Theory of the Yield and Fano Factor of Electron-Hole Pairs Generated in Semiconductors by High-Energy Particles

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Yield of secondaries and ratio of variance to yield (Fano factor) are associated with a simple statistical model. This model applies almost independently of details of the physics, though introduction in it of refinements from band structure would be straightforward. The model is described in terminology of an equivalent problem which is formulated and called "crazy carpentry." Analytical and Monte Carlo results agree well. The statistics are largely determined at secondary energies of just a few ionization thresholds, above which yield increases linearly with energy, with an apparent threshold the same as the average final energy, or energy of a secondary when first it cannot ionize further. Limiting relative yield, equal to ionization threshold divided by average energy per electron-hole pair, and Fano factor are given in their dependence on losses to optical phonons. The limiting case of given relative energy loss to very many phonons of very small energy generally provides close approximations, so that the results apply also whatever kinds of phonons are produced. Illustrative simple cases are also treated, including the one-dimensional space-filling problem known as the "parking problem." The theoretical yield is used to fit data of Vavilov on the enhanced quantum yield in silicon, and some physical implications are considered.

1. INTRODUCTION AND DISCUSSION

HIGH-ENERGY particles incident on a semiconductor produce secondary electrons and holes through a branching process. With this absorber sufficiently thick to stop all the incident particles, the yield of secondaries serves to determine initial energy. The accuracy of this determination is limited by the variance of the yield, or the mean-square departure from the mean of yields in repeated measurements.[1] The ratio[2] of variance to yield, or Fano factor, equals unity for the case of the Poisson distribution, which applies if the final ionizations are essentially independent events. This independence may in principle come about through loss mechanisms in the generation process of a type resulting in expected yield that is relatively quite small; the Poisson distribution is that for fixed expectation value of an outcome of very small probability in very many trials. But loss mechanisms are in practice not of this type, and interdependence rather than independence is the essential stochastic character of the ionizations in a thick absorber. Fano factors are accordingly less than unity, values less than 0.5 having been observed in silicon, for example, by Blankenship and Mruk[3] and by others.

Yield, variance, and Fano factor are, indeed, associated with a simple phenomenological statistical model.[4] This model applies almost independently of details of the physics. More exact treatment based on band structure would yield only refinement of numerical results and may suitably be left as a separate and supplementary problem.[5] The model is conveniently described in the terminology of an equivalent problem which is formulated and called "crazy carpentry." It is shown that the crazy-carpentry model accounts for Fano factors considerably smaller than unity and is consistent with the proportionality of observed yield to incident energy over a wide range, independently of the nature of the incident particles. It confirms the surmise that yield and Fano factor must both be strongly influenced simply by the circumstance that all the energy is absorbed in the semiconductor. That is, when the branching process has continued until most of the secondaries have energies of only a few to several times the ioniza-

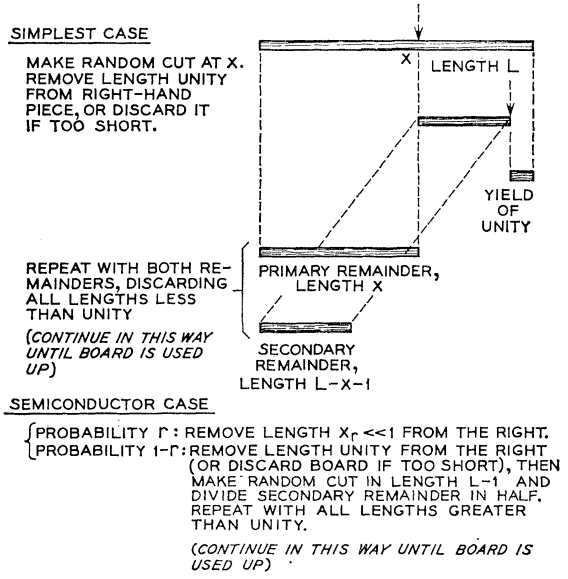

FIG. 1. Rules for two cases of crazy carpentry, of which more formal statements are given in Secs. A1 and 2.3. For the semiconductor case, secondary remainders may be divided in unequal ratio; the equal ratio specified represents the simplest assumption.

[1] Observed full linewidth at half-maximum for Gaussian line shape is 2.35 times the square root of the variance.
[2] U. Fano, Phys. Rev. 72, 26 (1947).
[3] J. L. Blankenship and W. F. Mruk, Bull. Am. Phys. Soc. 9, 49 (1964).
[4] W. van Roosbroeck, Bull. Am. Phys. Soc. 9, 274 (1964).
[5] Such treatment would entail elaborate calculation to obtain certain more exact probability distributions whose introduction into the model in place of simple idealized distributions would be entirely straightforward.

tion threshold, then there is in effect an energy constraint. This results in appreciable generation of further secondaries of energies less than the ionization threshold, so that the average yield of secondaries per collision drops. Thus, yield and Fano factor are determined largely in the final stages of the branching process when, moreover, most of the secondaries are generated. Most of the separate ionizations must therefore be considered as events that are interdependent in a certain specific manner.

Crazy carpentry is explained in Fig. 1. A simple case for secondaries of one kind only is considered first. Board length L represents incident energy in units of the ionization threshold. Boards of unit length are obtained by the ultimately wasteful procedure of first making a random cut, then cutting a board of unit length from the right-hand piece, if this is long enough, then making random cuts in the two remainders, and so on, discarding all lengths less than unity, until the board is used up. The right-hand length represents energy absorbed in a collision, namely, a random fraction of the incident energy; the left-hand length, or primary remainder, represents energy left in the incident particle; the unit length obtained represents absorption of the ionization threshold energy, hence yield of a secondary; and the resulting secondary remainder represents energy of this secondary. It is evident that for large L, the waste is negligible in the initial stages, and becomes appreciable only when the remainders are mostly just a few units in length.

For semiconductors, this simple case is extended to take into account electron and hole secondaries and losses to optical phonons. Then, as shown at the bottom of the figure, crazy carpentry involves probability r that a small piece of length x_r is cut off the right-hand end. This cut represents generation of an optical phonon, $x_r \equiv E_R/E_i$ being the Raman phonon energy divided by the ionization threshold energy. Length unity is removed from the right-hand end with probability $1-r$, representing yield of a secondary, and a random cut is made. The resulting secondary remainder is now, under the simplest assumption, divided in half, representing equal energies for the secondary electron and hole. Probability r properly applies, of course, only for lengths greater than unity, since it is the probability in a collision of optical phonon generation by an energetic parricle that can produce secondaries.

Two approaches to the crazy-carpentry problems are employed: Analysis based on mathematical formulation; and the Monte Carlo method consisting in having a computer do the crazy carpentry itself. With principal interest attaching to expected yield and variance, or to the first two moments of the distribution of yields, the mathematical formulation employing the moment-generating function—rather than probabilities of the various yields for given L—is simplest and most direct. Specifically sought are the asymptotic or limiting yield and variance which are proportional to L for large L, that is, the limiting slopes for yield and variance as functions of L. The Fano factor is given by the ratio of these slopes. Methods of analysis are devised that provide analytical approximations for these quantities. These methods must in general be used, though an exact expression is found for the limiting yield in the simplest case of Fig. 1. Results from the Monte Carlo procedure and from the analysis are in good agreement.[6]

In the analysis, it is somewhat more convenient to deal with $w(L)$, the expected waste for length L, rather than with the expected yield, $y(L)=L-w(L)$. In the simplest case, the analysis gives $w(L)$ for $L\geq 1$ in terms of an integral from zero to $L-1$ whose integrand contains this function itself as a factor, a relationship that follows from a "retarded" first-order differential equation for $w(L)$ involving $w(L-1)$. In view of the upper limit of the integral, $w(L)$ can in principle be obtained analytically for successive unit intervals in L, starting with what is tantamount to the definition of the ionization threshold, namely, $w(L)=L$ in the initial interval $0\leq L<1$. The variance $v(L)$ starts with $v(L)=0$ in this interval, and the corresponding relationship for it has considerably more formal complexity. In the semiconductor case, what would otherwise be a highly intractable formulation is brought within the scope of the methods of approximate analysis by derivation of approximate retarded differentio-integral relationships for $L\geq 1$ that the first and second moments, respectively, satisfy, together with the boundary values for $L=1$. These relationships apply because the Raman phonon energy is small compared to the ionization threshold, so that expansion to the first order in the ratio x_r of these energies provides a good approximation. In these relationships, phonon losses are taken into account through a single parameter, K, which is the ratio of expected values in a collision of energy given up, respectively, to optical phonons and to secondaries. Analytical solutions for $w(L)$ and $v(L)$ can in principle be obtained for the successive unit intervals in L; each of the two relationships generates, in effect, a sequence of "nested" differential equations for these intervals of which the equation for any specified interval itself depends on the solutions for all the preceding intervals. The solutions depend on the probability r of phonon generation, which enters through the boundary conditions, as well as on K; and r and K together determine x_r. The number N of phonons per ionization is $r/(1-r)$, which is also the ratio of the mean free path for ionization to that for scattering by the optical phonons.

