c) For $\mathcal{E}>\mathcal{E}_c(T)$, where $\mathcal{E}_c(T)$ is given by (26), the vortex ring immediately captures the ion to form a charged vortex ring, whose velocity then assumes a value solely determined by $\mathcal{E}$ and T, i.e., the smaller root of the Eq. (24). For $\mathcal{E}<\mathcal{E}_c(T)$, the created n ring does not capture the ion and eventually disappears, while the ion resumes its motion in accordance with (29) with initial velocity nv_c.

4. INTERPRETATION OF EXPERIMENTS

A. Rayfield and Reif

In the RR region we can neglect the presence of the normal fluid. Hence the ion essentially moves in a pure superfluid at $T=0$°K. Let us follow a positive

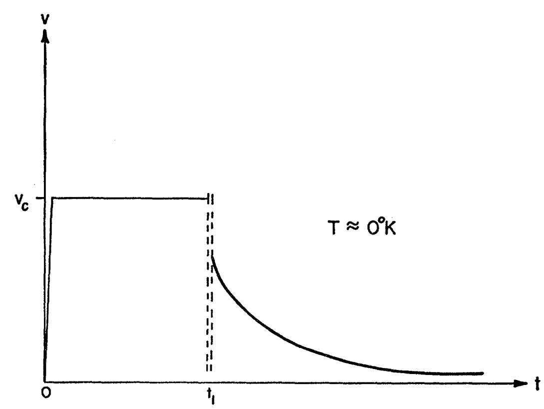


FIG. 7. History of an ion traveling through a pure superfluid in a constant uniform electric field. Between $t=0$ and $t=t_1$ it creates turbulence, which at t_1 becomes a vortex ring of $n=1$. The latter immediately captures the ion to form a charge vortex ring. For $t>t_1$ the ion remains bound to the vortex ring.

ion through the liquid in a field of 10 V/cm. The ion velocity is sketched as a function of time in Fig. 7. It accelerates from rest, and reaches $v_c = 5.2$ m/sec after a negligible time, whereupon it begins to create turbulence in the superfluid. Remaining at this velocity, it supplies energy to the growing turbulence at the rate of 5.2×10^3 eV/sec. After $t_1 = 4.3\times10^{-5}$ sec, a 1 ring is created with energy 0.22 eV, and radius 98 Å. The vortex ring immediately captures the ions to become a charged vortex ring, which is stable because there is no energy loss. The charged vortex ring contains all the energy absorbed by the ion from the electric field up to this moment, and therefore has an energy, radius, and velocity uniquely determined by its position. Hereafter the charged vortex ring decelerates as it gains energy in the electric field. When it reaches the end of L its velocity is related to its energy $e\mathcal{E}L$ in a way characteristic of a free vortex ring, for the kinetic energy of the ion is negligible. Due to the absence of energy loss, the ion is always captured, and remains captured, by the first ring it creates. Thus, only charged vortex rings of $n=1$ can be created, no matter how large the electric field.

The distance traveled by the ion during the creation of a 1-ring is $0.22/\mathcal{E}$, where $\mathcal{E}$ is in V/cm. To put the basic hypothesis of our model to direct experimental test, one might repeat the experiments of RR at such a small field that $0.22/\mathcal{E} > L$, or $\mathcal{E} < 0.73$ V/cm, for $L = 0.3$ cm. Under this circumstance the creation process is never completed, and the ion should always arrive at the detector with the unique velocity v_c. One might even be able to study the trail of turbulence created by the ion.

B. Careri, Cunsolo, and Mazzoldi

When energy loss is large, as in the CCM region, the behavior of an ion is very different from that in the RR region, because (a) the terminal velocity $v_\infty(\mathcal{E},T)$ can be fairly low, and (b) a charged vortex ring is unstable unless $\mathcal{E} > \mathcal{E}_c(T)$, where $\mathcal{E}_c(T)$ is fairly high. Let us follow the motion of an ion through liquid helium at $T \approx 1°$K in a constant uniform field $\mathcal{E}$. It is important to remember that $\mathcal{E}$ is fixed for any run of the experiment. Changing the value of $\mathcal{E}$ means that we repeat the whole experiment with a different constant uniform field.

