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Analysis of Triple-Correlation Measurements

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The formalism for the analysis of angular-correlation measurements of the radiative decay of aligned nuclei is expressed in a manner such as to simplify computational procedures involved in cases where several continuous parameters (multipolarity mixing ratios and population numbers or statistical tensors) and their error matrix must be determined from the experimental data. Two specific analysis procedures are presented. The discussion is based upon triple-correlation formulas which include the possibility of one or more intermediate unobserved radiations in multiple cascades. Parallel developments are presented in terms of population parameters and statistical tensors of the aligned states. It is shown that a statistical-tensor representation leads to a more elegant and convenient formulation for general multiple cascades.

INTRODUCTION

THE theory of angular correlations of successive radiative transitions from isolated, aligned nuclear states is well developed. Excellent treatments of the subject have been given by Biedenharn and Rose,[1] Rose,[2] Devons and Goldfarb,[3] Litherland and Ferguson,[4] and many others. Attention has recently been concentrated upon problems of applying the formalism to the determination from experimental data of the spins of states involved in the decay and the continuous parameters relating to the emitted radiation. Methods of analysis of data have been treated, among others, by Ferguson and Rutledge,[5] Smith[6,7] and Ferguson.[8]

The analysis of data on angular correlations of, for example, two successive radiations in cascade from a state formed by capture of a nonzero-spin particle is, in general, complicated by the nonlinear dependence of the correlation upon several continuous angular-momenta mixing parameters in addition to the quantized spin variables. The formation of a resonance state by capture may be characterized by three continuous variables; the channel spin-mixing parameter and the parameters relating to the mixing of orbital angular momenta in each channel. There may also be a multipolarity mixing parameter for each of the radiative transitions which follow the formation of the resonance giving a total of five continuous variables to be extracted from the data.

The method of Ferguson and Rutledge[5] is primarily applicable to radiations from states formed by capture of a bombarding particle. The method requires that the mixing parameters relating both to the formation and to the decay of the resonance state be simultaneously fitted to the data. In cases where only two mixing parameters appear, e.g., when the target nucleus has zero spin, contour diagrams have proven very useful and unique solutions have been obtained.[9] Cases in which more than two mixing parameters are required are frequently encountered, but few such problems have been completely analyzed because of their complexity when approached by the usual analytic or graphical techniques.

The introduction by Litherland and Ferguson[4] and by Smith[6] of analysis techniques involving a formula-

[1] L. C. Biedenharn and M. E. Rose, Rev. Mod. Phys. **25**, 729 (1953).
[2] M. E. Rose, ORNL-2516, Office of Technical Services. U. S. Department of Commerce, Washington D. C., 1958 (unpublished).
[3] S. Devons and L. J. B. Goldfarb in *Encyclopedia of Physics*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 42, p. 362. See also the article of G. R. Satchler, Phys. Rev. **94**, 1304 (1954).
[4] A. E. Litherland and A. J. Ferguson, Can. J. Phys. **39**, 788 (1961).
[5] A. J. Ferguson and A. R. Rutledge, Atomic Energy of Canada, Ltd., CRP-615, AECL-420, 1962 (unpublished).
[6] P. B. Smith, in *Nuclear Reactions*, edited by P. M. Endt and P. B. Smith (North-Holland Publishing Company, Amsterdam, 1962), Vol. II.
[7] P. B. Smith, Can. J. Phys. **42**, 1101 (1964).
[8] A. J. Ferguson, *Angular Correlation Methods in Gamma-Ray Spectroscopy* (North-Holland Publishing Company, Amsterdam, 1965).
[9] G. I. Harris and L. W. Seagondollar, Phys. Rev. **131**, 787 (1963).

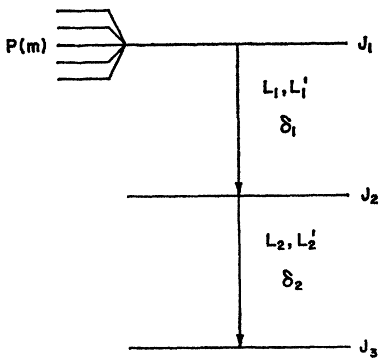

FIG. 1. Decay scheme for a double gamma-ray cascade from an aligned nuclear state. The quantum numbers and parameters of Eq. (1) are shown.

tion in terms of the statistical tensors or population numbers of the state formed in the bombardment process has resulted in a more efficient approach to the general problem. The technique consists essentially of replacing the, in general, nonlinear parameters related to the formation of a state by linear parameters which specify the alignment of the state with respect to the incoming beam. Two important advantages accrue from this formulation: (1) The analysis of correlations of radiations emitted by the aligned state can be conducted without detailed knowledge of the mechanism of formation of the state. (2) Linear, least-squares computer programs can be used to obtain "best values" of the population numbers (or statistical tensors) where the nonlinear multipolarity mixing ratios for the emitted radiations are treated as parameters (see Smith[7]). These techniques have already found extensive use in cases where only a primary radiation from the aligned state is mixed. At present, only a few cases have been investigated in which both a primary and a secondary radiation in cascade are mixed because the entire range of both mixing ratios must be searched for possible acceptable solutions.[7]

The purpose of the present paper is to present the triple correlation formulas in a "factored" form which provides for a convenient extension to multiple cascades, a more efficient tabulation of necessary coefficients, and more efficient and flexible computer analysis techniques. The triple correlation formula is presented in such a form for double cascades and multiple cascades with one or more unobserved radiations, and is developed in terms of the population densities (parameters) and statistical tensors of the aligned state. A brief discussion is then given of linear, least-squares analysis techniques which utilize the greater efficiency of the factored form of the equations. The angular-correlation coefficients necessary for the analysis of data are being issued in a separate report. There is no claim to originality with respect to angular-correlation theory except perhaps for the specific manner of factorization and its utilization in analysis techniques. The intent is to provide a more practical and usable formulation for the more difficult situations now consistently being encountered by the experimentalist.

TRIPLE-CORRELATION FORMULA

No Unobserved Radiations

We shall employ the notation of Smith[6,7] for the intensity correlation of two successive radiations from an *aligned* state and show how it may be extended to cases involving one or more intermediate unobserved radiations. The extended form will then serve as the basic form in the discussion of analysis.

The intensity correlation expressed as a function of the angles θ_1, θ_2, and ϕ, where θ_1 and θ_2 are the angles between the propagation vectors of the incoming beam and of the primary and secondary radiations, respectively, and where ϕ is the azimuth angle between the two radiations, is given by[6]

$$W(\theta_1,\theta_2,\phi)=\sum_{KMN} A_{KM}{}^N Q_K Q_M X_{KM}{}^N(\theta_1,\theta_2,\phi). \quad (1)$$

The decay scheme to which this formulation is applicable is shown in Fig. 1. The Q_K and Q_M are introduced in order to take the finite solid angle of the detectors into account. The functions $X_{KM}{}^N(\theta_1,\theta_2,\phi)$ are defined by

$$X_{KM}{}^N(\theta_1,\theta_2,\phi)=\left[\frac{(2M+1)(2K+1)(K-N)!(M-N)!}{(K+N)!(M+N)!}\right]^{1/2} \times P_K{}^N(\cos\theta_1)P_M{}^N(\cos\theta_2)\cos N\phi. \quad (2)$$

In the population-parameter representation, the expression for $A_{KM}{}^N$ is

$$A_{KM}{}^N=\sum_{m\geq 0} P(m) \sum_{L_1L_1'L_2L_2'} \delta_1{}^{p_1}\delta_2{}^{p_2}C_{KM}{}^N, \quad (3)$$

where

$$C_{KM}{}^N=\beta(-)^f\hat{J}_1{}^2\hat{J}_2{}^2\hat{L}_1\hat{L}_1'\hat{L}_2\hat{L}_2'(L_11L_1'-1|K0)(L_21L_2'-1|M0)W(J_2L_2J_2L_2';J_3M)$$
$$\times\sum_k(-)^{J_1-m}(J_1mJ_1-m|k0)(K-NMN|k0)\begin{Bmatrix} J_2 & L_1 & J_1 \\ J_2 & L_1' & J_1 \\ M & K & k \end{Bmatrix}. \quad (4)$$