It is not feasible to extend the analytical solutions piecewise in this way very far; the calculations rapidly become unwieldy, even in the simplest case. But solutions for the first few unit intervals themselves indicate,

[6] Crazy-carpentry runs on 200 boards of length 100 provide the expected yield or the first moment with good accuracy. A number of boards at least 1 order of magnitude greater would be required for second moments of comparable accuracy, especially for the range of considerable losses to phonons.

and Monte Carlo results verify, that the limiting straight lines are quickly approached. Related to this circumstance is the efficacy of the analytical method by which closely approximate determination of the slopes of the straight lines may be accomplished. Important for this determination are comparatively simple exact relationships that the theory provides between slopes and intercepts on an axis of the limiting lines for the yield and variance. In the semiconductor case, the relationship for the line for the yield involves K, while that for the line for the variance involves the limiting relative yield as well. With these relationships, slope and intercept can be found if just one point substantially on a limiting line is determined. They are used to make correction for the intercepts in obtaining the limiting slopes from Monte Carlo yields and variances. With this correction, yields and variances need not be determined for large L; moderate L suffices at which the theoretical curves are close to the lines. One of the analytical methods for finding the slopes, though not the best one, is based directly on these results.

In advance of formulation and results for the semiconductor case given in Sec. 2, it may be well to illustrate the present discussion briefly by results for simpler cases, theory for which is given in Sec. A of the Appendix.

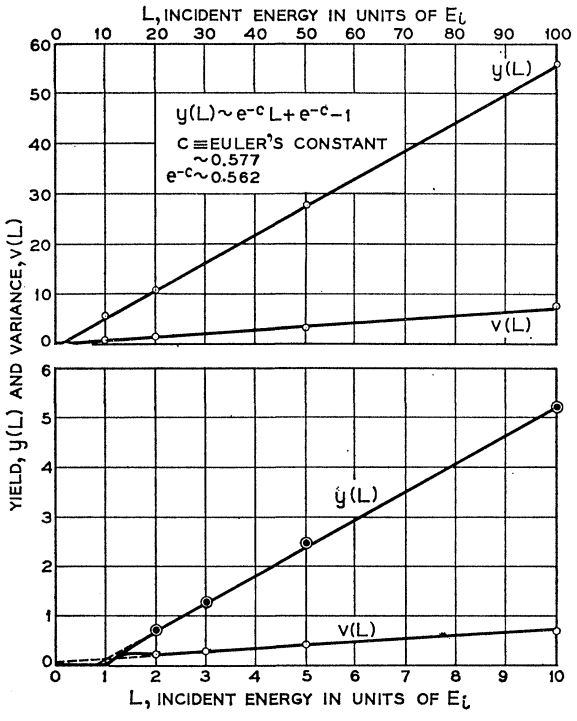

FIG. 2. Dependence of theoretical and Monte Carlo yields and variances on L for the simplest case of crazy carpentry. This case corresponds to ionizing secondaries of one kind, random fractions of whose entire energies are absorbed in collisions. Length unity represents the ionization threshold energy. The curves are theoretical and the points are Monte Carlo. Standard deviations for four of the Monte Carlo yields are given roughly by radii of inner disks.

Theoretical curves as well as Monte Carlo points for the simplest case of Fig. 1 are shown in Fig. 2. These curves and points for yield $y(L)$ and variance $v(L)$ are in close agreement. The lower part of the figure is a tenfold enlargement of the lower left-hand corner of the upper part, and the radii of the inner disks of the points for the yield about equal the standard deviations for the mean. The quick approach of the curves to the limiting straight lines is evident. The limiting yield depends in a simple manner on Euler's constant, as indicated in the figure, and equals about 56%. In conjunction with the slope of the line for the variance obtained from closely approximate analysis, a value for the Fano factor of 0.1205 is obtained.[6a]

For this case, the waste comes about in part through cuts within unit lengths at the right-hand ends. The physical implication is that the uniform probability distributions include the range below the ionization threshold, so that the probability, in effect, of phonon production by a secondary of energy above this threshold increases in a certain definite manner with decrease in this energy. Besides requiring division of the secondare remainders, the rules for the semiconductor case give the probability distribution below the ionization threshold as a delta function of specified probability at the Raman phonon energy. For comparison, it is thus well to consider also the modified simple case which differs from that of Fig. 2 in that energies absorbed on collision less than the ionization threshold are ruled out, so that there is always yield of a secondary if energy exceeds the ionization threshold. This case is clearly the semiconductor case with the probability r set equal to zero and with no division of secondary remainders, corresponding to no phonon losses and secondaries of one kind only. As may readily be verified, the rules then turn out to be equivalent to those for a certain one-dimensional random space-filling problem[7] which has been called the "parking problem." For this, the relative yield for large L is 0.748. This value was obtained by numerical integration of a certain integral[8]; the analysis, though rather similar to that for the case of Fig. 2, is formally less simple in its results. Figure 3 shows theoretical curves and Monte Carlo

[6a] *Note added in proof.* This case applies in the theory of radiation damage [E. I. Blount (private communication)]; the yield is then that of displaced secondary atoms: See F. Seitz and J. S. Koehler, in *Solid State Physics*, edited by F. Seitz and D. Turnbull (Academic Press Inc., New York, 1956), Vol. 2, pp. 380–383. The effect of unsharp threshold energy has been considered by A. E. Fein, Phys. Rev. **109**, 1076 (1958) and by W. S. Snyder and J. Neufeld, *ibid.* **103**, 862 (1956).

[7] E. N. Gilbert (private communication); A. Rényi, Magyar Tud. Akad. Mat. Kut. Int. Közlemények **3**, 109 (1958); Math. Revs. **21**, 577 (1960) [English transl.: TR 64-186 (Bell Telephone Laboratories, Inc., 1964)]. Cars of unit length are parked successively along a street of given length, a specified end of each being placed at a randomly selected point in the space available for parking until no more is left. Relative yield is the expected number of cars that can be parked per unit length.

[8] A. Rényi, Ref. 7. See also P. E. Ney, Ann. Math. Statistics **33**, 702 (1962). Of some interest for the present context is that the generalization of this latter paper to cars of random lengths was originally for theory of binary cascades.

points for this modified simple case. The points are in good agreement with the curves, which involve exact and approximate analysis given in Sec. A2. A Fano factor of only about 0.05 is found.

In comparison, results that are presented for the semiconductor case show the effect of division of the secondary remainders, as for secondaries of two kinds but no phonon losses: This gives a moderate decrease of limiting relative yield to 0.64, but an appreciable increase of Fano factor to about 0.13. Phonon losses further reduce yield and increase Fano factor. The latter asymptotically approaches unity as phonon losses increase indefinitely, but over a fairly wide range of moderate phonon losses Fano factors obtain that are appreciably smaller than unity.

The crazy-carpentry model purports to apply in principle as long as all incident unreflected energy is ultimately accounted for by secondary electrons and holes and by phonons, whatever may be the complexity of the generation process.[9] Energy losses in a thick absorber that cannot be so accounted for may be obviated through suitable restriction of incident energy. Thus, the high-energy incident particles here considered are particles, restriction of whose energy to less than, say 10 or 100 MeV may be necessary to rule out rapid radiation damage during a pulse, certain nuclear transformations, knock-on collisions, and related effects. For the present context, energy may also be restricted in practice by available absorber thickness, since this thickness is to exceed the range of the incident particles in the material.

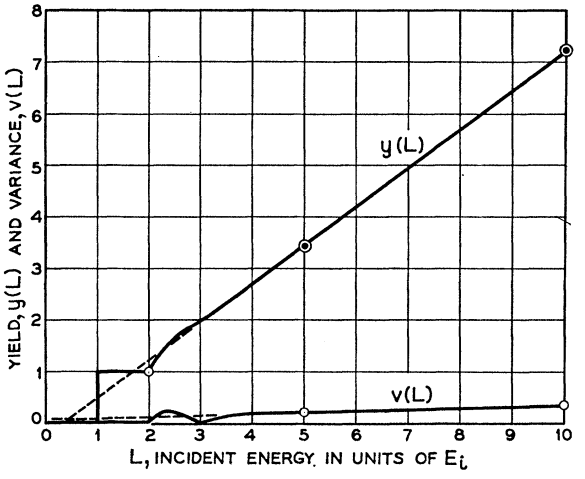

FIG. 3. Dependence of theoretical and Monte Carlo yields and variances on L for the case of the parking problem. This case corresponds to ionizing secondaries of one kind, random fractions of whose energies in excess of the ionization threshold energy are absorbed in collisions. The theoretical curves here shown only up to $L=10$ also agree closely with Monte Carlo points for larger values given in Table III of Sec. A2. Standard deviations for two of the Monte Carlo yields are given roughly by radii of inner disks.

[9] For discussion of mechanisms involved see, for example, G. Dearnaley and D. C. Northrup, *Semiconductor Counters for Nuclear Radiations* (E. and F. N. Spon, Ltd., London, 1963).

Incident heavy charged particles generate energetic secondary electrons and holes, the so-called delta rays, along ionization tracks. However, an appreciable fraction of incident energy may go into secondaries of energies only a few to several electron volts that are produced through plasma oscillations.[10] The threshold for ionization by the heavy particles[11] is anisotropic in the crystal and considerably larger than that for ionization by the secondaries. If the latter is anisotropic, then the threshold E_i for the branching process is of course its effective value for essentially random directions.

Incident gamma or x rays produce electron and hole secondaries mainly through Compton scattering and the photoelectric effect. Incident electrons may ionize directly without there being involved radiation that produces further secondaries. But if they are sufficiently energetic, they may give bremsstrahlung; or they may excite electrons from atomic levels below the valence band, with radiation then resulting from transitions involving empty levels so produced.[12] In practice, with an absorber thick enough to stop all incident particles, there appears generally to be no significant energy loss through escape of secondary radiation.

