Muon Capture and Inelastic Electron Scattering in C^{12} and O^{16}†

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The dipole (first-forbidden) contribution to the muon-capture matrix elements, M_V, M_A, and M_P, is calculated using wave functions computed in the particle-hole theory. It is found that $(M_V^2)_D=(M_A^2)_D=(M_P^2)_D$ to within 13%. The assumption that the dipole part of the nuclear matrix element may be expressed as the unretarded dipole matrix element multiplied by the elastic form factor is found to hold to about 1%. Calculations of inelastic electron scattering from the 2^-, $T=1$ states in these nuclei predict large cross sections for some of the states at about 100 MeV/c momentum transfer. These "giant-magnetic-quadrupole states" are identified with observed levels found in recent 180° electron-scattering experiments.

I. INTRODUCTION

An assumption which has often been made in calculations of total-muon capture ratio is that all the nuclear matrix elements are equal[1]; $M_V^2=M_A^2=M_P^2$. In particular the results of Luyten, Rood, and Tolhoek,[2] and Foldy and Walecka[3] rely on this assumption. In addition, Foldy and Walecka assumed the dipole-vector matrix element may be written as the unretarded dipole-vector matrix element multiplied by the ground-state elastic form factor evaluated at the resonant neutrino momentum; $(M_V^2)_D=(M_V^2)_{UD}|F_{\mathrm{el}}(\nu_{\mathrm{res}})|^2$. Both of these assumptions have some theoretical justification. In particular, Foldy and Walecka have used the particle-hole wave functions of Lewis[4] for the 0^-, 1^-, and 2^-, $T=1$ states of O^{16} to compute the unretarded dipole[5] contributions of M_A^2 and M_V^2. They found $(M_V^2)_{UD}=(M_A^2)_{UD}$ to within 12%.

In this paper we expand their calculation in the following ways: In addition to $(M_V^2)_{UD}$ and $(M_A^2)_{UD}$ we also compute $(M_P^2)_{UD}$, and we have calculated $(M_V^2)_D$, $(M_A^2)_D$, and $(M_P^2)_D$. That is, the matrix elements have been evaluated at the correct neutrino momenta instead of in the long-wavelength limit. Finally, we have also carried out these calculations for C^{12} both with and without the inclusion of ground-state correlations. The ground-state correlations were included by using the random-phase approximation (RPA) as formulated by Lewis.[4] Since C^{12} is not a doubly magic nucleus, the inclusion of such correlations could be important. The wave functions for the 1^-, $T=1$ states in C^{12} and O^{16} have been computed by Lewis and Walecka.[6,7] Using the formalism of Ref. 6, we have calculated wave functions for the 0^- and 2^-, $T=1$ states in C^{12}. This method is merely a reformulation of the particle-hole theory developed by Brown and his co-workers.[8–10] However, the residual two-particle interaction was taken from low-energy nucleon-nucleon scattering, leaving no adjustable parameters in the theory.

We find that $(M_V^2)_D=(M_A^2)_D=(M_P^2)_D$ to within 13% in all the cases considered. This result is also found to hold in the unretarded limit. In fact the ratios $(M^2)_D/(M^2)_{UD}$ for $M=M_V$, M_A, and M_P are almost equal, with the average ratio being about equal to $|F_{\mathrm{el}}(\nu_{\mathrm{res}})|^2$. For all cases $(M^2)_D=(M^2)_{UD}|F_{\mathrm{el}}(\nu_{\mathrm{res}})|^2$ to about 1%. Our calculated results for $(M_V^2)_{UD}$ tended to be about twice those obtained by Foldy and Walecka who found $(M_V^2)_{UD}$ by integrating over the experimental photoabsorption cross section.