The factor $(-)^f$ determines the phase, where

$$f=J_3-J_2+L_1'-L_2+L_2'+M+N,$$

and the exponents p_1 and p_2 take on the values 0, 1, or 2 for pure L, mixed L, L', or pure L' radiation, respectively. The quantities δ_1 and δ_2 are defined to be the ratio of the transition matrix elements for L'-pole to L-pole radiation. The quantities in Eq. (4) written with a caret ($\hat{q}$) represent $(2q+1)^{1/2}$. The population parameters $P(m)$ represent the population of the magnetic substates m and $-m$ taken together. The factor β provides for the term multiplicity due to the restrictions $N\geq 0$, $L_1'\geq L_1$, $L_2'\geq L_2$. Explicitly, $\beta=(2-\delta_{N,0})(2-\delta_{L_1,L_1'})(2-\delta_{L_2,L_2'})$. Standard notations are employed for the Clebsch-Gordan, Racah, and 9-j coefficients. The indices K and M take on positive even integral values (and zero), and N takes on both even and odd positive integral values such that N does not exceed either K or M.

We now observe by inspection of Eq. (4) that the $C_{KM}{}^N$ coefficient defined by Smith can be written as

$$C_{KM}{}^N(J_1J_2J_3L_1L_1'L_2L_2'm)=\frac{C_{KM}{}^N(J_1J_2\mathcal{J}_3L_1L_1'\Lambda_2\Lambda_2m)}{(-)^{\mathcal{J}_3}\bar{Z}_1(\Lambda_2J_2\Lambda_2J_2;\,\mathcal{J}_3M)}(-)^{J_3}(2-\delta_{L_2,L_2'})\bar{Z}_1(L_2J_2L_2'J_2;\,J_3M)\,, \tag{5}$$

where the coefficient $\bar{Z}_1$ defined and tabulated by Ferguson[8] is related to the Z_1 coefficient tabulated by Sharp *et al.*[10] as follows:

$$\begin{aligned}\bar{Z}_1(LbL'b';\,cM)&=(-)^{M-L+L'-1}\hat{L}\hat{L}'\hat{b}\hat{b}'(L1L'-1\,|\,M0)W(LbL'b',cM)\,,\\&=(-)^{L'-L-M/2}Z_1(LbL'b';\,cM)\,.\end{aligned} \tag{6}$$

The ratio $C_{KM}{}^N/(-)^{\mathcal{J}_3}\bar{Z}_1$ is independent of the quantum numbers L_2, L_2', and J_3. Hence, they are replaced by the "dummy indices" Λ_2 and $\mathcal{J}_3$ in Eq. (5) in order to stress this property of the ratio. Λ_2 and $\mathcal{J}_3$ can be chosen at will provided all triangle conditions which involve them are satisfied. For convenience, we define the quantities $E_{KM}{}^N$ and h_M, where

$$\begin{aligned}E_{KM}{}^N(J_1J_2L_1L_1'm)&=(-)^{J_2-\mathcal{J}_3}\hat{J}_2\frac{C_{KM}{}^N(J_1J_2\mathcal{J}_3L_1L_1'\Lambda_2\Lambda_2m)}{\bar{Z}_1(\Lambda_2J_2\Lambda_2J_2;\,\mathcal{J}_3M)}\\&=(-)^{L_1'+N+1}(2-\delta_{N,0})(2-\delta_{L_1,L_1'})\hat{J}_1^2\hat{J}_2\hat{L}_1\hat{L}_1'(L_11L_1'-1\,|\,K0)\\&\quad\times\sum_k(-)^{J_1-m}(J_1mJ_1-m\,|\,k0)(K-NMN\,|\,k0)\begin{Bmatrix}J_2&L_1&J_1\\J_2&J_1'&J_1\\M&K&k\end{Bmatrix},\end{aligned} \tag{7}$$

and

$$\begin{aligned}h_M(J_2J_3L_2L_2')&=(\hat{J}_2)^{-1}(-)^{J_3-J_2}\\&\quad\times(2-\delta_{L_2,L_2'})\bar{Z}_1(L_2J_2L_2'J_2;\,J_3M)\,.\end{aligned} \tag{8}$$

Hence,

$$\begin{aligned}&C_{KM}{}^N(J_1J_2J_3L_1L_1'L_2L_2'm)\\&\quad=E_{KM}{}^N(J_1J_2L_1L_1'm)h_M(J_2J_3L_2L_2')\,.\end{aligned} \tag{9}$$

We note that $E_{KM}{}^N$ depends only upon parameters of the primary radiation, while h_M depends only upon parameters of the secondary (or final) radiation. $E_{KM}{}^N$ is normalized such that $E_{00}{}^0=\delta_{L_1,L_1'}$; similarly h_M is normalized such that $h_0=\delta_{L_2,L_2'}$.

The coefficient $A_{KM}{}^N$ in Eq. (1) may now be written as follows:

$$\begin{aligned}A_{KM}{}^N=\sum_m P(m)&\sum_{L_1L_1'}(1+\delta_1^2)^{-1}\delta_1^{p_1}E_{KM}{}^N(J_1J_2L_1L_1'm)\\&\times\sum_{L_2L_2'}(1+\delta_2^2)^{-1}\delta_2^{p_2}h_M(J_2J_3L_2L_2')\,.\end{aligned} \tag{10}$$

[The distinction between the Kronecker delta function and the symbol for multipolarity mixing ratios should be evident in Eqs. (7) and (8) and in those following.] As suggested by Smith,[7] we have inserted the factors $1+\delta_1^2$ and $1+\delta_2^2$ into the denominator in order to provide for more elegant programming.

We now define two new functions G_{mM} and H_M as follows:

$$\begin{aligned}G_{mM}(\delta_1,\Theta)=&\sum_{L_1L_1'}(1+\delta_1^2)^{-1}\delta_1^{p_1}\\&\times\sum_{KN}E_{KM}{}^N(J_1J_2L_1L_1'm)Q_KQ_MX_{KM}{}^N(\Theta)\,,\end{aligned} \tag{11}$$

and

$$H_M(J_2J_3\delta_2)=\sum_{L_2L_2'}(1+\delta_2^2)^{-1}\delta_2^{p_2}h_M(J_2J_3L_2L_2')\,, \tag{12}$$

where Θ stands for the set of angles θ_1, θ_2, ϕ. Because of the normalization of h_M, we have $H_0(\delta_2)=1$.

The triple correlation formula [Eq. (1)] appears in terms of the newly defined quantities G_{mM} and H_M as

[10] W. T. Sharp, J. M. Kennedy, B. J. Sears, and M. G. Hoyle, Atomic Energy of Canada, Ltd., CRT-556, AECL-97, 1957 (unpublished).

follows:

$$W(\Theta)=\sum_{mM} P(m)G_{mM}(\delta_1,\Theta)H_M(\delta_2). \tag{13}$$

One or more Intermediate Unobserved Radiations

Recent experience with angular-correlation studies on multiple-step gamma-ray cascades following the formation of a resonance state by proton capture has indicated the usefulness of measurements in which the intensity correlation of *any* pair of emitted radiations (relative to the incoming beam) is observed, rather than to restrict measurements to only the first two members of the cascade as has usually been done.[11,12] The possibility exists of having unobserved radiations emitted prior to the first of the two observed radiations, or emitted after the first but before the second observed radiation. Both situations may also, of course, occur in a given problem. The emission of unobserved radiations after the last observed radiation has no effect on the analysis and, thus, is not considered.