The crazy-carpentry model should constitute a good approximation whenever it properly applies. This conclusion derives in part from the model itself. It might first be noted that, consistently with this conclusion, the principal idealizations of the model are that no distinction is made between electrons and holes, particle energy only being considered, and that uniform probability distributions are assumed to obtain for energies absorbed in collisions. The distributions for the larger energies may exhibit maxima corresponding to ionizing secondaries most probably retaining a major fraction of their energy.[13] Furthermore, since depth of the valence band limits energy of mobile holes, the larger secondary energies are carried mostly by electrons. In principle, there are of course probability distributions, which depend on band structure, also for the division of energy between the secondary electrons and holes.[14]

The statistics should, however, not be much affected by these idealizations. In connection with the division of energy between electrons and holes, the case of the parking problem with Fano factor of 0.05 and the cor-

[10] W. M. Gibson, G. L. Miller, and A. G. Chynoweth (private communication).

[11] H. E. Schweinler, Natl. Acad. Sci.—Natl. Res. Council, Publ. 871, pp. 91–94.

[12] W. L. Brown, IRE Trans. Nucl. Sci. 8, 2 (1961); G. A. Baraff (private communication).

[13] E. O. Kane (private communication).

[14] Simplified calculation of distributions in energy after collision for the ionizing electron or hole and its two first-generation secondaries would provide qualitative insight. Estimates were accordingly sought based on densities of states and relative volume in wave-number space under conditions of energy and momentum conservation, for equal effective masses, parabolic band edges, and energies near these edges. Evaluation of the cumbersome multiple integral with two inequalities restricting the variables of integration that resulted, even with these simplifying assumptions, did not seem justified.

responding semiconductor case with no phonon losses with Fano factor of 0.13 represent extreme cases of all energy to one secondary and of energy equally divided. Use of a fixed ratio for the energy division—an arbitrary ratio is used in the analytical formulation—is no doubt an adequate approximation, and the Fano factor should in general not differ very much from the value for energies equal. In connection with the energies absorbed in collisions, the proper probability distributions depend in part on densities of states, and it seems likely that the maxima they may exhibit for the larger energies would extend over several ionization threshold energies, while for the smaller energies, the entire distributions should be fairly uniform. That is, any structure the distributions may exhibit should, like the valence band, extend over an energy range large compared with the energy range in which the statistics are determined.

Quantitative measures of this range at the end of the branching process are readily obtained from the yield function $y(L)$. The intercept of the straight-line approximation to $y(L)$ on the L axis represents an apparent threshold for ionization, the yield being essentially proportional to energy in excess of the corresponding apparent threshold in energy. The apparent threshold in L is given by a simple expression derived from the relationship between slope and intercept. This threshold is found to be the same as the average value L_f of "final length" or length (less than unit length) of a waste piece. In physical terms, if the incident particles are electrons, that is, the same as secondaries they produce, then the apparent threshold for ionization is the same as the average value of the final energy, or the energy of a secondary when first it cannot produce further secondaries. The effect of L_f on relative yield for at least moderate values of L can readily be specified: The relative yield for an L for which $y(L)$ is substantially on the limiting line is $(1-L_f/L)$ times the limiting relative yield Y. Since L_f is less than unity, relative yield $Y(L)/L$ then rapidly approaches Y with increasing L. For $L=3$, for example, relative yield is $0.74Y$ for the simplest case of Fig. 2, for which L_f is 0.78, and for the parking-problem case of Fig. 3, for which L_f is 0.34, it is $0.89Y$. For $L=4$ in the semiconductor case with no phonon losses, it is $0.93Y$ from $L_f=0.28$, and $0.95Y$ from the exact solution. Thus, for this semiconductor case, 95% of the yield from energetic particles is obtainable with particles of energy only four ionization thresholds. Reasonable energy losses to phonons do not reduce this ratio much: A value of about 89% applies for $K=1$, that is, if as much energy is expended on optical phonons as on ionization.

These high ratios and rapid approach of relative yield to its limiting value result in part from comparatively small L_f. Figure 4 gives a Monte Carlo frequency distribution in final length or energy for the semiconductor case with no phonon losses. As the figure shows, the distribution—whose average agrees with the theoretical L_f—is not at all uniform, but is sharply peaked towards the origin. The frequency distribution and L_f depend of course on phonon losses, but even for K of order unity the qualitative behavior that the figure illustrates presumably still obtains. It is evident that for very large K the distribution should be peaked towards length unity, since if nearly all collisions produce phonons, then the average final energy will nearly equal the ionization threshold.

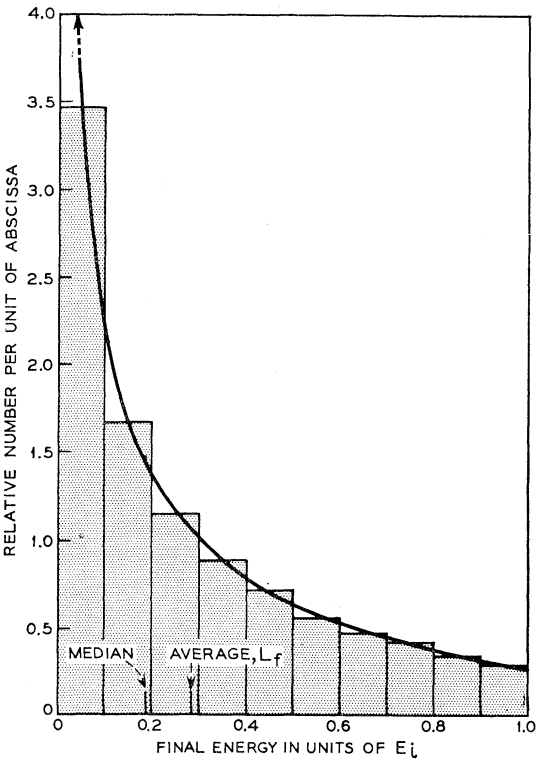

FIG. 4

FIG. 4. Monte Carlo frequency distribution in final length or relative final energy for the semiconductor case with no phonon losses. The results shown are from 200 boards of length 100, and the average number of final lengths per board in the respective intervals in L of length 0.1 are 44.66, 21.52, 14.88, 11.35, 9.29, 7.31, 6.06, 5.47, 4.38, and 3.81, whose total is 128.73. This divided into the average waste per board of 36.74 gives $L_f=0.285$.

Rapid approach to limiting relative yield of course results in part also from the yield curve going rapidly into the limiting straight line. This behavior of the curve is properly examined in terms of energies above the apparent threshold: The relative yield obtained by dividing $y(L)$ by $L-L_f$ rather than by L is very nearly the limiting relative yield Y for a $y(L)$ very nearly on the line. For the simplest case of Fig. 2, $y(L)/(L-L_f)$ is $0.88Y$ for $L-L_f=0.5$ and $1.03Y$ for $L-L_f=1.0$. For the semiconductor case with no phonon losses, $y(L)/(L-L_f)$ is $0.91Y$ for $L=2$, the smallest value of L which it is appropriate to consider, since, as is readily evident from the rules, $y(L)$ is unity over the

range $1 \leq L < 2$. Comparatively rapid approach to the line still obtains with phonon losses, which round off this step in $y(L)$. These results further emphasize the preponderant influence of the very final stages of the branching process.

The average energy ϵ required to produce an electron-hole pair is simply the ionization threshold E_i divided by Y. It is also the sum of three energies[15]: E_i; KE_i, the average energy per pair expended on phonon losses; and $2L_fE_i$, twice the average final energy. Thus, L_f is simply related to Y and K. It depends, of course, on K alone, and this dependence must be taken into account if ϵ is evaluated theoretically from the three energies.

Results for the semiconductor case are given in detail for equal energies to secondary electrons and holes, and for an x_r that is reasonable for silicon, to facilitate comparisons with work on the determination of ϵ and on carrier multiplication under high fields in this material.[16–18] It is found that Y and F in their dependence on K and r are changed by no more than 3% in going to the limiting case of negligible x_r and correspondingly large N, for which r is unity. Thus, Y and F depend essentially on K, though they would be modified somewhat by assuming another ratio for the division of energy between electrons and holes. The results are therefore not restricted to Raman phonons, but apply whatever phonons may be involved, and probably even if the losses differ for electrons and for holes. For energies of the order of E_g, losses to acoustical phonons may in fact be the greater.[19]

An illustrative application of the theory is made in Sec. 2.2 to the analysis of experimental results of Vavilov on quantum yield in silicon at 293°K from photons of energies $h\nu$ ranging from near the energy gap E_g to about 5 eV, in the ultraviolet.[17] These results show the enhanced quantum yield from impact ionization up to an energy of about $2E_i$, so that the known exact theoretical yield for $1 \leq L < 2$ applies. The apparent threshold for this enhanced yield in its dependence on $h\nu$ is not the same as the average final energy, since photon energy above E_g is (for direct transitions) divided between the photoelectron and photohole. It is readily shown to equal $E_g + 2L_fE_i$ however the energy $h\nu - E_g$ may be divided. It is found that the data are fitted well by assuming equal division of energy and $E_g = 1.1$ eV, $E_i = 1.0$ eV, and $\epsilon = 3.0$ eV, from which $K = 1.12$ results. Because the ratio E_i/ϵ fixes L_f as well as Y and K, the values $E_i = 1.1$ eV and[15,17,18] $\epsilon = 3.6$ eV, which otherwise seem generally to be good values for silicon, give a yield curve that departs from the data points by more than the estimated experimental error. The data could be fitted reasonably well by assuming not quite equal division of energy with E_i equal to or slightly larger than E_g, but this elaboration does not seem justified in the present context and without more detailed evidence. The surface-barrier detector used in the experiment involved heavy doping with decreased and depth-dependent mobility.[17,20] The collection efficiency α could therefore not be calculated, and the quantity treated as quantum yield $Q(h\nu)$ was accordingly $\alpha Q(h\nu)$ normalized to unity over an initial range of $h\nu$ above E_g. It may well be that there was appreciable concentration of impurities with deep energy levels with which somewhat lowered values of E_i and ϵ are associated.