Consider first $\mathcal{E} < \mathcal{E}_c(T)$. In Fig. 8 we sketch the velocity v of the ion as a function of time for three different values of $\mathcal{E}$. Figure 8(a) shows what happens if $\mathcal{E}$ is such that $v_\infty(\mathcal{E},T) < v_c$. The ion accelerates almost instantly to v_∞ and keeps this velocity. No vortex ring can be created because v never reached v_c. Figure 8(b) corresponds to a value of $\mathcal{E} < \mathcal{E}_c(T)$ such that $v_c < v_\infty(\mathcal{E},T) < 2v_c$. The ion accelerates up to v_c, and remains at this velocity for a time t_1 to create a 1 ring. Since for this field strength a charged vortex ring is unstable, the ion is not captured into a stable bound

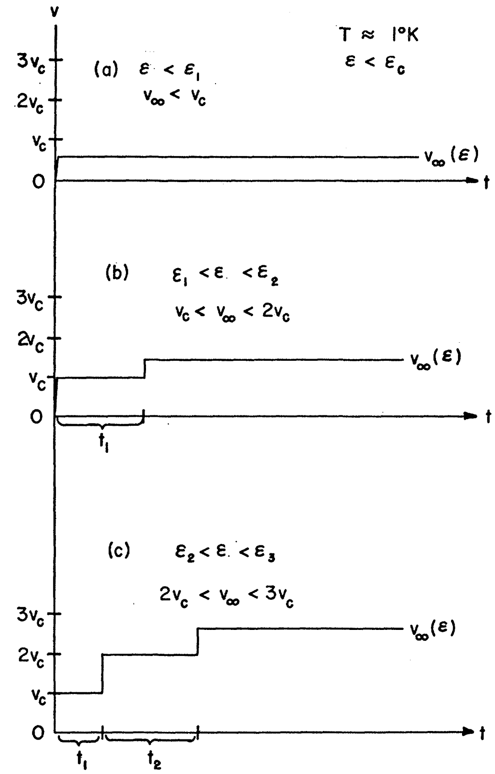


FIG. 8. History of an ion traveling through liquid helium at $T \approx 1°$K in a constant uniform electric field, which is less than the critical field required for a charged vortex ring to be stable. The ion creates a total of n vortex rings if $\mathcal{E}_n < \mathcal{E} < \mathcal{E}_{n+1}$, but none of them can capture the ion into a stable bound state.

state. Almost immediately after t_1, it resumes the acceleration to v_∞. The 1 ring created is very short-lived because of energy loss. Figure 8(c) should by now be self-explanatory. The ion creates a total of two vortex rings, a 1 ring at $v = v_c$, and a 2 ring at $v = 2v_c$, neither of which captures the ion.

We now present a quantitative description. Let $\mathcal{E}_n$ be defined by

$$v_\infty(\mathcal{E}_n,T) \equiv nv_c. \tag{30}$$

Using (18) we obtain

$$\mathcal{E}_n = -\mathcal{E}_0(T)\ln[1-(nv_c/v_0)]. \tag{31}$$

Comparison with (19) shows that $e\mathcal{E}_n = F_2(nv_c,T)$. Consider an ion moving at velocity nv_c, in the process of creating turbulence which will eventually become an n-ring. The power loss due to the viscous force F_2 is $nv_ce\mathcal{E}$. Therefore the power supplied to the turbulence is $nv_ce(\mathcal{E}-\mathcal{E}_n)$. Since by (21) the energy of an n ring at creation is $0.22\,n^2$ eV, the time required to create an n ring is

$$t_n(\mathcal{E},T) = nE_0/[v_ce(\mathcal{E}-\mathcal{E}_n)], \quad E_0 = 0.22 \text{ eV}. \tag{32}$$

Consider the range of electric field $\mathcal{E} < \mathcal{E}_c(T)$ and $\mathcal{E}_n < \mathcal{E} < \mathcal{E}_{n+1}$, for which $nv_c < v_\infty < (n+1)v_c$. In such a

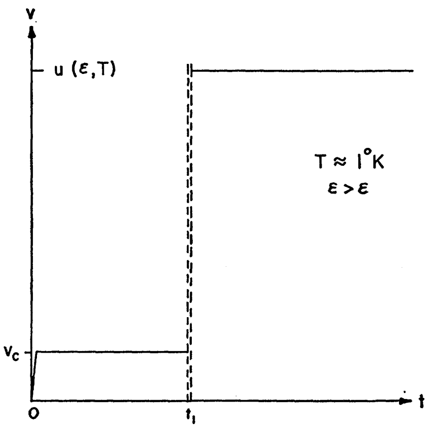


FIG. 9. Same as Fig. 8 except that the electric field is high enough for a charged vortex ring to be stable. The ion is captured at t_1 into a stable bound state by the first and only vortex ring it created. After t_1 the charged vortex ring moves with a steady state velocity u determined solely by $\mathcal{E}$ and T.

field the ion successively creates n vortex rings, and the total distance L traveled by the ion is related to the total transit time τ by

$$L=(t_1+2t_2+\cdots+nt_n)v_c+[\tau-(t_1+t_2+\cdots+t_n)]v_\infty, \quad (33)$$

or

$$\tau=v_\infty^{-1}\{L+(E_0/v_c)\sum_{l=1}^{n}[l(v_\infty-lv_c)/(\mathcal{E}-\mathcal{E}_l)]\}. \quad (34)$$