Muon capture takes place predominantly through the 0^-, 1^-, and 2^-, $T=1$ states. The transverse-electromagnetic-interaction matrix elements are very similar to those used in muon capture. Thus, these states (except for 0^-) may be used in a similar way to calculate photon processes or 180° electron scattering (which involves only the transverse matrix elements). The latter method is particularly useful since the momentum transfer can be varied, and the functional dependence of the theory on this parameter can be checked.

Lewis and Walecka,[6] and Lewis[7] have calculated the transverse form factors for inelastic electron scattering from 1^-, $T=1$ states in C^{12} and O^{16}. In particular, they find that the sum of the squared form factors in the giant-resonance region agrees quite well with the square of the experimental form factor for the giant resonance. The experiments were done at 180° scattering angle so that only the transverse components contributed. At low-momentum transfer such as in photoabsorption the contribution of higher multipoles is negligible, but when the momentum transfer is about 100 MeV/c we expect a fairly sizeable contribution. In particular we are inter-

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[1] The nuclear matrix elements are defined in Sec. III.
[2] J. R. Luyten, H. P. C. Rood, and H. A. Tolhoek, Nucl. Phys. **41**, 236 (1963).
[3] L. L. Foldy and J. D. Walecka, Nuovo Cimento **34**, 1026 (1964).
[4] F. H. Lewis, Jr., thesis, Stanford University, 1964 (unpublished); and private communication.
[5] By "dipole" we mean first forbidden, that is, the $L=1$ term in the multipole expansion of the neutrino wave function.
[6] F. H. Lewis, Jr., and J. D. Walecka, Phys. Rev. **133**, B849 (1964).
[7] F. H. Lewis, Jr., Phys. Rev. **134**, B331 (1964); **138**, AB5 (E) (1965).
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[10] N. Vinh-Mau and G. E. Brown, Nucl. Phys. **29**, 89 (1962).

ested in the modifications the 2^-, $T=1$ states make to the calculations mentioned above.

The calculations of the form factors for the 2^-, $T=1$ states are carried out using the same wave functions used for the muon-capture computations. For C^{12} there are three 2^- states, one of them in the giant-resonance region. However, the form factor for this state is fairly small, leading to a small modification of the results of Lewis and Walecka. However, the theory predicts a strong 2^- state at 20.76 MeV. Since our predicted levels tend to be about 1 or 2 MeV high, we identify this with an experimental level seen at 19.2 MeV.[11,12] At large momentum transfers this state becomes quite prominent as we predict. Furthermore the q (momentum transfer) dependence we predict is consistent with experiment; however, our squared form factor is a factor of 2 too large. The remaining state is negligible for q less than 200 MeV/c.

For O^{16} the results are similar in that one state has most of the strength, at least below 200 MeV/c. However, the state is at 21.34 MeV which is in the giant-resonance region, producing a large modification of the results of Lewis. Recent experiments[13] have been able to resolve some of the fine structure of the giant resonance. A prominent feature in these experiments is a state at 20.2 MeV which is quite strong at the higher momentum transfers. Identifying this as a 2^- state, we find the q-dependence of the form factor is in good agreement with experiment. One of the other 2^- states also occurs in the giant resonance at 24.52 MeV having approximately one-third the strength of the 21.34-MeV state. Of the remaining three 2^- states, the one at 13.85 MeV is the strongest and is identified with the observed 13-MeV state. In all cases where identification with our observed state is possible, the calculated squared form factor tends to be a factor of 2 larger than experiment but otherwise in fairly good agreement.

In Sec. II the wave functions are given. Section III contains the muon-capture calculation, and in Sec. IV the results of the scattering from the 2^- states are presented. Section V is a summary and discussion.