2. THE SEMICONDUCTOR CASE

2.1 Results from Theory and from Monte Carlo Computation

For the semiconductor case, numerical results that lend themselves to comparatively brief presentation are given here in advance of the theory for them given in Sec. 2.3. For the dependence on phonon losses, the value 0.035 is used for $x_r \equiv E_R/E_i$. This is a good value for silicon[21] for E_i equal to $1.5E_g$. If E_i is smaller, then x_r is corresponding increased and the average number N of phonons per ionizing collision decreased, since the dependence on phonon losses is determined essentially by the parameter $K = Nx_r$.

Figure 5 shows theoretical curves and Monte Carlo points for yield $y(L)$ and variance $v(L)$ for no phonon losses and also for $N = 20$, or $K = 0.7$. For the curves, exact solutions apply that give $w(L)$ and $v(L)$ for $1 \leq L < 2$, $w(L)$ for $2 \leq L < 3$, and $w(L)$ for $N = 0$ and $3 \leq L < 4$. The limiting straight lines, which are given by analytical approximations, are quickly approached. The curves and associated points are in generally good agreement.[22] For no phonon losses, $Y = \frac{2}{3}$ is the theoretical limiting slope for $y(L)$ according to the approximation employed, and this is slightly larger than the Monte Carlo value of 0.64, the line for which is shown dashed. This limiting slope is reduced to 0.40 by the phonon losses for $K = 0.7$. These phonon losses increase the theoretical F from 0.117 to 0.22. Such moderate phonon losses increase F mainly through the decrease in Y, the limiting slope of the line for $v(L)$ not changing very much.[23] While the curves are shown only up to $L = 10$,

[15] W. Shockley, Solid-State Electron. **2**, 35 (1961), Czech. J. Phys. **B11**, 81 (1961); A. G. Chynoweth, pp. 95–98 in the text of Ref. 11.

[16] See P. A. Wolff, Phys. Rev. **95**, 1415 (1954); G. A. Baraff, *ibid.* **128**, 2507 (1962); C. A. Lee, R. A. Logan, R. L. Batdorf, J. J. Kleimack, and W. Wiegmann, *ibid.* **134**, A761 (1964); E. Baldinger and W. Czaja, Nucl. Instr. Methods **10**, 237 (1961); E. Baldinger, W. Czaja, and J. Gutman, Helv. Phys. Acta **35**, 559 (1962) and other references therein and in Ref. 15.

[17] V. S. Vavilov, J. Phys. Chem. Solids **8**, 223 (1959).

[18] With a surface-barrier detector made from 21 000 Ω-cm *n*-type silicon, differing values of ϵ of 3.79 eV for 365-keV electrons and of 3.61 eV for 5.477-MeV alpha particles have been found: C. Bussolati, A. Fiorentini, and G. Fabri, Phys. Rev. **136**, A1756 (1964).

[19] E. M. Conwell, Phys. Rev. **135**, A1138 (1964).

[20] Phosphorus was diffused into *p*-type silicon.

[21] C. A. Lee (private communication); see C. A. Lee *et al.*, Ref. 16.

[22] See also Table I in Sec. 2.3.

[23] This slope $C = YF$ increases slightly from the value for no phonon losses to a maximum near $K = 1$ and then decreases asymptotically to zero. See Table II in Sec. 2.3.

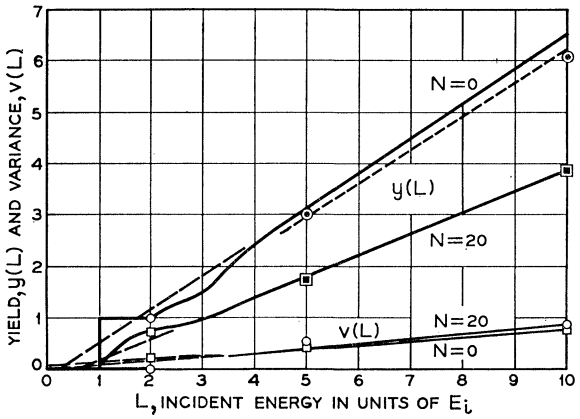

FIG. 5. Dependence of theoretical and Monte Carlo yields and variances on L for $N=0$ or no phonon losses and for $N=20$ in the semiconductor case. For $N=0$, the dashed line to the right with slope $Y=0.64$ and intercept $L_f=0.28$ is determined by Monte Carlo points given for values of L up to 100 in Table I, Sec. 2.3. Some standard deviations for Monte Carlo yields are indicated roughly by inner disks and squares.

their extrapolations are in correspondingly good agreement with Monte Carlo results[22] up to $L=100$. Through the equations that correct for the intercepts of the lines, the Monte Carlo data provide values[22] of Y and F essentially independent of L for L in excess of quite moderate values.

Figure 6 shows Y and F in their dependence on phonon losses, the curves being obtained from the theory and the points from Monte Carlo data. There is good agreement, though the values for F exhibit appreciable scatter for the larger phonon losses.[24] The curves, indicate, for example, that for an average energy ϵ per electron-hole pair of 3.6 eV, as for silicon, the ionization threshold E_i, which is ϵY, would equal 1.65 eV or 1.5 E_g for silicon for $Y=0.459$, hence for $K=0.51$ and $N=14.5$.

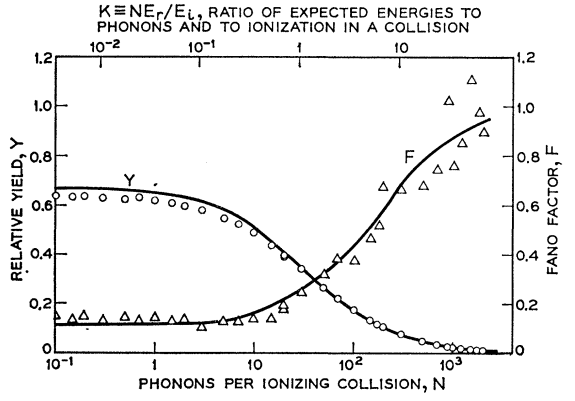

FIG. 6. Dependence of theoretical and Monte Carlo limiting relative yields and Fano factors on phonon losses. The more significant parameter is K, to which N is related by the value of 0.035 assumed for the relative optical phonon energy $x_r\equiv E_R/E_i$.

[24] See also Table II in Sec. 2.3.

Or, E_i would equal 1.1 eV, which is E_g for silicon, for $Y=0.31$, hence for $K=1.35$ and $N=38/1.5\sim25$.

2.2 Application to Results of Vavilov on Enhanced Quantum Yield in Silicon

Suppose energy $h\nu-E_g$ of a photon above the energy gap is divided so that the photoelectron has energy $c_n(h\nu-E_g)$ and the photohole, $c_p(h\nu-E_g)$, with $c_n+c_p=1$. As long as the excess energies are less than E_i, the quantum yield is unity. In an initial range of enhanced quantum yield, one of the excess energies exceeds E_i and the enhanced yield is $y(L)=1-r\times\exp[-(L-1)/K]$, in accordance with the solution for $w(L)$ for $1\leq L<2$, with L the excess energy divided by E_i. If both excess energies exceed E_i and do not differ too much, the enhanced yield is the sum of two such expressions. In particular, for the case of $c_n=c_p=\frac{1}{2}$, the total quantum yield in an initial range of enhanced yield is $1+2y[\frac{1}{2}(h\nu-E_g)/E_i]$, with $y(L)$ given as above, or[25] $3-2r\exp\{-[\frac{1}{2}(h\nu-E_g)/E_i-1]/K\}$.

In Fig. 7 are shown the results of Vavilov[17] fitted by this single-threshold formula, with $E_g=1.1$ eV, $E_i=1.0$ eV, and $\epsilon=3.0$ eV. The last two values determine Y. Then $K=1.12$ is determined by the theoretical result for the dependence of Y on K and r or by Fig. 6, as is $N=19.4$ and $r=N/(N+1)=0.95$. The apparent threshold $E_g+2L_fE_i$ in 1.98 eV, and the limiting line for the enhanced yield is shown dashed. The curve determined by the otherwise more plausible values, $E_i=E_g=1.1$ eV and $\epsilon=3.6$ eV, which does not fit the data so well, is also shown. For it, K is 1.35 and the apparent threshold 2.11 eV. The initial jump each curve exhibits at the enhanced-yield threshold E_g+2E_i that corresponds to $L=1$ is here of amount $2(1-r)$ and is associated, in effect, with nonzero energy of an optical phonon or finite number of phonons per ionization. If no phonons were produced, then the total quantum yield would be constant and equal to 3 over the initial range above the threshold up to $L=2$.