Consider the decay scheme depicted in Fig. 2 and suppose that only the radiation corresponding to L_i, L_i' and L_e, L_e' is observed. Then according to Satchler,[3] the intensity correlation can be written, with a trivial change in normalization to provide agreement with that of Eq. (13), as follows:

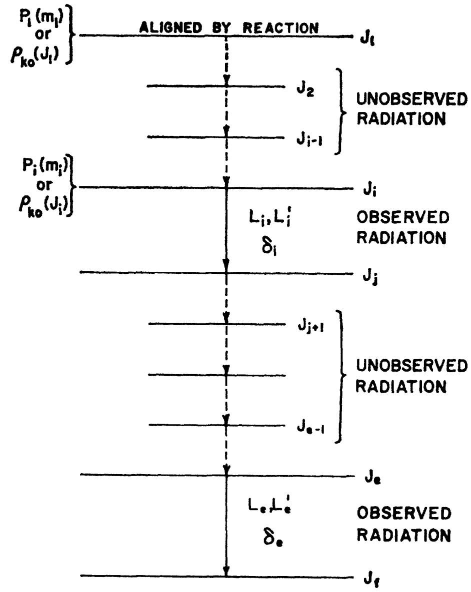

FIG. 2. Decay scheme for a generalized multiple cascade from an aligned nuclear state. The quantum numbers and parameters of Eq. (14) are shown. The first observed transition is taken to be between states (*i*) and (*j*), and the second observed transition is between states (*e*) and (*f*). The degree of alignment of state (1) is characterized by the statistical tensors or population parameters of that state.

$$W(\Theta)=\sum_{kKM} A_k(J,J_1J_1)A_M(J_eJ_e,J_f)R_{kKM}(J_iJ_iJ_jJ_j)Q_KQ_MS_{kKM}(\Theta)u_k(L_1L_1'J_1J_2,\delta_1)u_k(L_2L_2'J_2J_3,\delta_2)\cdots$$
$$\times u_k(L_{i-1}L_{i-1}'J_{i-1}J_i,\delta_{i-1})u_M(L_jL_j'J_jJ_{j+1},\delta_j)u_M(L_{j+1}L_{j+1}'J_{j+1}J_{j+2},\delta_{j+1})\cdots u_M(L_{e-1}L_{e-1}'J_{e-1}J_e,\delta_{e-1}), \tag{14}$$

where Θ now refers to the set of angles θ_i, θ_e, ϕ_{ie}. The factors u_k or u_M for the unobserved radiations are given by

$$u_l(J_aJ_b,\delta)=\frac{U_l(LJ_aJ_b)+\delta^2U_l(L'J_aJ_b)}{1+\delta^2}, \tag{15}$$

where $U_l(LJ_aJ_b)=(-)^{J_a+J_b-L}\hat{J}_a\hat{J}_bW(J_aJ_aJ_bJ_b;lL)$. The factor $A_M(J_eJ_e,J_f)$ which pertains to the final observed radiation is given as follows:

$$\begin{aligned}A_M(J_eJ_e,J_f)&=(1+\delta_e^2)^{-1}\sum_{L_eL_e'}\delta_e^{p_e}(2-\delta_{L_e,L_e'})\hat{J}_e\hat{L}_e\hat{L}_e'(-)^{J_f-J_e-L_e+L_e'+M-1}(L_e1L_e'-1|M0)W(L_eJ_eL_e'J_e;J_fM)\\&=(\hat{J}_e)^{-1}(1+\delta_e^2)^{-1}(-)^{J_f-J_e}\sum_{L_eL_e'}(2-\delta_{L_e,L_e'})\delta_e^{p_e}\bar{Z}_1(L_eJ_eL_e'J_e;J_fM)\\&=H_M(J_eJ_f\delta_e).\end{aligned} \tag{16}$$

We note that $H_M(J_eJ_f\delta_e)$ has the same form as Eq. (9), but the quantum numbers have been replaced by those of the final observed radiation. The factor R_{kKM} which pertains to the first observed, emitted radiation is given by

$$R_{kKM}(J_iJ_iJ_jJ_j)=(\hat{M})^{-1}\hat{J}_i\hat{J}_j(1+\delta_i^2)^{-1}\sum_{L_iL_i'}(2-\delta_{L_i,L_i'})\delta_i^{p_i}\hat{L}_i\hat{L}_i'(-)^{L_i'-1}(L_i1L_i'-1|K0)\begin{Bmatrix}J_j & L_i & J_i\\ J_j & L_i' & J_i\\ M & K & k\end{Bmatrix}. \tag{17}$$

[11] H. Van Rinsvelt and P. B. Smith, Physica **30**, 59 (1964).
[12] G. I. Harris, in Proceedings of the Conference on the Structure of Low-Medium Mass Nuclei, University of Kansas, 1964 (unpublished).

The angle function S_{kKM} is related to the $X_{KM}{}^N$ defined in Eq. (2) through

$$S_{kKM}(\Theta)=\hat{M}\sum_{N\geq 0}(-)^N(2-\delta_{N,0})(K-NMN|k0)X_{KM}{}^N(\Theta). \tag{18}$$

Finally, the factor A_k which relates to the formation of the state labeled by J_1, m_1 by the incoming beam is given by

$$A_k(J,J_1J_1)=\rho_{k0}(J_1). \tag{19}$$

Here $\rho_{k0}(J_1)$ is just the zeroth component of the statistical tensor of rank k for the formation of the state J_1. For convenience, the normalization of A_k has been chosen so that $A_0=\rho_{00}=1$. The population parameters are related to the statistical tensors by[3]

$$\rho_{k0}(J_1)=\hat{J}_1\sum_{m_1}(-)^{J_1-m_1}(J_1m_1J_1-m_1|k0)P_1(m_1). \tag{19a}$$

Now, in order to cast Eq. (14) in the general form of Eq. (13), we define the quantity

$$\bar{G}_{m_1M}(\delta_1\cdots\delta_i,\Theta)=\hat{J}_1\sum_k(-)^{J_1-m_1}(J_1m_1J_1-m_1|k0)[u_k(\delta_1)\cdots u_k(\delta_{i-1})]T_{kM}(\delta_i,\Theta), \tag{20}$$

where

$$T_{kM}(\delta_i,\Theta)=\sum_K Q_KQ_MR_{kKM}(\delta_i)S_{kKM}(\Theta). \tag{21}$$

We find that T_{kM} is related to the G_{mM} defined in Eq. (11) by

$$T_{kM}(\delta_i,\Theta)=(\hat{J}_i)^{-1}\sum_{m_i\geq 0}(2-\delta_{m_i,0})(-)^{J_i-m_i}(J_im_iJ_i-m_i|k0)G_{m_iM}(\delta_i,\Theta), \tag{22}$$

where account is taken of the fact that k is even. We can now express $\bar{G}_{m_iM}$ in terms of G_{m_iM}:

$$\bar{G}_{m_1M}(\delta_1,\cdots,\delta_i,\Theta)=\hat{J}_1(\hat{J}_i)^{-1}\sum_k(-)^{J_1-m_1}(J_1m_1J_1-m_1|k0)[u_k(\delta_1)\cdots u_k(\delta_{i-1})]$$
$$\times\sum_{m_i\geq 0}(-)^{J_i-m_i}(2-\delta_{m_i,0})(J_im_iJ_i-m_i|k0)G_{m_iM}(\delta_i,\Theta). \tag{23}$$

The factor $\bar{H}_M$ is defined as follows:

$$\bar{H}_M(\delta_j,\cdots,\delta_e)=[u_M(\delta_j)\cdots u_M(\delta_{e-1})]H_M(\delta_e). \tag{24}$$

Equation (14) for the triple correlation may now be written in the same form as Eq. (13) as follows:

$$W(\Theta)=\sum_{m_1M}P_1(m_1)\bar{G}_{m_1M}(\delta_1,\cdots,\delta_i,\Theta)\bar{H}_M(\delta_j,\cdots,\delta_e). \tag{25}$$

Unfortunately, however, the simple appearance of this expression is misleading because of the complexity, in general, of the factor $\bar{G}_{m_1M}$. Equation (25) is, for that reason, very difficult to apply in practice as it stands.

