Finite-Sample-Size Effects in Nuclear Cross-Section Fluctuations*

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The question of bias and error due to finite sample size is considered, and it is suggested that these effects may be computed by replacing the excitation function with a number of independent cross sections. It is argued that the effective number of independent cross sections is given by $n=(\Delta E/\pi\Gamma_0)+1$ as long as the energy range ΔE is not too great. The results of some Monte Carlo calculations are also given.

THE problem of biases and errors due to finite sample size is a major difficulty in the study of fluctuating nuclear excitation functions.[1,2] If a reliable estimate of the effective number of independent points involved can be made, the problems of making corrections and estimating errors will be partially solved. One possible method of making such an estimate is described below.

It has been shown[3] that the variance of the average cross section is

$$\mathrm{Var}(\langle\sigma\rangle)=\bar{\sigma}^2\left[\frac{2\tan^{-1}S}{S}-\frac{\ln(1+S^2)}{S^2}\right],$$

where $\bar{\sigma}$ is the ensemble average of the cross section, $\langle\sigma\rangle$ is the average of the cross section over an energy span ΔE, and S is the ratio of energy span to coherence width. It is also known that if an average from n independent points is computed, the variance of this average will be given by

$$\mathrm{Var}(\langle\sigma\rangle)=\bar{\sigma}^2/n,$$

so that it is natural to make the correspondence

$$n=\frac{S}{2\tan^{-1}S-[\ln(1+S^2)]/S},$$

where n is now the effective number of independent energy points in the excitation function. For values of S less than about 50 this function is very well approximated by

$$n\simeq(S/\pi)+1.$$

By using n as the number of independent points, the bias and uncertainty of each quantity associated with the type of analysis under consideration may be calculated. On this basis, the Monte Carlo method was used with n determinations of the cross section each chosen from a χ^2 distribution with $2N$ degrees of freedom. A minimum of 5000 such determinations were employed for each n. The expectation value of the autocorrelation function for zero argument, $R(0)$, was found for small N to satisfy

$$NR(0)=\frac{(n-1)(4n-4+N)}{4n^2}.$$

This expression is a good representation of the Monte Carlo results within about 1% for $n>4$ and $N=1$, 2, or 3. For larger N this formula is expected to become a poorer approximation as can be seen by comparison with the analytic expression obtainable for $n=2$.

The numbers obtained for $\mathrm{Var}[R(0)]$ are in reasonable agreement with the approximate expression given by Hall.[4] For $N=1$ or 2, the variances given by this expression are consistently greater than the Monte Carlo results, but the difference is less than 20% when moderately large ($n\geq 20$) sample sizes are used. The variances calculated for $N=1$, 2, and 3 are plotted in Fig. 1.

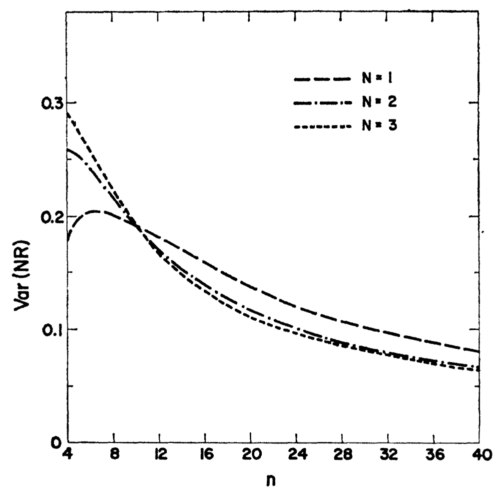

FIG. 1. The variances of $R(0)$ as calculated by the Monte Carlo method described in the text. These curves are estimated to be accurate to within 5%.

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[1] T. Ericson, Phys. Rev. Letters **5**, 430 (1960); Advan. Phys. **9**, 425 (1960); in *Proceedings of the International School of Physics "Enrico Fermi,"* Course 23, 1961 (Academic Press Inc., New York, 1963), p. 142; Ann. Phys. (N. Y.) **23**, 390 (1963); and Phys. Letters **4**, 258 (1963).

[2] D. M. Brink and R. O. Stephen, Phys. Letters **5**, 77 (1963) and D. M. Brink, R. O. Stephen, and N. W. Tanner, Nucl. Phys. **54**, 577 (1964).

[3] M. Böhning, in *Comptes Rendus du Congrès International de Physique Nucléaire* (Centre National de la Recherche Scientifique, Paris, 1964), Vol. II, p. 697 and also reported by T. Mayer-Kuckuk in the 1964 Hercegnovi Lectures, Max Planck Institute report (unpublished).