2.3 Theoretical Analysis

The general semiconductor case is best approached by first neglecting phonon losses. The rules for the crazy carpentry without these losses, more formally stated than in Fig. 1, are as follows: *If initial length is L, then $L<1\rightarrow$ zero contribution to $y(L)$, or $L>1\rightarrow$ contribution unity to $y(L)$ and three new initial lengths: x, $\beta(L-x-1)$, and $\gamma(L-x-1)$, where x is a random length in the interval $0<x<L-1$ and β and γ are constants satisfying $\beta+\gamma=1$. Repeat from the beginning with new initial lengths. (Continue this procedure until there are no further initial lengths.)*

Here, x is the primary remainder representing remaining energy of the initial particle, and $L-x-1$ is the secondary remainder. The fractions β and γ

[25] This differs from Shockley's expression of Ref. 15 only by the factor r that multiplies the exponential.

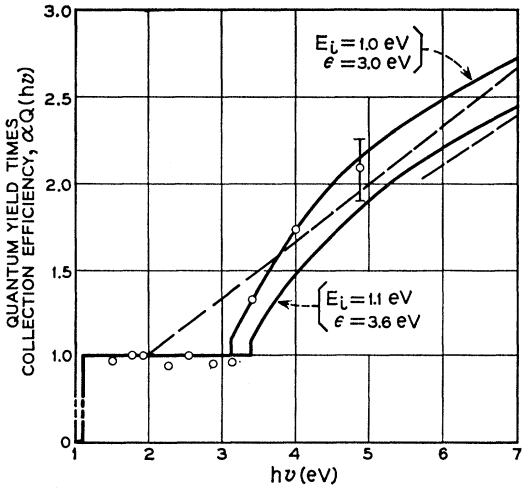

FIG. 7. Results of Vavilov on enhanced quantum yield in silicon fitted by theoretical curves from the crazy-carpentry model. The points are reproduced from Ref. 17, and the two dashed lines are the limiting lines for the yields which the respective theoretical curves approach asymptotically.

correspond to electrons and to holes and give the division of energy between them, the specialization $\beta=\gamma=\frac{1}{2}$ being the one that will be considered in detail.

Denote by $\lambda^{(L)}$ the total length wasted for initial length L in one crazy-carpentry procedure carried to completion. The superscript notation is suitable because $\lambda^{(L)}$, as a statistical variable, is not a function of L in the usual strict sense. The moment-generating function[26] for this initial length is then

$$\psi(L,s)\equiv\mathbf{E}(\exp[s\lambda^{(L)}]), \tag{1}$$

where $\mathbf{E}$ denotes the expected value. The moments are obtained by differentiation with respect to the parameter s, the first two, the expected values of λ and of λ^2, being given by

$$w(L)\equiv\mathbf{E}(\lambda)=[\partial\psi(L,s)/\partial s]|_{s=0}, \tag{2}$$

and

$$u(L)\equiv\mathbf{E}(\lambda^2)=[\partial^2\psi(L,s)/\partial s^2]|_{s=0}. \tag{3}$$

Thus, the mean waste for length L is $w(L)$ and the corresponding variance is

$$v(L)=u(L)-[w(L)]^2. \tag{4}$$

It is readily seen that the rules give

$$0\leq L<1, \qquad \psi(L,s)=\exp(sL), \tag{5}$$

since $\lambda=L$ holds for $L<1$, and also

$$1\leq L, \quad \psi(L,s)=\int_0^{L-1}(L-1)^{-1}\times\mathbf{E}(\exp[s(\lambda^{(x)}+\lambda^{(\beta(L-x-1))}+\lambda^{(\gamma(L-x-1))})])dx, \tag{6}$$

[26] If this function is known, then the probability distribution for λ can in principle be found by Fourier transformation.

or, with Eq. (1),

$$1\leq L, \quad (L-1)\psi(L,s)=\int_0^{L-1}\psi(x,s)\times\psi(\beta(L-x-1),s)\psi(\gamma(L-x-1),s)dx. \tag{7}$$

Equation (6) states, in effect, that all the waste comes from length $L-1$ divided into the three new initial lengths. Note that for x governed by some probability distribution other than the uniform one, the appropriate function of x and L would appear as a factor in the integrand. If, in addition, β or $\gamma=1-\beta$ were not constant but a statistical variable, then a double integral with the product of two distribution functions in the integrand would result.

Differentiations of Eqs. (5) and (7) in accordance with Eqs. (2) and (3) give the relationships

$$0\leq L<1, \qquad w(L)=L, \quad y(L)=0, \tag{8}$$

and

$$1\leq L, \quad (L-1)w(L)=\int_0^{L-1}\{w(x)+w(\beta x)+w(\gamma x)\}dx \tag{9}$$

for the first moment, and

$$0\leq L<1, \qquad u(L)=L^2, \quad v(L)=0, \tag{10}$$

and

$$1\leq L, \quad (L-1)u(L)=\int_0^{L-1}\{u(x)+u(\beta x)+u(\gamma x)+2w(L-x-1)[w(\beta x)+w(\gamma x)]+2w(\beta x)w(\gamma x)\}dx \tag{11}$$

for the second moment.

Losses to optical phonons may now readily be taken into account: The rules that apply under $L>1$ are extended by specifying that $L>1$ implies probability r that the two new initial lengths x_r and $L-x_r$ result and probability $1-r$ that there is contribution unity to $y(L)$ and three new initial lengths result as specified originally. Length x_r of course contributes to $w(L)$. The moment-generating function is thus given by Eq. (5) and by

$$1\leq L, \quad \psi(L,s)=r\mathbf{E}(\exp[s(x_r+\lambda^{(L-x_r)})])+(1-r)\times\left\{\int_0^{L-1}(L-1)^{-1}\mathbf{E}(\exp[s(\lambda^{(x)}+\lambda^{(\beta(L-x-1))}+\lambda^{(\gamma(L-x-1))})])dx\right\} \tag{12}$$

or

$$1\leq L, \quad \psi(L,s)=r[\exp(sx_r)]\psi(L-x_r,s)+(1-r)(L-1)^{-1}\times\int_0^{L-1}\psi(x,s)\psi(\beta(L-x-1))\psi(\gamma(L-x-1))dx, \tag{13}$$

from which follows

$$1\leq L,\quad w(L)=r[x_r+w(L-x_r)]+(1-r)(L-1)^{-1}\times\int_0^{L-1}\{w(x)+w(\beta x)+w(\gamma x)\}dx \quad (14)$$

for the first moment, and

$$1\leq L,$$
$$u(L)=r[x_r^2+2x_rw(L-x_r)+u(L-x_r)]+(1-r)(L-1)^{-1}\times\int_0^{L-1}\{u(x)+u(\beta x)+u(\gamma x)+2w(x)w(\beta(L-x-1))+2w(x)w(\gamma(L-x-1))+2w(\beta x)w(\gamma x)\}dx \quad (15)$$

for the second moment. For $0\leq L<1$, Eqs. (8) and (10) apply.

With $x_r\ll 1$, functional differentio-integral relationships that are good approximations for $1\leq L$ may be derived from Eqs. (14) and (15). Expanding to the first order in x_r gives

$$dw(L)/dL+K^{-1}w(L)=1+K^{-1}(L-1)^{-1}\times\int_0^{L-1}\{w(x)+w(\beta x)+w(\gamma x)\}dx,\quad 1\leq L \quad (16)$$

and

$$du(L)/dL+K^{-1}u(L)=2w(L)+K^{-1}(L-1)^{-1}\int_0^{L-1}\{u(x)+u(\beta x)+u(\gamma x)+2w(L-x-1)[w(\beta x)+w(\gamma x)]+2w(\beta x)w(\gamma x)\}dx,\quad 1\leq L, \quad (17)$$

with K the parameter defined by

$$K\equiv rx_r/(1-r)=rE_R/(1-r)E_i. \quad (18)$$

Note that K is the ratio of expected values of energy in a collision associated, respectively, with optical-phonon generation and with ionization. Though x_r is small, K may be small or large, depending on the value of

$$N\equiv r/(1-r)=K/x_r, \quad (19)$$

the expected number of phonons per ionizing collision.

The functions $w(L)$ and $u(L)$ are discontinuous at $L=1$: Eqs. (8) and (10) give $w(1-)=u(1-)=1$, while the correct initial values for Eqs. (16) and (17) may be derived from Eqs. (14) and (15), which give

$$w(1+)=u(1+)=r. \quad (20)$$

The discontinuities may be attributed to nonzero energy of an optical phonon, since r approaches unity as x_r approaches zero for given K.

For large L, limiting linear solutions hold for the expected waste and variance, namely

$$w(L)=AL+B \quad (21)$$

and

$$v(L)=CL+D, \quad (22)$$

with A, B, C, and D constants. From these equations and Eq. (4), the corresponding approximation for $u(L)$ is

$$u(L)=A^2L^2+(2AB+C)L+B^2+D. \quad (23)$$

The Fano factor F is given by

$$F=C/Y=C/(1-A). \quad (24)$$

The problem is essentially that of evaluating the constants of the limiting linear solutions. The initial solutions of Eqs. (8) and (10) used in the integrands of Eqs. (16) and (17) give differential equations for $1\leq L<2$ which may be solved for the initial values of Eqs. (20). These solutions may then in turn be used with the initial solutions in the integrands to give differential equations for $2\leq L<3$, and so on. It is especially in connection with Eq. (17) for $u(L)$ that calculations according to this procedure quickly become unwieldy. Also, in this equation, a contribution to the integrand has $w(L-x-1)$ as a factor. For this contribution, solutions for $w(L)$ over initial ranges in L must be considered near the upper limit of the integral as well as near the lower, and, for large L, one factor of this contribution is large when the other is small. Since L occurs in the integrand, this difficulty cannot be circumvented by differentiating the equation. Note that Eq. (16) for $w(L)$ does not have L in the integrand, so that the integral may be eliminated between this equation and its derivative.