We shall consider two changes in Eq. (25) which simplify the application to actual problems. The first change is to place the emphasis on the population parameters of the state labeled by J_i, m_i instead of the state J_1, m_1 (see Fig. 2). The second and more fundamental change is to replace the population parameter representation by a statistical tensor representation. The latter change will be considered in the following subsection.

The shift in emphasis from the populations of state J_1,m_1 to those of state J_i,m_i is accomplished by making use of Eq. (23) to put Eq. (25) in the following form:

$$W(\Theta)=\sum_{m_iM}\{\hat{J}_1(\hat{J}_i)^{-1}\sum_{m_1\geq 0}P_1(m_1)\sum_k(-)^{J_1-m_1+J_i-m_i}(2-\delta_{m_i,0})(J_1m_1J_1-m_1|k0)(J_im_iJ_i-m_i|k0)$$
$$\times[u_k(\delta_1)\cdots u_k(\delta_{i-1})]\}G_{m_iM}(\delta_i,\Theta)\bar{H}_M(\delta_j,\cdots,\delta_e). \tag{26}$$

By comparison with Eq. (13), we can identify the quantity in the curly brackets with $P_i(m_i)$. Hence, we have

$$W(\Theta)=\sum_{m_iM}P_i(m_i)G_{m_iM}(\delta_i,\Theta)\bar{H}_M(\delta_j,\cdots,\delta_e), \tag{27}$$

where

$$P_i(m_i)=\hat{J}_1(\hat{J}_i)^{-1}\sum_{m_1\geq 0}P_1(m_1)\sum_k(-)^{J_1-m_1+J_i-m_i}(2-\delta_{m_i,0})(J_1m_1J_1-m_1|k0)$$
$$\times(J_im_iJ_i-m_i|k0)[u_k(\delta_1)\cdots u_k(\delta_{i-1})]. \tag{28}$$

Equation (27) will be taken as the standard form of the triple-correlation formula in the population parameter representation. The relative simplicity of the factor G_{m_1M} as compared with $\bar{G}_{m_1M}$ provides for straightforward practical application in complex situations. It can be readily seen that Eq. (27) reduces to the proper form [Eq. (13)] for the case of no intermediate unobserved radiations.

Equation (28), which relates the population of the state J_i,m_i to those of state J_1,m_1, could be applied in the analysis after the $P_i(m_i)$ (and their error matrix) have been determined by a least-squares analysis of Eq. (27). We prefer, however, to show that Eq. (28) reduces to the more familiar form in which appear the squares of the vector-coupling coefficients, and in which is involved a summation only over the magnetic substates and not also over the index k.

It is evident from the form of Eq. (28) that $P_i(m_i)$ can be expressed in terms of the population parameters $P_{i-1}(m_{i-1})$ of the preceding state in the decay scheme as follows:

$$P_i(m_i)=\hat{J}_{i-1}(\hat{J}_i)^{-1}(2-\delta_{m_i,0})\sum_{m_{i-1}\geq 0} P_{i-1}(m_{i-1})\sum_k(-)^{J_{i-1}-m_{i-1}+J_i-m_i} \times(J_{i-1}m_{i-1}J_{i-1}-m_{i-1}|k0)(J_im_iJ_i-m_i|k0)u_k(\delta_{i-1}), \quad (29)$$

or

$$P_i(m_i)=(2-\delta_{m_i,0})\sum_{m_{i-1}\geq 0} f(m_{i-1},m_i,\delta_{i-1})P_{i-1}(m_{i-1}), \quad (30)$$

where

$$f(m_{i-1},m_i,\delta_{i-1})=(1+\delta_{i-1}^2)^{-1}[F(m_{i-1},m_i,L_{i-1})+\delta_{i-1}^2F(m_{i-1},m_i,L_{i-1}')], \quad (31)$$

$$F(m_{i-1},m_i,L_{i-1})=\sum_k V_k(m_{i-1},m_i,L_{i-1}), \quad (32)$$

and

$$\begin{aligned}V_k(m_{i-1},m_i,L_{i-1})&=\hat{J}_{i-1}(\hat{J}_i)^{-1}(-)^{J_{i-1}-m_{i-1}+J_i-m_i}(J_{i-1}m_{i-1}J_{i-1}-m_{i-1}|k0)(J_im_iJ_i-m_i|k0)U_k(L_{i-1}J_{i-1}J_i)\\&=(-)^{-L_{i-1}-m_{i-1}-m_i}(\hat{J}_{i-1})^2(J_{i-1}m_{i-1}J_{i-1}-m_{i-1}|k0)(J_im_iJ_i-m_i|k0)W(J_{i-1}J_{i-1}J_iJ_i;kL_{i-1}). \quad (33)\end{aligned}$$

Equation (15) and the definition of $U_k(LJ_aJ_b)$ have been utilized in defining the above quantities.

Now consider the quantity $F(m_{i-1},m_i,L_{i-1})$. We note that the index k in all the preceding equations must be even as a result of the assumed definite spin and parity for the decaying state. With the observation that $V_k(-\alpha,\beta,L)=(-)^kV_k(\alpha,\beta,L)$, the sum over even k in the definition of $F(m_{i-1},m_i,L_{i-1})$ can be extended to a sum over all (positive) k by writing

$$\begin{aligned}F(m_{i-1},m_i,L_{i-1})&=\tfrac{1}{2}\sum_{\text{all }k\geq 0}[1+(-)^k]V_k(m_{i-1},m_i,L_{i-1})\\&=\tfrac{1}{2}\sum_{\text{all }k\geq 0}[V_k(m_{i-1},m_i,L_{i-1})+V_k(-m_{i-1},m_i,L_{i-1})]. \quad (34)\end{aligned}$$

One can now show, by means of the sum rule,[13]

$$\sum_f \hat{e}\hat{f}(a\alpha f\beta+\delta|c\alpha+\beta+\delta)(b\beta d\delta|f\beta+\delta)W(abcd;ef)=(a\alpha b\beta|e\alpha+\beta)(e\alpha+\beta d\delta|c\alpha+\beta+\delta), \quad (35)$$

that

$$\sum_{\text{all }k} V_k(m_{i-1},m_i,L_{i-1})=(L_{i-1}m_i+m_{i-1}J_i-m_i|J_{i-1}-m_{i-1})^2, \quad (36)$$

and similarly for $\sum V_k(-m_{i-1},m_i,L_{i-1})$. We have, thus,

$$F(m_{i-1},m_i,L_{i-1})=\tfrac{1}{2}[(L_{i-1}m_i+m_{i-1}J_i-m_i|J_{i-1}m_{i-1})^2+(L_{i-1}m_i-m_{i-1}J_i-m_i|J_{i-1}-m_{i-1})^2]. \quad (37)$$

A similar expression holds for $F(L_{i-1}')$. It now follows that, if in Eq. (30) the sum is extended to all m_{i-1} such that $-J_{i-1}\leq m_{i-1}\leq J_{i-1}$, we get

$$P_i(m_i)=2^{\delta(m_{i-1},0)-\delta(m_i,0)}\sum_{\text{all }m_{i-1}} P_{i-1}(m_{i-1})\bar{f}(m_{i-1},m_i,\delta_{i-1}), \quad (38)$$

where now,

$$\bar{f}(m_{i-1},m_i,\delta_{i-1})=(1+\delta_{i-1}^2)^{-1}[(L_{i-1}m_i+m_{i-1}J_i-m_i|J_{i-1}m_{i-1})^2+\delta_{i-1}^2(L_{i-1}'m_i+m_{i-1}J_i-m_i|J_{i-1}m_{i-1})^2]. \quad (39)$$

[13] L. J. B. Goldfarb, in *Nuclear Reactions*, edited by P. M. Endt and M. Demeur (North-Holland Publishing Company, Amsterdam, 1959), Vol. I.