II. WAVE FUNCTIONS

The excited states of a closed-shell nucleus in the particle-hole model are obtained by diagonalizing the Hamiltonian between states in which one nucleon is in an excited state leaving a hole in a shell. To obtain the negative parity states that are considered here, the nucleon must be excited to at least the next oscillator level which is an energy of about 15 MeV. The matrix element of the Hamiltonian may be divided into two parts, the first of which, E_0, is diagonal. E_0 may be identified as the energy of the hole and particle without an interaction between them and thus can be calculated from the energy levels of the neighboring $A\pm1$ nuclei. The remaining, nondiagonal, term may be thought of as a residual particle-hole interaction. Since the interaction potential is attractive, the particle-hole interaction is repulsive for $T=1$ states which raises the perturbed energy levels several MeV. This allows identification of some of the 1^- states with the giant-dipole resonance.[8] A derivation of E_0 and the particle-hole interaction matrix elements for $T=1$ state has been given by Lewis and Walecka.[6]

For the internucleon potential we use a nonsingular potential that fits low-energy nucleon-nucleon scattering. It has been shown that, at least for doubly magic $+2$ nuclei, this is a fairly good approximation.[14,15] We use a Serber-force Yukawa well with

$$^1V_0=-46.87\text{ MeV},\quad {}^1\mu=0.8547\text{ F}^{-1},$$
$$^3V_0=-52.13\text{ MeV},\quad {}^3\mu=0.7261\text{ F}^{-1},$$

which is obtained from Ref. 14. We take our single-particle states to be harmonic oscillator wave functions with an oscillator parameter $b'=b\sqrt{2}$, where $b=(\hbar/M\omega)^{1/2}$, with $\hbar\omega$ the oscillator energy. By fitting Coulomb energy differences in mirror nuclei, b is found to be 1.6 F for C^{12} and 1.67 F for O^{16}.[16] The values of E_0 are the same as those given in Refs. 6 and 7.

Our results for the 0^- and 2^- states combined those of Lewis and Walecka[6] for the 1^- states in C^{12}, and are shown in Table I. We have also computed the wave functions for C^{12} using the random-phase approximation as formulated by Lewis.[4] These wave functions are not included here since they gave results which are quite similar to those obtained using the wave functions in Table I (for example see Figs. 2 and 3, and Tables III and IV). That inclusion of ground-state correlations for these states is not very important was first pointed out by Vinh-Mau and Brown.[10] The wave functions for O^{16} calculated by Lewis[4] are given in Table II.

TABLE I. Energies and wave functions for J^-, $T=1$ states in C^{12}.

J	E (MeV)	$2s_{1/2}(1p_{3/2})^{-1}$	$1d_{5/2}(1p_{3/2})^{-1}$	$1d_{3/2}(1p_{3/2})^{-1}$	$1p_{1/2}(1s_{1/2})^{-1}$
0	25.66			0.931	−0.364
	35.78			0.364	0.931
1	19.57	0.977	−0.168	0.133	−0.016
	23.26	0.194	0.952	−0.211	0.106
	25.01	−0.088	0.252	0.933	−0.243
	35.80	−0.027	−0.044	0.260	0.964
2	18.91	0.937	−0.349	−0.025	
	20.76	0.342	0.928	−0.149	
	23.94	0.075	0.131	0.988	

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[13] G. Vanpraet (to be published); and private communication.
[14] J. F. Dawson, I. Talmi, and J. D. Walecka, Ann. Phys. (N. Y.) **18**, 339 (1962).
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[16] B. C. Carlson and I. Talmi, Phys. Rev. **96**, 436 (1954).

TABLE II. Energies and wave functions for J^-, $T=1$ states in O^{16}.