[4] I. Hall, Phys. Letters **10**, 199 (1964).

If the coherence width Γ is calculated by comparing $R(\epsilon)$ with a Lorentzian form near $\epsilon=0$, the bias and variance of the value so obtained may be calculated. Near $\epsilon=0$, the assumption is made that the bias in $R(\epsilon)$ is independent of ϵ. This leads to the formulas

$$\Gamma=\Gamma_0\left[\frac{(n-1)(4n-4+N)}{4n^2}\right]^{1/2}$$

and

$$\mathrm{Var}(\Gamma)=(\Gamma_0^4/4\Gamma^2)\,\mathrm{Var}[NR(0)]$$

for the expected value Γ of the coherence width and the variance of the value obtained.

This paper is a summary of some of the results of a report[5] which contains plots of further Monte Carlo results as well as a fuller treatment of the questions considered here. Also treated are questions not considered here, such as the effect of finite sample size on the frequency distribution function.

[5] W. R. Gibbs, Los Alamos Scientific Laboratory Report LA 3266, 1965 (available from Clearing House for Federal Scientific and Technical Information, National Bureau of Standards, U. S. Department of Commerce, Springfield, Virginia).

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Remarks on Charged-Particle Scattering

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The scattering of charged particles from a shielded Coulomb potential is reviewed. The limit as the shielding radius becomes infinite is discussed. A method of determining reaction cross sections, recently introduced by the authors is treated in detail and applied to the scattering of protons from He^4 and H^2 at 40 MeV.

INTRODUCTION

THE study of forces between charged particles is complicated by the presence of the infinite-ranged Coulomb force. For a potential which falls off as slowly as r^{-1} we cannot apply the usual scattering boundary condition to the wave function, viz.,

$$\psi(\mathbf{r})\sim\exp(i\mathbf{k}\cdot\mathbf{r})+r^{-1}f(\theta)\exp(ikr) \qquad (1)$$

since even at very large distances the incident wave is distorted. In a practical laboratory scattering experiment this problem does not arise, however, because the charges are shielded so that the potential vanishes beyond some shielding radius R. If the radiation source is placed well beyond the shielding radius, an initial state approximating a plane wave can be prepared. We shall consider the case in which the shielding radius is very large and both the source and detector are located very far outside that shielding radius. We idealize this to the case where $R\to\infty$, still maintaining the condition that the source and detector are located far beyond R.

I. THE SHIELDED COULOMB FIELD

Let us consider a point charge located at the origin shielded by a double layer of charge such that the potential energy is given by

$$\begin{aligned} V(r)&=zz'e^2/r, \quad r<R \\ &=0, \quad r>R. \end{aligned} \qquad (1.1)$$

We may treat the scattering problem with the shielded Coulomb potential Eq. (1.1) by means of the angular-momentum expansion. We write

$$\psi(\mathbf{r})=(kr)^{-1}\textstyle\sum_l a_l i^l(2l+1)F_l(k,r)P_l(\cos\theta). \qquad (1.2)$$

The radial function $F_l(k,r)$ is a solution of the radial equations

$$\frac{d^2F_l}{d\rho^2}+\left[1-\frac{l(l+1)}{\rho^2}-\frac{\eta}{\rho}\right]F_l=0, \quad r<R \qquad (1.3)$$

$$\frac{d^2F_l}{d\rho^2}+\left[1-\frac{l(l+1)}{\rho^2}\right]F_l=0, \quad r>R. \qquad (1.4)$$

Here $\rho=kr$ and $\eta=zz'e^2/\hbar v$. The regular solution of Eq. (1.3) takes the asymptotic form

$$F_l\approx A\sin[kr-l\pi/2+\sigma_l-\eta\ln(2kr)] \qquad (1.5)$$

for $kr\gg l$, where $\sigma_l=\arg\Gamma(l+1+i\eta)$. The solutions of Eq. (1.4) take the asymptotic form

$$F_l\approx\sin(kr-l\pi/2+\delta_l) \qquad (1.6)$$

for $kr\gg l$. Thus, if $R\gg l/k$, we may equate the logarithmic derivatives of the two solutions at the shielding radius R to obtain the result

$$\begin{aligned} &\cot(kR-l\pi/2+\delta_l) \\ &\quad=(1-\eta/kR)\cot[kR-l\pi/2+\sigma_l-\eta\ln(2kR)]. \end{aligned} \qquad (1.7)$$

The phase shift δ_l given by Eq. (1.7) has the form,

$$\delta_l=\sigma_l-\eta\ln(2kR)+\gamma_l \qquad (1.8)$$

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