The factor $2^{\delta(m_{i-1},0)-\delta(m_i,0)}$ arises from the fact that both P_i and P_{i-1} are here defined to be the populations of the respective positive and negative magnetic substates taken together.

By induction from Eq. (38), we can now express the populations $P_i(m_i)$ in terms of $P_1(m_1)$ as follows:

$$P_i(m_i)=2^n \sum_{m_{i-1},m_{i-2},\cdots,m_1} P_1(m_1)[\bar{f}(m_1,m_2,\delta_1)\cdots\bar{f}(m_{i-1},m_i,\delta_{i-1})], \tag{40}$$

where $n=\delta_{m_1,0}-\delta_{m_i,0}$. Equation (40) is a generalization of equivalent expressions given earlier by one of the authors[12] and by Nordhagen.[14] The physical interpretation of this expression is more evident than that of Eq. (29) since the squared vector-coupling coefficients in the factors $\bar{f}$ are just the relative transition probabilities between the indicated substates.

Formulation in Terms of Statistical Tensors

Two undesirable characteristics of the population-parameter formulation presented above make it worthwhile to present also the parallel formulation in terms of the statistical tensors of the aligned states. One undesirable feature of population parameters is evident in Eq. (40). There is not a one-to-one correspondence between the population parameters of successive levels. A summation must be performed over the magnetic substates of each level independently. This feature would also cause difficulty in the general analysis procedure to be discussed. Another undesirable feature has been reported recently by Nordhagen.[15] A more general statement of his rule may be given as follows:

When the spin of the initially formed state (J_1) has a value larger than $L_i'+L_e'$ (see Fig. 2), the triple correlation is independent of those statistical tensors which have an order k larger than $2(L_i'+L_e')$. In this case the population parameters of the states labeled by J_1, $J_2\cdots$, J_i cannot be determined by angular correlation measurements. In cases where a knowledge of the formation of the initial state, or where a comparison between parameters determined by analyzing angular correlations of more than one gamma-ray cascade emitted from the same level, is required to obtain a unique solution, the population-parameter analysis is of little value when the situation is governed by the above rule. However, the set of statistical tensors which are determined by the correlation can be compared with formation theory or with those obtained from other cascades. The statistical-tensor formulation is therefore useful if only to signal certain limitations which are not immediately evident in the population-parameter formulation.

The general expression for one or more intermediate unobserved radiations given in Devons and Goldfarb[3] is actually written in terms of the statistical tensor notation. The term $A_k(J,J_1J_1)$ in Eq. (14) is just $\rho_{k0}(J_1)$ [see Eq. (19)], the zeroth component of the statistical tensor for the formation of the state labeled by J_1. Now, in analogy with the development following Eq. (18), we define the quantity $\bar{T}_{kM}$, where

$$\begin{aligned}\bar{T}_{kM}(\delta_1,\cdots,\delta_i,\Theta) &=\sum_K R_{kKM}(J_iJ_iJ_jJ_j,\delta_i)\\ &\quad\times[u_k(\delta_1)\cdots u_k(\delta_{i-1})]Q_KQ_MS_{kKM}(\Theta)\\ &=[u_k(\delta_1)\cdots u_k(\delta_{i-1})]T_{kM}(\delta_i,\Theta).\end{aligned} \tag{41}$$

The "tensor" quantity T_{kM} is given in terms of previously defined quantities in Eqs. (21) and (22).

A more explicit expression can be written for T_{kM} using Eqs. (7), (11), and (22). We obtain,

$$\begin{aligned}T_{kM}(\delta_i,\Theta)&=(1+\delta_i^2)^{-1}\sum_{L_i,L_i'}\delta_i^{p_i}\\ &\quad\times\sum_{KN}\bar{E}_{KM}{}^N(J_iJ_jL_iL_i',k)Q_KQ_MX_{KM}{}^N(\Theta),\end{aligned} \tag{42}$$

where

$$\begin{aligned}\bar{E}_{KM}{}^N(J_iJ_jL_iL_i',k)&=(-)^{L_i'+N+1}\\ &\quad\times 2^{2-\delta(L_i,L_i')-\delta(N,0)}\hat{J}_i\hat{J}_j\hat{L}_i\hat{L}_i'(L_i1L_i'-1|k0)\\ &\quad\times(K-NMN|k0)\begin{Bmatrix}J_j & L_i & J_i\\ J_j & L_i' & J_i\\ M & K & k\end{Bmatrix}.\end{aligned} \tag{43}$$

The coefficient $\bar{E}_{KM}{}^N$ plays the same role in the statistical-tensor formulation as does $E_{KM}{}^N$ in the population-parameter representation. The coefficients are related to each other through the transformation

$$E_{KM}{}^N(m)=\hat{J}\sum_k(-)^{J-m}(JmJ-m|k0)\bar{E}_{KM}{}^N(k). \tag{44}$$

The counterpart of Eq. (25) in the statistical-tensor formulation is

$$W(\Theta)=\sum_{kM}\rho_{k0}(J_1)\bar{T}_{kM}(\delta_1,\cdots,\delta_i,\Theta)\bar{H}_M(\delta_j,\cdots,\delta_e). \tag{45}$$

In analogy with the arguments following Eq. (25), we will also find it useful here to place the emphasis on the statistical tensors of the state J_i instead of the state J_1. We thus present the counterpart of Eq. (27) as

[14] R. Nordhagen, Proton Capture Formation Tables, Fysisk Institutt, Universitetet i Oslo, 1964 (unpublished).

[15] R. Nordhagen, Nucl. Instr. Methods **26**, 353 (1964). This reference cites a communication from the Chalk River Group which indicates that a similar limitation of population parameters has been found by G. Kaye. See also R. Nordhagen and A. Tveter, [Nucl. Phys. **63**, 529 (1965)] for further discussion of this rule and an introduction to the application of statistical tensor analysis.

follows:

$$W(\Theta)=\sum_{kM} \rho_{k0}(J_i)T_{kM}(\delta_i,\Theta)\bar{H}_M(\delta_j,\cdots,\delta_e). \quad (46)$$

At this point, a nice feature of this representation becomes clear. Whereas the expression, Eq. (40), relating the population parameters of state J_i with those of J_1 is not very simple and is not easily obtained from Eqs. (27) and (28), a relation between the statistical tensors of states J_i and J_1 is immediately evident from Eqs. (41), (45), and (46), and is given by

$$\rho_{k0}(J_i)=[u_k(\delta_1)\cdots u_k(\delta_{i-1})]\rho_{k0}(J_1). \quad (47)$$

Thus a one-to-one correspondence exists between the statistical tensors of successive states in the cascade. This feature alone would seem to make the statistical tensor representation more desirable than the population-parameter representation for the analysis of complex angular-correlation problems.

In addition to the limitation reported by Nordhagen and extended to multiple cascades at the beginning of this subsection, further limitations on the information available from triple-correlation measurements are evident upon a closer examination of Eqs. (46) and (47). Let $J_p(\min)$ be the smallest member of the set $[J_1, J_2\cdots, J_i, L_i'+L_e', L_i'+J_I(\min)]$, where $J_I(\min)$ is the smallest member of $[J_j,J_{j+1},\cdots J_e]$. Then for any state J of the set $[J_1,J_2,\cdots,J_i]$ which has a spin $J>J_p(\min)$, the $\rho_{k0}(J)$ with $k>2J_p(\min)$ will be indeterminant. In addition, all population parameters of such a state will be indeterminant.