Approximate analytical expressions for Y and F whose calculation represents a reasonable limit of feasibility may, however, be obtained by using the exact solutions for the functions in the integrands only up to the value 2 of their arguments, and the limiting linear solutions beyond. Fortunately, with the rapid approach to these limiting solutions, the expressions so obtained turn out to be reasonably good approximations.

Though this calculation subsumes them, it appears best to derive first certain results pertaining to the yield. Consider Eq. (16) in the limit of large L, so that Eq. (21) applies. The two equations then give

$$(1+K)A-2B=K, \quad (25)$$

which holds for any division of energy between electrons and holes, as specified by β or γ. For given K, this equation fixes B in terms of A, thus also fixes the apparent threshold L_f in terms of Y:

$$L_f=B/(1-A)=B/Y=\tfrac{1}{2}[Y^{-1}-(1+K)]. \quad (26)$$

Note that L_fE_i is the average final energy, since Eq.

(26) is tantamount to

$$\epsilon = E_i/Y = (1+K+2L_f)E_i, \tag{27}$$

and KE_i is the energy loss to phonons per ionization.

Equations (21) and (25) may be solved simultaneously for A and B in terms of the coordinates of a point on the line, whence follow

$$Y = [y(L)+\tfrac{1}{2}]/[L+\tfrac{1}{2}(1+K)] \tag{28}$$

and

$$L_f = \tfrac{1}{2}[L-(1+K)y(L)]/[y(L)+\tfrac{1}{2}]. \tag{29}$$

Thus, to obtain both constants of the straight-line approximation or both Y and L_f for any given K it suffices essentially to determine $w(L)$ or $y(L)=L-w(L)$ at a single point on the curve that very nearly lies on the line. Though Eq. (28) has its main application in obtaining Y from Monte Carlo data that give $y(L)$, later in this section are given illustrative examples of approximate expressions so obtained for Y in terms of K and r from theoretical values of $y(2)$ and $y(3)$.

Calculation of the approximate analytical expression for Y involves use of the solution[27]

$$1 \leq L < 2, \quad w(L) = L-1+r\exp[-(L-1)/K], \tag{30}$$

which follows from Eqs. (16) and (20). Then, Eq. (16) is written for the case $\beta=\gamma=\frac{1}{2}$ and large L is assumed so that the linear approximation of Eq. (21) is used in it except where the exact solutions of Eqs. (8) and (30) are used for the functions in the integrand for values less than 2 of their arguments. Integrating and multiplying both sides by $K(L-1)$ then gives an equation formally a quadratic in L; but the terms in L^2 cancel identically. Requiring that the terms in L also cancel gives the relationship between A and B of Eq. (25). The condition that the constant term vanish gives

$$(9-K)A+12B = 5-K+5rK[1-\exp(-1/K)], \tag{31}$$

whence, with Eq. (25), follows the result

$$Y = 1-A = (2-J)/(3+K), \tag{32}$$

with

$$J \equiv rK[1-\exp(-1/K)]. \tag{33}$$

The parameters r, J, K, and N are of course interrelated: From Eqs. (18), (19), and (33) it is evident that all start for no phonon losses together at zero, and as K and N increase indefinitely, $r=K/(K+x_r)=N/(N+1)$ and J both approach unity. Thus, Y according to the approximation of Eq. (32) is $\frac{2}{3}$ for no phonon losses and approximtely $(3+K)^{-1}$ for large K.

In calculation of the corresponding expression for F, use is made also of the solution

$$1 \leq L < 2, \quad u(L) = (2L-1)r\exp[-(L-1)/K] + (L-1)^2, \tag{34}$$

which follows from Eqs. (17) and (20). Then, Eq. (17) is also written for the case $\beta=\gamma=\frac{1}{2}$, and large L is assumed so that the approximations of Eqs. (21) and (23) are used in it except where the exact solutions of Eqs. (8), (10), (30), and (34) are used for the functions in the integrand for values less than 2 of their arguments. It may suffice simply to point out that the function $u(x)$ gives rise to three distinct integrals between respective limits 0, 1, 2, and $L-1$; the two functions of $\frac{1}{2}x$ each give three integrals between limits 0, 2, 4, and $L-1$; and $4w(\frac{1}{2}x)w(L-x-1)$ gives five integrals between limits 0, 2, 4, $L-3$, $L-2$, and $L-1$. Evaluating the integrals and multiplying both sides by $K(L-1)$ gives an equation formally a cubic in L, but in which the terms in L^3 cancel identically. Requiring that the terms in L^2 cancel gives Eq. (25). Requiring that the terms in L cancel gives

$$(1+K)C-2D = 2\{5-K+5rK[1-\exp(-1/K)]\}A + 2KB+(2K-17)A^2-2(K+15)AB+8B^2. \tag{35}$$

The condition that the constant term vanish gives

$$\begin{aligned}(9-K)C+12D = {} & 6+rK\{[5+18K][1-\exp(-1/K)] \\ & -18\exp(-1/K)\}+2r^2K[1-\exp(-2/K)] \\ & -\{31+2rK([14+9K][1-\exp(-1/K)] \\ & -9\exp(-1/K))\}A \\ & +\{12-2K+12rK[1-\exp(-1/K)]\}B \\ & +43A^2+2(1+K)AB-50B^2. \end{aligned} \tag{36}$$

The left-hand members of these equations are formally the same as those of Eqs. (25) and (31), A and B being replaced, respectively, by C and D. Investigation of what other properties the pairs of equations may have in common reveals that the right-hand member of Eq. (35) vanishes with K, which is the right-hand member of Eq. (25). Thus, $D=\frac{1}{2}C$ holds, as does $B=\frac{1}{2}A$, for no phonon losses. Indeed, Eq. (35) simplifies to

$$(1+K)C-2D = K^2(1-A)^2 = K^2Y^2 \tag{37}$$

upon eliminating the exponential from it by use of Eq. (31) and then eliminating B by use of Eq. (25). With this equation, $y(L)$ and $v(L)$ very nearly on their limiting lines determine F in accordance with

$$F = [v(L)+\tfrac{1}{2}K^2Y^2]/[y(L)+\tfrac{1}{2}] \tag{38}$$

a result analogous to Eq. (28) for Y.

It appears that Eqs. (25) and (37) and their consequences, Eqs. (28) and (38), hold quite generally, even though Eqs. (37) and (38) have been derived in connection with a particular approximation. Equation (25) holds as well for a somewhat different crazy-carpentry model—not treated here—which is a more direct formal extension of the simplest case than is the semiconductor case for secondaries of two kinds and phonon losses. Also, for secondaries of one kind only, equations similar to Eqs. (25) and (37) but with coefficients of B and D

[27] Note that $w(L)$ for $L<2$ does not depend on β or γ, which may be expected, since then $L-1<1$. That is, if a pair of secondaries is produced, neither secondary will be energetic enough to produce further secondaries.

TABLE I. Dependence of Monte-Carlo yields, variances, and Fano factors for the semiconductor case on L.

N	L	$y(L)$	$y(L)/L$	$v(L)/y(L)$	Y	F
0	2	1.000	0.5000	0		
	5	3.060	0.6120	0.18183	0.6473	0.15629
	10	6.070	0.6070	0.13923	0.6257	0.12863
	20	12.486	0.6243	0.13053	0.6335	0.12550
	50	31.595	0.6319	0.10554	0.6355	0.10380
	100	63.88	0.6388	0.12660	0.6406	0.12561
20	2	0.7200	0.3600	0.2800	0.4281	0.2020
($K=0.7$)	5	1.7800	0.3560	0.2312	0.3897	0.19684
	10	3.865	0.3865	0.2036	0.4023	0.18932
	20	7.905	0.3953	0.17152	0.4032	0.16179
	50	19.810	0.3962	0.19707	0.3994	0.19414
	100	39.75	0.3975	0.18068	0.3991	0.17940

unity should obtain, $A=B$ and $C=D$ being shown to hold in Sec. A1 for the simplest case and in Sec. A2 for the parking-problem case.

The Fano factor may now be obtained in accordance with Eq. (24) by solving Eqs. (36) and (37) for C. In view of the form of Eq. (37), it is advantageous, by use of Eqs. (25), (32), and (33), first to express A, B, and the exponentials in Eq. (36) in terms of Y and the parameters K and r. The result for F may be written as

$$F=(7/60)(1-\tfrac{1}{2}J)^{-1}(1+\tfrac{1}{3}K)^{-2}\times[1-(1/7)(12J/K-50-24r+67J)J+(2/21)(9-108r+129J+78rJ-101J^2)K+(1/21)(49-72r+24J+44rJ-35J^2)K^2]. \quad (39)$$

In accordance with this result, F equals 7/60 for no phonon losses and, as may be expected, approaches unity as K increases indefinitely. For the limiting case of zero x_r, probability r of a phonon-producing collision may be set equal to unity, since an indefinitely large number N of phonons per ionizing collision is then required to realize any given value of K.

Table I gives Monte Carlo yields and variances in their dependence on L for $N=0$ and $N=20$. These appear in part as the points in Fig. 5. Included also are Y and F obtained from these data by means of Eqs. (28) and (38). The Monte Carlo values on each line of Tables I and II represent crazy carpentry on 200 boards, of length 100 for Table II, which gives the dependence of Y and F on K obtained from Monte Carlo results together with corresponding values entirely from theory. The Monte Carlo Y and F appear as the points in Fig. 6. The theoretical values for $x_r=0.035$ are from Eqs. (32),

TABLE II. Dependence of limiting relative yield and Fano factor on phonon losses.