J	E (MeV)	$2s_{1/2}(1p_{1/2})^{-1}$	$2s_{1/2}(1p_{3/2})^{-1}$	$1d_{5/2}(1p_{1/2})^{-1}$	$1d_{3/2}(1p_{1/2})^{-1}$	$1d_{5/2}(1p_{3/2})^{-1}$	$1d_{3/2}(1p_{3/2})^{-1}$
0	14.41	0.998					0.070
	27.28	−0.070					0.998
1	14.61	0.991	−0.063		−0.011	−0.113	−0.029
	18.65	−0.051	−0.145		0.897	−0.339	0.038
	21.01	0.030	0.942		0.002	−0.302	0.143
	23.89	0.119	0.245		0.322	0.881	0.214
	26.63	0.012	−0.165		−0.301	−0.072	0.936
2	13.85		0.069	0.974	−0.025	0.201	0.076
	18.69		−0.008	−0.020	0.965	0.248	−0.077
	20.01		0.730	0.086	0.177	−0.654	0.017
	21.34		0.672	−0.178	−0.185	0.673	−0.173
	24.52		0.100	−0.110	0.042	0.135	0.979

III. MUON CAPTURE

With certain assumptions[3] the muon capture rate may be written as

$$\Lambda_{\mu c}=(\nu_\mu{}^2(|\phi_\mu|^2)_{\rm av}/2\pi\hbar^2c)[G_V{}^2M_V{}^2+3G_A{}^2M_A{}^2+(G_P{}^2-2G_PG_A)M_P{}^2]+\Lambda_{\mu c}'. \tag{1}$$

The G's are effective coupling constants. $\Lambda_{\mu c}'$ contains nucleon recoil corrections, and $\nu_\mu\equiv(m_\mu c)/\hbar$. In particular we are interested in $M_V{}^2$, $M_A{}^2$, and $M_P{}^2$ which are defined as follows:

$$M_V{}^2=\sum_a{}'\sum_b\left(\frac{\nu_{ab}}{\nu_\mu}\right)^2\int\frac{d\hat{\nu}}{4\pi}|(b|\sum_{i=1}^A\tau^{(-)}(i)\exp[-i\boldsymbol{\nu}_{ab}\cdot\mathbf{x}(i)]|a)|^2,$$

$$M_A{}^2=\tfrac{1}{3}\sum_a{}'\sum_b\left(\frac{\nu_{ab}}{\nu_\mu}\right)^2\int\frac{d\nu}{4\pi}|(b|\sum_{i=1}^A\tau^{(-)}(i)\boldsymbol{\sigma}(i)\exp[-i\boldsymbol{\nu}_{ab}\cdot\mathbf{x}(i)]|a)|^2, \tag{2}$$

$$M_P{}^2=\sum_a{}'\sum_b\left(\frac{\nu_{ab}}{\nu_\mu}\right)^2\int\frac{d\hat{\nu}}{4\pi}|(b|\sum_{i=1}^A\tau^{(-)}(i)\hat{\nu}\cdot\boldsymbol{\sigma}(i)\exp[-i\boldsymbol{\nu}_{ab}\cdot\mathbf{x}(i)]|a)|^2,$$

where $\boldsymbol{\nu}_{ab}\equiv\mathbf{p}_\nu/\hbar$ is the neutrino wave number, and $|a)$ and $|b)$ are the initial and final states of the nucleus, and $\sum'$ means an average over the states $|a)$.

Provided isotopic spin is a good quantum number and the initial state has $T=0$, we may change $\tau^{(-)}$ to $\tau^{(3)}$ by substituting