ANALYSIS OF EXPERIMENTAL DATA

Summary of Formalism

In this section we discuss general features of analysis techniques which are similar in principle to the method discussed in detail by Smith,[7] but which take full advantage of the "factored" forms of the generalized triple-correlation formulas given in the preceding section. It will be assumed here that the correlation has been measured at a set consisting of A points over the surface of a sphere. The desired size and distribution of the set of A points to be sampled has been discussed by Smith. The analysis method considered by him is restricted to the determination of the spins and one, or both, of the unquantized radiation mixing ratios in a double gamma-ray cascade following the formation of a state characterized by the population numbers $P(m)$. The population numbers, which are also determined by the analysis, can then be compared with formation theory for the complete analysis.

The method proposed by Smith consists in treating the (one or two) mixing ratios as parameters in a least-squares solution of Eq. (1). In his notation, the sum over K, M, N, L_1, L_1', L_2, and L_2' of Eq. (1) is called $S_{ma}(\delta_1,\delta_2)$, and the resulting set of A equations with P (the number of magnetic substates which can be populated in the reaction) unknowns, is given by

$$W_a=\sum_m P(m)S_{ma}(\delta_1,\delta_2)\,, \quad (a=1,2,\cdots,A). \quad (48)$$

A computer program calculates the best values of the $P(m)$ for assigned values of δ_1 and/or δ_2, and the corresponding values of Q^2, defined as

$$Q^2=\frac{1}{A-P-q}\sum_a(W_a-W_a^*)^2\omega_a^2\,, \quad (49)$$

where the asterisk indicates that the value of a quantity has been calculated from Eq. (48) in which the least-squares solutions for the $P(m)$ have been resubstituted. The weight factor ω_a is the inverse of the standard deviation of W_a. If an assumed spin combination is in agreement with the data, then a minimum in the neighborhood of unity will be found in Q^2 as δ_1 and δ_2 are varied over the range $-\infty\leq\delta\leq\infty$. This minimum is referred to as χ^2. The quantity $(A-P-q)$ in Eq. (49) is the number of degress of freedom. The q is the number of mixing parameters varied.

After the correct spin combination and the values of the mixing parameters near the minimum of Q^2 have been found, the program of Smith automatically finds the exact minimum under the assumption that Q^2 is a quadratic function of δ_1 and δ_2 in the neighborhood of the minimum. If an acceptable value of χ^2 is found for only one spin combination, then the values of δ_1, δ_2, and the $P(m)$ represent the solution. The error matrix corresponding to these quantities is then determined using quantities which are already available. The treatment of errors will be discussed below in connection with the analysis method based upon the correlation function forms used in this paper.

The two basic forms we wish to consider here are Eq. (27) (population-parameter representation) and Eq. (46) (statistical-tensor representation). For convenience of comparison, and for purposes of summary, they are presented in a form similar to Eq. (48) as follows:

$$W_a=\sum_{m_iM} P_i(m_i)G_{m_iM}{}^a(\delta_i)\bar{H}_M(\delta_j,\cdots,\delta_e) \quad (50)$$

in the population-parameter (PP) representation, and

$$W_a=\sum_{kM} \rho_{k0}(J_i)T_{kM}{}^a(\delta_i)\bar{H}_M(\delta_j,\cdots,\delta_e) \quad (51)$$

in the statistical-tensor (ST) representation. The normalization of these equations is such that, in addition to $\bar{H}_0=1$, $E_{00}{}^0=\delta_{L_i,L_i'}$, and $\bar{E}_{00}{}^0=\delta_{k,0}\delta_{L_i,L_i'}$ in the definitions of G and T [Eqs. (11) and (42)], we have $\rho_{00}(J_i)=\sum_{m_i}P_i(m_i)=1$. Equations (50) and (51), in contrast to Eq. (48), are applicable to multiple cascades involving one or more intermediate unobserved radiations. They also, of course, can be applied to the usual double cascade in which there are no unobserved intermediate radiations. In this special case, $i=1$ and $\bar{H}_M(\delta_j,\cdots,\delta_e)$ becomes $H_M(\delta_2)$. H and $\bar{H}$ are defined by Eqs. (12) and (24), respectively.

The relation between the population parameters of successive states in the multiple cascade is given by Eq. (40), and the corresponding relation for the statistical tensors is given by Eq. (47). The transformation relating T_{kM} and G_{m_iM} is given by Eq. (22), and the transformation between the coefficients $\bar{E}$ and E which are required for the computation of T and G, respectively, is given by Eq. (44).

The coefficients necessary for analysis based upon Eqs. (50) and (51) are tabulated by Watson.[16] The coefficients $E_{KM}{}^N(J_1J_2L_1L_1'm)$ and $\bar{E}_{KM}{}^N(J_1J_2L_1L_1'k)$ are tabulated for half-integral $J_1\leq 11/2$, $J_2\leq 15/2$; for integral $J_1\leq 5$, $J_2\leq 7$; and for mixed dipole-quadrupole or quadrupole-octupole transitions. The coefficients $h_M(J_2J_3L_2L_2')$, Eq. (8), required for the H_M and $\bar{H}_M$ are tabulated for half-integral $J_2\leq 15/2$, $J_3\leq 21/2$; for integral $J_2\leq 7$, $J_3\leq 10$; and for multipolarities through octupole. In addition, the coefficient $U_l(LJ_aJ_b)$, Eq. (15), which refer to the intermediate unobserved radiations are tabulated for half-integral $J_a,J_b\leq 21/2$; integral $J_a,J_b\leq 10$; and the octupole limitation is again employed. The coefficients $\bar{f}(m_im_j,\delta_i)$, Eq. (39), which relate the population parameters of successive states in the cascade, may be obtained from the tabulations of Nordhagen[14] or directly from available tabulations of Clebsch-Gordan coefficients.

Solution of Eqs. (50) and (51)

In what follows, we present a generalized version of procedures which are being employed in this laboratory for solving Eqs. (50) and (51). These procedures are analogous to the method of Smith reviewed above. However, the method given below allows the general problem to be treated in "stages," each of which involve a search for minima in Q^2 for only one (or two) of the mixing parameters. If, for an assumed spin combination, no acceptable minimum Q^2 is obtained in any one of the "stages" of analysis, then the assumed combination is said not to be in agreement with the data. Further analysis based upon that combination is unnecessary. The method is found to be very efficient in the case of a double cascade where only one stage is required. In many cases, an assumed spin combination can be discarded after a cursory investigation of the first stage for minima in Q^2, and the succeeding stages need not be considered. The method shares with that of Smith the advantage that the possibility of convergence to a secondary minimum, or that an equally good minimum might exist and not be recognized, is eliminated. There is full assurance that all acceptable solutions will be found. An essential feature of these methods is the utilization of the variance-covariance (error) matrix[8,17] in order to account for correlation between parameters in the successive stages of the analysis and to provide errors for the determined quantities consistent with the experimental errors of the input data. Two approaches, to the solution of Eq. (50) or (51), which are formally similar yet sufficiently different in application to justify separate discussion, are being used. The first, and more general, approach consists of performing separate linear analysis for each transition in the cascade. Thus we think of one analysis stage for each mixed radiation. It is possible that some assumed spin combinations will be eliminated in even the first stage since acceptable Q^2 values may not be attained. The second approach, and apparently the more practical for the most common two-step cascade, is to combine a search for minima in Q^2 over the range of values of the mixing ratios of both observed radiations into the first stage of analysis. This approach is very similar to the solution of the two-parameter problem proposed by Smith, but is designed for multiple cascades and makes efficient use of the factored forms.

Single-Parameter Method

The basic step in this approach is effected by combining the population parameters (or statistical tensors) and the $\bar{H}_M$ into a single parameter $I_{\alpha M}$ defined as

$$I_{\alpha M}(\delta_j,\cdots,\delta_e)=\pi_\alpha\bar{H}_M(\delta_j,\cdots,\delta_e)\,, \tag{52}$$

where π_α stands for $P_i(m_i)$ in the PP representation, and for $\rho_{k0}(J_i)$ in the ST representation. The quantity $I_{\alpha M}$ is referred to as the "information parameter" since it contains all information about the degree of alignment of the state J_i and the properties of all transitions in the cascade below the state J_j (see Fig. 2).[18] The basic equation to be solved in the first stage of the analysis (in either representation) is given by

$$W_a=\sum_{\alpha M} I_{\alpha M}(\delta_j,\cdots,\delta_e)\Gamma_{\alpha M}{}^a(\delta_i)\,, \qquad (a=1,2,\cdots,A)\,, \tag{53}$$

[16] D. D. Watson and G. I. Harris, to appear in the form of an unpublished ARL Technical Documentary Report, Aerospace Research Laboratories, Wright-Patterson AFB, Ohio.