The average drift velocity is therefore given by[11]

$$\langle v\rangle=v_\infty[1+(E_0/e\mathcal{E}L)(v_\infty/v_c)g_n(\mathcal{E},T)]^{-1}, \quad (\mathcal{E}<\mathcal{E}_c, \text{ and } \mathcal{E}_n<\mathcal{E}<\mathcal{E}_{n+1}), \quad (35)$$

where

$$g_n(\mathcal{E},T)=\sum_{l=1}^{n}[l(1-lv_c/v_\infty)/(1-\mathcal{E}_l/\mathcal{E})]. \quad (36)$$

Equation (35) is valid for $\mathcal{E}<\mathcal{E}_c(T)$, corresponding to $\langle v\rangle<v_1$, where v_1 is about 33 m/sec, and is independent of the temperature.[12]

For $\mathcal{E}>\mathcal{E}_c(T)$, a charged vortex ring is stable. Therefore, an ion is always captured by the very first ring created, which has $n=1$. In Fig. 9 we sketch the ion velocity as a function of time for this case. The ion accelerates rapidly to v_c, at which it creates a 1 ring, after maintaining this velocity for a time t_1. It is then immediately captured by the vortex ring to form a charged vortex ring, which has a unique velocity $u(\mathcal{E},T)$ determined solely by $\mathcal{E}$ and T. The charged vortex ring moves at the constant velocity $u(\mathcal{E},T)$ hereafter. The time t_1 may be neglected compared to the total transit time. Hence

$$\langle v\rangle=u(\mathcal{E},T), \quad (\mathcal{E}>\mathcal{E}_c) \quad (37)$$

where $u(\mathcal{E},T)$ is the smaller root of the equation

$$\ln[8R(u)/a]-\ln(1-u/v_0)-\tfrac{1}{4}=e\mathcal{E}/\alpha(T), \quad (38)$$

where $R(u)$ is the radius of a vortex ring of translational velocity u, and is tabulated in Table I.

To compare with the experimental results of CCM, we calculate $\langle v\rangle$ at $T=0.889$°K. At this temperature the relevant parameters are, by the various formulas given earlier

$$\begin{aligned}\alpha&=63\text{ eV/cm},\\ \mu_0&=19.5\text{ cm}^2/\text{V sec},\\ \mathcal{E}_0&=189\text{ V/cm},\\ \mathcal{E}_c&=300\text{ V/cm},\\ v_1&=33\text{ m/sec}.\end{aligned} \quad (39)$$

Table II gives the calculated values of $\mathcal{E}_n$, and $\langle v\rangle$ at $\mathcal{E}=\mathcal{E}_n$, which agree quite well with the corresponding experimental values shown. Note that the values of $\langle v\rangle$ at $\mathcal{E}=\mathcal{E}_n$ are not equally spaced. The model predicts that for $\mathcal{E}<\mathcal{E}_c$ the ion creates up to six vortex rings, but present data are not sufficiently accurate to check this number. A complete theoretical curve of $\langle v\rangle$ for $0<\mathcal{E}<400$ V/cm is shown in Fig. 3, with the experimental points of CCM on the same graph. A similar plot of $\langle\mu\rangle$ versus $\langle v\rangle$ is shown in Fig. 4.

The behavior of the theoretical curve in Fig. 3 in the neighborhood of $\mathcal{E}=\mathcal{E}_c$ may be explained as follows. At $\mathcal{E}=\mathcal{E}_c-\epsilon$, the ion created a succession of six rings, and was in the process of creating a seventh when it reached the detector. At $\mathcal{E}=\mathcal{E}_c+\epsilon$, the ion was captured by the first and only ring it created, and the resulting charged vortex ring adjusted its velocity to the electric field present. In Fig. 3, the theoretical value for $\langle v\rangle$ should jump discontinuously from 32 m/sec to 33 m/sec at $\mathcal{E}=\mathcal{E}_c$. On the high side of $\mathcal{E}=\mathcal{E}_c$ the slope of $\langle v\rangle$ is infinite.

TABLE II. Comparison between theory and experimental results of CCM at $T=0.889$°K. $\mathcal{E}_n$ is the field at the nth discontinuity of slope of $\langle v\rangle$.