		Monte Carlo			Theoretical $x_r=0.035$		Theoretical $r=1$		
N	$y(L)/L$	$v(L)/y(L)$	Y	F	Y	F	Y	F	K
0	0.6388	0.12660	0.6406	0.12562	0.6667	0.11667	0.6677	0.11667	0
0.10	0.6382	0.14947	0.6400	0.14831	0.6658	0.11666	0.6647	0.11638	0.0035
0.15	0.6339	0.13502	0.6357	0.13397					
0.2	0.6337	0.15156	0.6355	0.15037	0.6647	0.11652	0.6628	0.11612	0.0070
0.3	0.6285	0.13490	0.6303	0.13384					
0.5	0.6230	0.14141	0.6248	0.14029	0.6609	0.11583	0.6570	0.11549	0.0175
0.7	0.6300	0.12319	0.6318	0.12222					
1.0	0.6197	0.14233	0.6215	0.14119	0.6532	0.11472	0.6474	0.11496	0.035
1.5	0.6069	0.12930	0.6087	0.12825					
2	0.5944	0.13286	0.5962	0.13177	0.6363	0.11424	0.6287	0.11569	0.070
3	0.5798	0.10884	0.5816	0.10794					
5	0.5483	0.12740	0.5501	0.12633	0.5841	0.12631	0.5750	0.12979	0.175
7	0.5230	0.13002	0.5247	0.12895					
10	0.4855	0.14045	0.4872	0.13931	0.5075	0.16467	0.4985	0.16774	0.35
15	0.4366	0.14718	0.4383	0.14611					
20	0.3985	0.18194	0.4001	0.18066	0.4035	0.2242	0.3967	0.2232	0.70
20	0.3975	0.18072	0.3991	0.17944					
30	0.3412	0.2488	0.3427	0.2471					
50	0.2667	0.3204	0.2680	0.3186	0.2638	0.3206	0.2607	0.3143	1.75
70	0.2216	0.3820	0.2228	0.3801					
100	0.17601	0.3780	0.17703	0.3782	0.17520	0.4365	0.17387	0.4296	3.50
150	0.13056	0.4661	0.13145	0.4665					
180	0.11461	0.5176	0.11540	0.5181					
200	0.10741	0.6793	0.10809	0.6745	0.10728	0.5899	0.10681	0.5844	7.0
300	0.07896	0.6660	0.07939	0.6677					
500	0.04956	0.6753	0.04994	0.6834	0.05024	0.7793	0.05015	0.7762	17.5
700	0.03587	0.7467	0.03625	0.7518					
900	0.02956	1.0482	0.02973	1.0234					
1000	0.02591	0.7513	0.02619	0.7657	0.02671	0.8760	0.02669	0.8743	35
1200	0.02176	0.8541	0.02202	0.8544					
1500	0.018863	1.1491	0.018827	1.1130					
1800	0.016213	0.9695	0.016070	0.9826					
2000	0.014663	0.8488	0.014511	0.8953	0.013803	0.9340	0.013796	0.9331	70

(33), and (39) and are represented by the curves in this figure. The table includes theoretical values for the limiting case of $r=1$. That these are within 3% of the values for $x_r=0.035$ attests to the applicability of the approximate differento-integral relationships, Eqs. (16) and (17), since these hold exactly in the limit of $r=1$.

With Eq. (26), theoretical values of L_f that correspond to ones of Y in Table II are $\frac{1}{4}$ for $N=0$, 0.3102 for $N=10$, 0.3892 for $N=20$, 0.6607 for $N=200$, and 0.7240 for $N=2000$; L_f approaches unity as N and K increase indefinitely.

The tabulated theoretical values of Y from Eqs. (32) and (33) are appreciably better than result from setting L to equal to 2 in Eq. (28) and evaluating $y(2)$ by use of Eq. (30). This procedure gives $Y=[3-2r\exp(-1/K)]/(5+K)$, and hence Y equal to $\frac{3}{5}$ for $N=0$ and 0.448 for $N=20$. The similar procedure with $L=3$ makes use of the solution

$$2\leq L<3,$$

$$w(L)=\left[\exp\left(\frac{-L}{K}\right)\right]\left\{r\exp\left(\frac{1}{K}\right)+\exp\left(\frac{2}{K}\right)+(L-2)\exp\left(\frac{L}{K}\right)+\left(r+\frac{1}{K}\right)\exp\left(\frac{1}{K}\right)\times\left[Ei\left(\frac{L-1}{K}\right)-Ei\left(\frac{1}{K}\right)\right]-r\ln(L-1)\cdot\exp\left(\frac{2}{K}\right)\right\}, \quad (40)$$

where

$$Ei(u)\equiv\int_{-\infty}^{u}u^{-1}\exp(u)du \quad (41)$$

is the exponential integral. The result is

$$Y=(7+K)^{-1}\left\{5-2r\exp\left(\frac{-2}{K}\right)-2\exp\left(\frac{-1}{K}\right)-2\left(r+\frac{1}{K}\right)\exp(-2/K)\cdot\left[Ei\left(\frac{2}{K}\right)-Ei\left(\frac{1}{K}\right)\right]+2(\ln 2)r\exp\left(\frac{-1}{K}\right)\right\}. \quad (42)$$

This approximation is less simple in form and not quite so good as the tabulated one. Note that it gives Y equal to $4/7\sim0.5714$ for $N=0$, 0.4044 for $N=20$, 0.10880 for $N=200$, and 0.02675 for $N=1000$.

For no phonon losses, Eq. (40) simplifies considerably, the yield being given by $y(L)=2-(L-1)^{-1}$ for $2\leq L<3$, as may also more readily be shown directly. For $3\leq L<4$, the exact yield $y(L)=5-[7+\ln(L-2)]/(L-1)$ obtains for this case.

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APPENDIX

A1. The Simplest Case of Crazy Carpentry

The rules for the case illustrated in Fig. 1, stated more formally than in this figure, are: *If initial length is L, then $L<1\rightarrow$ zero contribution to $y(L)$ or $L>1\rightarrow$ new initial length x, where x is a random length in the interval $0<x<L$; then $L-x<1\rightarrow$ second new initial length $L-x$ or $L-x>1\rightarrow$ contribution unity to $y(L)$ and second new initial length $L-x-1$. Repeat from the beginning with new initial lengths. (Continue this procedure until there are no further initial lengths.)*

With $\lambda^{(L)}$ the total length wasted in the crazy carpentry on a single board of initial length L, the moment-generating function of Eq. (1) is given by Eq. (5) and by

$$1\leq L$$

$$\psi(L,s)=\int_0^{L-1}L^{-1}\mathbf{E}(\exp[s(\lambda^{(x)}+\lambda^{(L-x-1)})])dx+\int_{L-1}^{L}L^{-1}\mathbf{E}(\exp[s(\lambda^{(x)}+L-x)])dx, \quad (43)$$

or

$$1\leq L,\quad L\psi(L,s)=\int_0^{L-1}\psi(x,s)\psi(L-x-1,s)dx+\int_{L-1}^{L}\psi(x,s)\exp[s(L-x)]dx. \quad (44)$$

With Eq. (2), differentiation of Eqs. (5) and (44) with respect to s gives Eq. (8) and

$$1\leq L,\quad Lw(L)=\int_0^{L}w(x)dx+\int_0^{L-1}w(x)dx+\tfrac{1}{2}. \quad (45)$$

Differentiating this equation with respect to L and integrating again, using $w(1)\equiv1$, results in

$$1\leq L,\quad w(L)=1+\int_0^{L-1}\frac{w(x)dx}{x+1}. \quad (46)$$

With Eq. (3), differentiation twice with respect to s gives Eq. (10) and

$$1\leq L,\quad Lu(L)=\int_0^L u(x)dx+\int_0^{L-1} u(x)dx+2\int_0^{L-1} w(x)w(L-x-1)dx+2\int_{L-1}^{L}(L-x)w(x)dx+\tfrac{1}{3}. \quad (47)$$

These results are due to E. N. Gilbert, as is also the derivation that follows of the asymptotic dependence of $w(L)$ on L for large L.

Write

$$q(L)\equiv w(L)-L-1. \quad (48)$$

Then

$$\begin{aligned} Ldq(L)/dL&=0, && 0\leq L<1\\ &=q(L-1), && 1\leq L. \end{aligned} \quad (49)$$

Laplace transformation of this equation, with

$$Q(p)\equiv\int_0^\infty q(L)\exp(-pL)dL, \quad (50)$$

gives

$$pdQ(p)/dp+[1+\exp(-p)]Q(p)=0. \quad (51)$$

The solution for $Q(p)$ is

$$Q(p)=-p^{-1}\exp\left[\int_p^\infty x^{-1}\exp(-x)dx\right]; \quad (52)$$

note that $Q(p)$ asymptotically $(-p^{-1})$ for large p follows directly from Eq. (50), since $q(L)=-1$ holds for $0\leq L<1$. For small values of p, the integral in Eq. (52) is approximately $\ln(1/p)-c$, where $c\sim 0.5772$ is Euler's constant. Hence $Q(p)$ is approximately $-\exp(-c)\cdot p^{-2}$ for small p and $q(L)$ for large L is asymptotic to the transform of this, and thus approximately $[-\exp(-c)]L$. It follows that $w(L)$ for large L is approximately $[1-\exp(-c)]L$, and the limiting relative yield Y is $\exp(-c)\sim 0.5618$. This result was also obtained by H. O. Pollack.