[17] A. H. Wapstra, G. J. Nijgh, and R. Van Lieshout, *Nuclear Spectroscopy Tables* (North-Holland Publishing Company, Amsterdam, 1959).

[18] An example of the information carried by the parameter $I_{\alpha M}$ is a simple proof of a theorem reported by Harris and Seagondollar (Ref. 9) and also discovered independently by Van Rinsvelt and Smith (Ref. 11); see also Ref. 12. A statement of the theorem is as follows: It is not possible to determine uniquely the mixing ratio δ of the terminal member of a gamma-ray cascade, by means of intensity-direction correlation measurements alone, if the spin of the state emitting the terminal gamma-ray is less than two. From Eqs. (24) and (52), it can be seen that all dependence of the angular correlation on the mixing ratio δ_e of the terminal member is contained in the factor $H_M(\delta_e)$. The functional dependence of H_M upon δ_e is given explicitly by Eq. (16) in which appears the coefficient $\bar{Z}_1(L_eJ_eL_e'J_e;J_fM)$. From the triangle conditions on $\bar{Z}_1$, we find that for $J_e<2$ the index M can assume, at most, the values 0 and 2. For $M=0$, $H=1$, and for $M=2$, H is quadratic in δ_e. Hence, in this case the dependence of the correlation upon δ_e is uniquely quadratic. There always exist, therefore, two solutions for δ_e; unless, of course, both solutions are identical.

where $\Gamma_{\alpha M}{}^{a}$ stands for $G_{m_i M}{}^{a}$ or $T_{kM}{}^{a}$ in the PP or ST representations, respectively. In analogy to the solution of Eq. (48) proposed by Smith, a computer program[19] calculates the best values of the $I_{\alpha M}$ of Eq. (53) for assigned values of δ_i and the corresponding values of Q^2 as defined by Eq. (49). The number of unknowns P in Eq. (49) must now be interpreted as the number of elements in the set of $I_{\alpha M}$. The same program also provides the error matrix for the set of quantities determined at the value of δ_i corresponding to a minimum in Q^2. The diagonal elements correspond to the variances in the $I_{\alpha M}$ and δ_i, and the off-diagonal elements correspond to the covariances between these quantities.

If, for the assumed spin combination, a δ_i is found which corresponds to an acceptable minimum in Q^2, the analysis may proceed to the second stage. The $I_{\alpha M}$, for which numerical values and an error matrix have been found in the first stage, are written explicitly using Eqs. (24) and (52) as follows:

$$I_{\alpha M}(\delta_j,\cdots,\delta_e)=\tau_{\alpha M}(\delta_j,\cdots,\delta_{e-1})H_M(\delta_e)\,, \quad (54)$$

where

$$\tau_{\alpha M}(\delta_j,\cdots,\delta_{e-1})=\pi_\alpha[u_M(\delta_j)\cdots u_M(\delta_{e-1})]\,. \quad (55)$$

The same formal procedure used in the first stage is now applied to Eq. (54). The best values of the $\tau_{\alpha M}$ are computed for assigned values of δ_e. A significant difference, however, is that whereas the error matrix $[\omega_a{}^2]$ for the first stage is diagonal (the measured values of W_a are uncorrelated), that corresponding to the second stage is the nondiagonal $I_{\alpha M}$ error matrix. For convenience, we denote the index combination αM by l, and the elements of the I_l error matrix by $(N^{-1})_{ll'}$. The quantity Q^2 appropriate to the correlated variables of the second stage is then given by

$$Q^2=\sum(I_l-I_l^*)(I_{l'}-I_{l'}^*)N_{ll'}\,, \quad (56)$$

where the asterisk indicates that the value of I_l has been calculated from Eq. (54) in which the least-squares solutions for the τ_l have been resubstituted.

Note that in the most common case of a double cascade, we would have $\tau_{\alpha M}=\pi_\alpha$ and the second stage would complete the analysis (except, of course, the analysis pertaining to the formation of state J_i). If, however, the general situation represented by Eq. (55) prevails, the analysis proceeds to the third stage in which the values of $\tau_{\alpha M}$ determined in the second stage are written explicitly as follows:

$$\tau_{\alpha M}(\delta_j,\cdots,\delta_{e-1})=\tau_{\alpha M}'(\delta_j,\cdots,\delta_{e-2})u_M(\delta_{e-1})\,, \quad (57)$$

where

$$\tau_{\alpha M}'(\delta_j,\cdots,\delta_{e-2})=\pi_\alpha[u_M(\delta_j)\cdots u_M(\delta_{e-2})]\,. \quad (58)$$

[19] The IBM-7094 programs for the analysis of Eqs. (53) and (59) were written by A. K. Hyder, Jr., and D. D. Watson of this Laboratory.

In analogy to the second stage, the best values of the $\tau_{\alpha M}'$ are computed for assigned values of δ_{e-1} along with appropriate values of Q^2 defined in analogy with Eq. (56). The pattern of solution is now clear and is continued until all parameters including the π_α are determined along with their corresponding error matrices.

Smith[7] has discussed in considerable detail a procedure which can be applied to the computation of the elements of the error matrix required for Eq. (56) of our "second stage." In particular, he shows how to compute directly the elements of an augmented error matrix which includes, in addition to the elements $(N^{-1})_{ll'}$, elements which would correspond in the present formalism to the variances in δ_i and the covariances between the δ_i and the I_l. The diagonal elements of this matrix provide the variances in the I_l and of δ_i. An equivalent procedure has been developed[20] which consists of augmenting the normal matrix $N_{ll'}$ of Eq. (53) such that the elements of the inverse of this augmented matrix provide the variances and covariances of the δ_i and the I_l. Similarly, an error matrix can be obtained for the third stage of analysis by augmenting and inverting the normal matrix of Eq. (54). The same procedure also applies to succeeding stages if required.

Two-Parameter Method

In this approach to the solution of the correlation problem, we begin with a modified version of Eq. (53); namely,

$$W_a=\sum_{\alpha M}\tau_{\alpha M}\Gamma_{\alpha M}{}^{a}(\delta_i)H_M(\delta_e)\,, \quad (59)$$

where $\tau_{\alpha M}$ is given by Eq. (55). A computer program[19] then calculates the best values of the set of $\tau_{\alpha M}$ of Eq. (59) for assigned values of both parameters δ_i, δ_e and the corresponding values of Q^2 defined by Eq. (49). The number of unknowns P now must be interpreted as the number of elements of the set of $\tau_{\alpha M}$.

The program first computes and stores all $H_M(\delta_e)$ for points equally spaced between $-90°$ and $+90°$ in the variable $\tan^{-1}\delta_e$. Then for a fixed value of $\tan^{-1}\delta_i$, say $-90°$, Q^2 and the best least-squares values of the $\tau_{\alpha M}$ and corresponding error matrix are computed using the stored values of $H_M(\delta_e)$ for each δ_e. The computer then automatically moves to a new value of $\tan^{-1}\delta_i$ and repeats the process. In this manner the entire δ_i, δ_e plane is covered by a grid typically as fine as 2° steps in $\tan^{-1}\delta$ in a few minutes by an IBM 7094 computer. The time depends, of course, critically upon the size of the set $\tau_{\alpha M}$ since this determines the size of the matrix which must be inverted at each grid point for the least-squares solution. A very significant reduction in computational time is obtained by the use of the "factored" form, Eq. (59). The set of $H_M(\delta_e)$ need only be computed once; the same values are used for each δ_i.