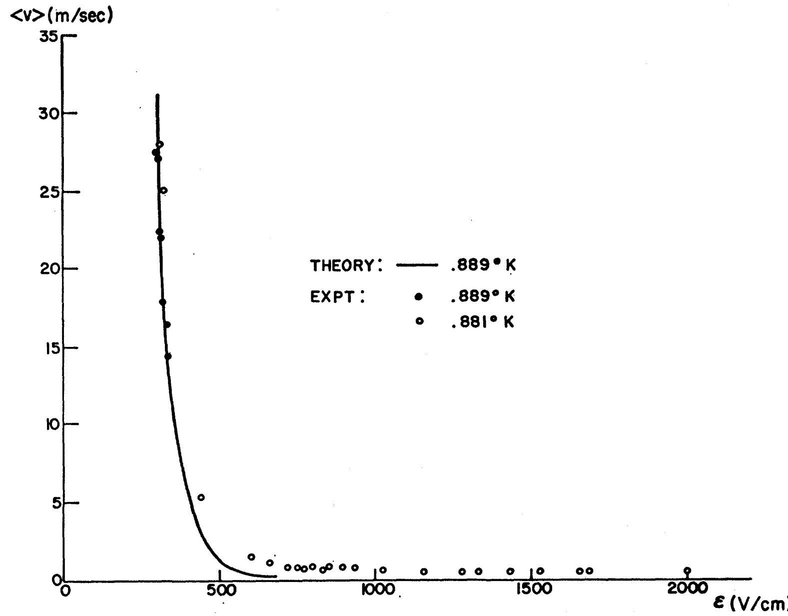
	$\mathcal{E}_n$ (V/cm)		$\langle v\rangle$ (m/sec)	
n	Theory	Expt.	Theory	Expt.
1	28.0	28	5.2[a]	5.2
2	58.5	60	10.3	10.3
3	93.0	93	15.1	15.0
4	132	130	19.7	20.6
5	176	180	24.0	24.5
6	230	b	28.0	b
7	294	b	32.0	b

[a] Fitted to experiment.
[b] Insufficient accuracy.

[11] Equation (33) does not hold if $\mathcal{E}$ is such that $\sum_{l=1}^{n-1}t_l<\tau<\sum_{l=1}^{n}t_l$. These values of $\mathcal{E}$ are contained in small disjoint intervals described by $\mathcal{E}=\mathcal{E}_n+\delta$, $(0\leqslant\delta\ll\mathcal{E}_n)$. In these intervals $\langle v\rangle$ is not given by (35), but should be independent of $\mathcal{E}$, and of such a value that the entire $\langle v\rangle-\mathcal{E}$ curve for $\mathcal{E}<\mathcal{E}_c(T)$ is continuous. Existing experiments are not sufficiently accurate to reveal these details.

[12] The prediction that v_1 is independent of the temperature has been verified by Careri in recent experiments (private communication).

FIG. 10. Average drift velocity at very high electric fields. The theoretical curve is Eq. (37), giving the steady-state velocity of a charged vortex ring of $n=1$. Discrepancy between theory and experiment in the region $400<\mathcal{E}<800$ V/cm is probably due to underestimation of the energy loss of the charged vortex ring. Discrepancy for higher fields is probably due to additional effects involving the walls of the helium chamber, for at $\mathcal{E}=1200$ V/cm, theory predicts that the charged vortex ring has a radius of 1 cm.



In Fig. 10 we show a theoretical curve of $\langle v\rangle$ versus $\mathcal{E}$ for $400<\mathcal{E}<2000$ V/cm, together with some data of Careri *et al.*[7] There is considerable discrepancy between theory and experiment, especially for $\mathcal{E}>1200$ v/cm. To understand these discrepancies, note that at these high fields the vortex ring has a very large radius, which, according to (38) increases exponentially with $\mathcal{E}$. It seems plausible that for a very large radius, the energy loss is greater than that given by (23). The reason is that a large ring is surrounded by a smaller velocity field, and hence a medium of higher pressure. (Cf. footnote 10). Hence the function $\alpha(T)$ in (23) should increase slowly with R.

The data shown in Fig. 10 indicates the existence of a minimum in $\langle v\rangle$ at about $\mathcal{E}=1200$ V/cm. Beyond that point $\langle v\rangle$ rises steadily with $\mathcal{E}$. It can be readily verified that according to the present model the radius of the charged vortex ring is 0.1 mm at $\mathcal{E}=800$ V/cm, but increases to 1 cm at $\mathcal{E}=1200$ V/cm. Here a new effect due to the containing walls of the helium chamber must enter. We shall not discuss this effect, because it may involve the geometry of the helium chamber used in the experiment.

As mentioned in Sec. 3A, the present model is not expected to be correct outside of small neighborhoods of $T=0$°K and $T=1$°K. Indeed, it is incorrect for $T\lesssim 0.7$°K, as shown by Fig. 5. According to the model, $\mathcal{E}_c(T)/\mathcal{E}_0(T)$ should be independent of temperature, but Fig. 5 shows that this is not so when $T\lesssim 0.7$°K. Another reflection of the breakdown of the model outside of a narrow range of temperatures about 1°K is the phenomenona of "metastability" observed by CCM when thermal disturbances are present in their apparatus. We believe, however, that the basic hypothesis (20) remains valid throughout $0\leqslant T\lesssim 1$°K, but the finite mean-free-path of the ion has to be taken into account properly.