Comparison might be made with the value of Y from the method of approximation employed in Sec. 2.3 for the semiconductor case, and this method then used also to determine F. Equations (46) and (47) are written for large L and exact solutions are used in integrands up to the value 2 of the arguments of the functions and the approximations for large L of Eqs. (21) and (23) beyond. Besides Eqs. (8) and (10), the exact solutions are

$$1\leq L<2,\qquad w(L)=L-\ln L, \quad (53)$$

and[28]

$$1\leq L<2,\qquad u(L)=L^2+\ln L-2L\ln L. \quad (54)$$

Equation (46) then gives

$$A=B \quad (55)$$

from the requirement that the terms in $\ln L$ cancel, and, with this result,

$$Y=1-A=\frac{1}{4}\left(1+\ln 3+\int_1^2\frac{\ln x}{x+1}dx\right)\sim 0.5613. \quad (56)$$

This Y is close to the exact theoretical value, being only about 0.1% smaller. Note that an apparent threshold $L_f\sim 0.78$ results, and this is presumably the average final length.

For the corresponding calculation of F, write

$$2\leq L,\quad Lu(L)=1+\int_1^L u(x)dx+\int_1^{L-1} u(x)dx+4\int_0^{\frac{1}{2}(L-1)} w(x)w(L-x-1)dx+2\int_{L-1}^{L}(L-x)w(x)dx, \quad (57)$$

which follows readily from Eqs. (10) and (47). Note that $w(L-x-1)$ may now be written as $A(L-x-1)+B$, since x does not exceed $\frac{1}{2}(L-1)$ in the integral in which this function occurs. An equation formally cubic in L results, but with cubic terms that cancel identically. The requirement that the quadratic terms cancel gives Eq. (55), while the linear terms cancel if

$$C-D=(13-8\ln 2)A-17A^2 \quad (58)$$

holds, and the constant terms vanish if

$$\tfrac{7}{2}C+5D=4(5/3-\ln 2)(1-2A)+\tfrac{1}{2}A^2 \quad (59)$$

holds.

The right-hand member of Eq. (58) very nearly vanishes for the A of Eq. (56); this gives (-0.0014) for $C-D$. It is thus a substantially consistent approximation to assume

$$C=D \quad (60)$$

which, moreover, probably holds exactly for secondaries of one kind only.[29] Equations (58) and (60) imply $A=(1/17)(13-8\ln 2)\sim 0.4385$, a value even closer to the exact theoretical one of about 0.4382 than is the value 0.4387 of Eq. (56). With Eq. (60) and the corresponding value of A, Eq. (59) gives

$$C=0.0674,\quad F=C/(1-A)=0.12048. \quad (61)$$

[28] This was first derived by Miss Li-Ying Fan.

[29] That this does not follow precisely with the present approximation is related to the explicit dependence of the integrand in Eq. (46) on x as well as on $w(x)$.

In Fig. 2, the limiting slope of the theoretical curve for $v(L)$ corresponds to these values.

With Eqs. (55) and (60), Y and F may be obtained from Monte Carlo $y(L)$ and $v(L)$ that are very nearly on the limiting lines by use of

$$Y=[y(L)+1]/(L+1) \tag{62}$$

and

$$F=v(L)/[y(L)+1]. \tag{63}$$

A Monte Carlo run in addition to those of the points of Fig. 2 was made for 200 boards of initial length 1000, and this gave $Y=0.5614$ and $F=0.1375$.

A2. The Case of the Parking Problem

The rules for this case are: *If initial length is L, then $L<1\rightarrow$ zero contribution to $y(L)$ or $L>1\rightarrow$ contribution unity to $y(L)$ and new initial lengths x and $L-x-1$, where x is a random length in the interval $0<x<L-1$. Repeat from the beginning with new initial lengths. (Continue until there are no further initial lengths.)*

The moment-generating function for this case is given by Eq. (5) and by

$$1\leq L,\quad \psi(L,s)=\int_0^{L-1}(L-1)^{-1}\times\mathsf{E}(\exp[s(\lambda^{(x)}+\lambda^{(L-x-1)})])dx \tag{64}$$

or

$$1\leq L,\quad (L-1)\psi(L,s)=\int_0^{L-1}\psi(x,s)\times\psi(L-x-1,s)dx. \tag{65}$$

Equations (5) and (65) give Eq. (8),

$$1\leq L,\qquad w(L)=2(L-1)^{-1}\int_0^{L-1}w(x)dx, \tag{66}$$

Eq. (10), and

$$1\leq L,\qquad u(L)=2(L-1)^{-1}\times\int_0^{L-1}[u(x)+w(x)w(L-x-1)]dx. \tag{67}$$

Equation (66) may be written as an equation in $y(L)$, namely

$$1\leq L,\qquad y(L)=1+2(L-1)^{-1}\int_0^{L-1}y(x)dx, \tag{68}$$

and in this form in readily seen to agree with the equation given by Rényi.[7]

Treating the parking-problem case by the method of approximation employed in Secs. 2.3 and A1 involves the exact solutions

$$1\leq L<2,\qquad w(L)=L-1,\quad u(L)=(L-1)^2, \tag{69}$$

which are readily derived. Then, Eq. (66) gives

$$A=B=\tfrac{1}{4},\quad Y=\tfrac{3}{4} \tag{70}$$

a value of Y that is quite close to the exact result of Rényi, namely,

$$Y=\int_0^{\infty}\exp\left\{-2\int_0^t\frac{1-\exp(-u)}{u}du\right\}dt\sim 0.748, \tag{71}$$

which was obtained by solving the retarded differential equation that is equivalent to Eq. (68) by Laplace transformation. Note that Eq. (70) results in $L_f=\frac{1}{3}$.

The method applied to Eq. (67) gives

$$C=D+4A(1-4A)=D \tag{72}$$

and

$$3C+5D=(4/3)-(26/3)A+4B+(37/3)A^2+4AB-15B^2, \tag{73}$$

whence

$$C=1/32,\quad F=C/(1-A)=1/24\sim 0.04167. \tag{74}$$

The approximate theoretical values of Y, L_f, and F are in good agreement with Monte Carlo results shown in Fig. 3 and given in Table III; the respective averages of the values of Y and F in the table are 0.7458 and 0.04838. Equations (62) and (63) apply also for the present case, and the Monte Carlo Y and F were obtained from $y(L)$ and $v(L)$ by use of these equations.

The method employed would of course give closer approximations if exact solutions were used in the integrands up to larger arguments. For the next stage of approximation, the required solutions are

$$2\leq L<3,\qquad w(L)=1-(L-2)(3-L)/(L-1),\quad u(L)=1-(L-2)^2(3-L)/(L-1). \tag{75}$$

As may be seen from Eqs. (66) and (67) and the forms of these solutions, Y and F calculated to the next approximation are not rational fractions, but involve logarithms as well. Similarly, the exact solutions for the next interval, $3\leq L<4$, contain terms in $\ln(L-2)$ and the corresponding further approximation gives equations that are not simply quadratic or cubic in L but which contain logarithmic terms. Requirements that terms in respective powers of L cancel and that coefficients of logarithmic terms vanish should, however, be consistent.

TABLE III. Dependence of Monte Carlo yields, variances, and Fano factors for the parking-problem case on L.

L	$y(L)$	$y(L)/L$	$v(L)/y(L)$	Y	F
2	1.0000	0.5000	0		
5	3.450	0.6900	0.07174	0.7417	0.05562
10	7.220	0.7220	0.05147	0.7473	0.04521
20	14.740	0.7370	0.04969	0.7495	0.04653
50	36.90	0.7379	0.04564	0.7430	0.04444
100	74.48	0.7448	0.05075	0.7473	0.05008

Equations (4) and (75) give

$$2\leq L<3, \qquad v(L)=2(L-2)(3-L)/(L-1)^2, \quad (76)$$

namely, a variance that vanishes at each end of the interval, where $w(L)$ and $u(L)$ both equal unity. This result is readily verified. Yield $y(L)$ unity and $v(L)$ zero for all L in the interval $1\leq L<2$ follow from Eqs. (69) and is otherwise evident from the rules. It is readily seen that the rules also imply that if $L=3$, then $y(L)$ is exactly 2 with $v(L)$ zero: After initial contribution to yield of unity, the random cut results in two new initial lengths just one of which provides a further and final contribution of unity. If L is slightly less than 3, then nonzero variance obtains, yields being 2 most of the time and unity a small part of the time, with $y(L)$ slightly less than 2. Similarly, if L is slightly greater than 3, then yields are 2 most of the time and 3 a small part of the time, with $y(L)$ slightly greater than 2 and variance nonzero.

With Eq. (62), the theoretical $y(4)$ furnishes Y quite accurately. The readily obtained solution

$$3\leq L<4, \quad y(L)=(L-1)^{-1}[7L-17-4\ln(L-2)] \quad (77)$$

gives $y(4)=2.7425$, whence follows $Y=0.7485$, which agrees with the three significant figures of Rényi's result of Eq. (71). A similar procedure based on Eq. (63) could be used to obtain F, but for specified accuracy it would be necessary to calculate $u(L)$ for an L appreciably larger than the L for which $y(L)$ would have to be calculated to give Y with the same accuracy.

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