[20] D. D. Watson, G. I. Harris, and L. W. Seagondollar (to be published).

Two output formats for the Q^2 values have proved useful. The first is simply an array consisting of Q^2 values at each grid point from which contour diagrams of Q^2 versus δ_i, δ_e are constructed. The location of possible solutions in the entire δ_i, δ_e plane are thereby immediately evident. The second output format is somewhat more subtle but very useful in practice. For each fixed δ_i, the δ_e which corresponds to the minimum Q^2 is located by the computer. This minimum Q^2 and corresponding δ_e are printed out. Thus one obtains a "shadow" in the Q^2, δ_i plane of the three-dimensional Q^2 surface. In a similar manner, a shadow plot of the surface is obtained in the Q^2, δ_e plane. The values of $\tau_{\alpha M}$ and their error matrices are also printed out for each point of the "shadow plots." A more detailed discussion of this method will be given in a later publication[20] in which it will also be shown that errors in δ_i or δ_e obtained from the "width" of dips in the shadow plots properly account for correlations between the two mixing ratios.

Once the δ_i and δ_e and corresponding values of $\tau_{\alpha M}$ and error matrix have been determined for an acceptable minimum Q^2, the analysis then proceeds to a second stage very similar to the third stage of the single-parameter method. For this stage the $\tau_{\alpha M}$ are written explicitly as in Eq. (57). The analysis then proceeds as discussed following Eq. (58).

Although for general multiple cascades the only significant formal difference between the two analysis methods outlined above consists in combining the first two stages of the single-parameter approach into the first stage of the two-parameter approach, experience has shown that some important practical advantages are gained from the latter method in most problems encountered to date. One must, for example, in either method measure the intensity correlation at a set of angles over the sphere which sufficiently overdetermines the unknowns $I_{\alpha M}$ or $\tau_{\alpha M}$. However, in the common case of a double cascade in which the $\tau_{\alpha M}$ become simply π_α, this condition becomes in the two-parameter method of much less importance since there are fewer π_α than $I_{\alpha M}$ to be determined. A somewhat more serious problem shared by both methods in the application to general multiple cascades is the existence of nonlinear constraints upon the solutions $I_{\alpha M}$ (or $\tau_{\alpha M}$) which are unaccounted for in the linear analysis techniques discussed. To see this, we recall that $I_{\alpha M}=\pi_\alpha \bar{H}_M$ from Eq. (52). As a condition for an acceptable solution, one has the set of equations

$$I_{\alpha M}/I_{\alpha 0}=I_{\alpha' M}/I_{\alpha' 0} \qquad (60)$$

for all $\alpha \neq \alpha'$ and $M \neq 0$. Similar relations must hold between the $\tau_{\alpha M}$. Because of the nonlinear nature of these "auxiliary" equations, they are not included in the linear least-squares analysis. In the few cases where the single-parameter method has been used, these relations have simply been treated as auxiliary conditions to be checked when otherwise acceptable Q^2 values are obtained. Their rigorous inclusion in the analysis will almost certainly require iterative techniques. It appears that the presence of these nonlinear conditions will always cause some difficulty in general multiple-cascade problems. They also appear in the standard double cascade if the single-parameter method is applied. However, in this standard problem the difficulty is circumvented by the two-parameter method since the only relations between the π_α are those which arise from the mechanism of formation of the state J_i. It has always been standard practice to consider such relations as auxiliary conditions on the entire problem.

Formation of the State J_i

In the above section, we have outlined a procedure for determining the population parameters or statistical tensors of the state J_i of Fig. 2 and the multipolarity mixing ratios of transitions following the formation of that state. For purposes of discussion of the methods considered in this paper, we adopt a viewpoint in which the mixing ratios of unobserved gamma radiations emitted prior to the first observed radiation of the cascade are considered as formation parameters. Thus, in the case of proton capture, these multipolarity mixing ratios are considered as formation parameters along with the channel spin and orbital angular-momenta mixing ratios which relate to the formation of state J_1 by proton bombardment. Although such a viewpoint is not essential, it is found to be most convenient if the PP representation is employed because of the rather inelegant relationship, Eq. (40), between the population parameters of states J_i and J_1. The analysis procedure discussed above does not lend itself to be extended in a natural way to the determination of the parameters $\delta_1,\cdots,\delta_{i-1}$ in the PP representation.

The situation is considerably different in the ST representation. In this case the statistical tensors of state J_i are related to those of state J_1 through Eq. (47) which does allow a natural extension of the "stage" analysis to the determination of the parameters $\delta_1,\cdots,\delta_{i-1}$. Specifically, if there do appear unobserved radiations prior to the first observed radiation, the procedure employed is to use Eq. (47) to write Eq. (53) in the form,

$$W_a=\sum_{kM} I_{kM}(\delta_1,\cdots,\delta_{i-1},\delta_j,\cdots,\delta_e)T_{kM}{}^a(\delta_i)\,, \quad (61)$$

where now,

$$I_{kM}=\rho_{k0}(J_1)[u_k(\delta_1)\cdots u_k(\delta_{i-1})]\bar{H}_M(\delta_j,\cdots,\delta_e)\,. \quad (62)$$

It is then evident that, in principle, the analysis procedure following Eq. (53) can now be continued until the $\rho_{k0}(J_1)$ are finally determined. At this point, the $\rho_{k0}(J_1)$ can be either converted to population parameters using Eq. (19a) or compared directly with formation theory as discussed for proton capture by Nordhagen.[14] Smith[7] has considered in detail the problem of error analysis when the population parameters $P_1(m_1)$ are compared with proton-capture formation theory.

SUMMARY AND CONCLUSION

A parallel development of the intensity direction ("triple") correlation of cascade radiations from aligned states, where the degree of alignment is specified by either the population numbers of the magnetic substates or by the statistical tensors of the state, has been presented. In addition, the usual double-cascade formulation has been extended to apply to cases where one or more intermediate unobserved radiations may be present. The equations are written in a "factored" form which allows the computational problem of data analysis to be undertaken in discreet stages. Two analysis procedures are presented and compared. The first involves a linear least-squares analysis for each transition of a multiple cascade, and the second involves a two parameter, least-squares analysis for the two observed radiations of the cascade and single parameter analysis for unobserved radiations. For a standard double cascade problem, the second method has been found more practical because of the smaller number of unknowns which must be determined and the absence of nonlinear auxiliary conditions which would have to be satisfied. For a general multiple cascade, however, there appears to be little advantage of one method over the other. Iterative techniques will be required in either case for the rigorous inclusion of the auxiliary equations. The suggested analysis procedures assure that all acceptable solutions will be found.

It has been found that the form of the equations are such that the necessary coefficients can be tabulated in a more efficient manner than in existing standard, triple-correlation-coefficient tabulations. For example, Watson[16] has tabulated the necessary coefficients for both the statistical-tensor and population-parameter representations of this paper in a volume considerably smaller than the tables of Smith.[6] Furthermore, whereas the table of Smith applies only to double cascades and is limited to dipole and quadrupole radiations, that of Watson can be used for multiple cascades and for radiations through octupole. (However, the present tables provide for the mixing of only two multipolarities in each transition.)

It is shown in the development that, owing to the simpler relation between statistical tensors of successive states in a cascade, a more elegant and more easily applied formalism is obtained using statistical tensors than if population parameters are employed. Even in the standard double-cascade case, it has been found by Nordhagen[15] that the statistical tensors are more useful when the spin of the aligned state is greater than the sum of the highest multipolarities of the two cascade radiations. A generalized version of his rule has been presented for multiple cascades. It thus seems more appropriate, even though both are presented, to use the statistical tensor rather then the population parameter formulation in the analysis of triple-correlation data. This is especially true when the more complex situations considered in this paper prevail.

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