$$\tau^{(\pm)}(j)=\mp\tfrac{1}{2}[T_\pm,\tau^{(3)}(j)],$$

where

$$T_\pm=\tfrac{1}{2}\sum_{j=1}^A[\tau^{(1)}(j)\pm i\tau^{(2)}(j)]. \tag{3}$$

$M_V{}^2$, for example, becomes

$$\begin{aligned}M_V{}^2=\tfrac{1}{4}\sum_a{}'\sum_{bb'b''}\left(\frac{\nu_{ab}}{\nu_\mu}\right)^2\int\frac{d\hat{\nu}}{4\pi}(a|\sum_{i=1}^A\tau^{(3)}(i)\exp[i\boldsymbol{\nu}_{ab}\cdot\mathbf{x}(i)]|b'')\\ \times(b''|T_+|b)(b|T_-|b')(b'|\sum_{i=1}^A\tau^{(3)}(i)\exp[-i\boldsymbol{\nu}_{ab}\cdot\mathbf{x}(i)]|a) \quad (4)\\ =\tfrac{1}{2}\sum_a{}'\sum_{b'}\left(\frac{\nu_{ab}}{\nu_\mu}\right)^2\int\frac{d\hat{\nu}}{4\pi}|(b'|\sum_{i=1}^A\tau^{(3)}(i)\exp[i\boldsymbol{\nu}_{ab}\cdot\mathbf{x}(i)]|a)|^2;\end{aligned}$$

b' is the $T_3=0$ component of the excited $T=1$ state of which b is the $T_3=-1$ component. Since b' and a refer to the same nucleus we can use our wave functions for C^{12} and O^{16} to calculate the muon capture in these nuclei.

TABLE III. Squared matrix elements for muon capture to individual states.

	C^{12}					C^{12} (RPA)					O^{16}			
J_i	E_i (MeV)	$(M_V^2)_D^{(i)}$	$(M_A^2)_D^{(i)}$	$(M_P^2)_D^{(i)}$	J_i	E_i (MeV)	$(M_V^2)_D^{(i)}$	$(M_A^2)_D^{(i)}$	$(M_P^2)_D^{(i)}$	J_i	E_i (MeV)	$(M_V^2)_D^{(i)}$	$(M_A^2)_D^{(i)}$	$(M_P^2)_D^{(i)}$
0	25.66		0.091	0.273	0	25.53		0.087	0.260	0	14.41		0.013	0.039
	35.78		0.001	0.004		35.37		0.002	0.006		27.28		0.077	0.230
1	19.57	0.023	0.013		1	19.76	0.021	0.012		1	14.63	0.016	0.024	
	23.26	0.472	0.017			23.08	0.424	0.026			18.65	0.016	0.040	
	25.01	0.000	0.197			24.95	0.002	0.173			21.01	0.002	0.021	
	35.80	0.072	0.010			35.61	0.059	0.015			23.89	0.638	0.060	
											26.63	0.184	0.147	
2	18.91		0.011	0.013	2	18.90		0.010	0.012	2	13.85		0.144	0.173
	20.76		0.210	0.251		20.67		0.190	0.228		18.69		0.000	0.000
	23.94		0.059	0.071		23.92		0.053	0.064		20.01		0.020	0.024
											21.34		0.302	0.362
											24.52		0.092	0.110

We make a multipole expansion of the matrix elements. With a 0^+ ground state we have[17]

$$
\begin{aligned}
M_V{}^2&=2\pi\sum_{b'}(\nu_{ab}/\nu_\mu)^2\left|\left(b'J_{b'},\,T=1\right\|\sum_{i=1}^{A}\tau^{(3)}(i)j_{J_{b'}}(\nu_{ab}x_i)Y_{J_{b'}}(\Omega_{\mathbf{x}_i})\left\|0^+,\,T=0\right)\right|^2,\\
M_A{}^2&=\frac{2\pi}{3}\sum_{b'L}(\nu_{ab}/\nu_\mu)^2\left|\left(b'J_{b'},\,T=1\right\|\sum_{i=1}^{A}\tau^{(3)}(i)j_L(\nu_{ab}x_i)\mathfrak{Y}_{J_{b'}L1}(\Omega_{\mathbf{x}_i})\cdot\boldsymbol{\sigma}(i)\left\|0^+,\,T=0\right)\right|^2,\\
M_P{}^2&=2\pi\sum_{b'L}\left(\frac{\nu_{ab}}{\nu_\mu}\right)^2(2L+1)\begin{pmatrix}1&L&J_{b'}\\0&0&0\end{pmatrix}^2\left|\left(b'J_{b'},\,T=1\right\|\sum_{i=1}^{A}\tau^{(3)}(i)j_L(\nu_{ab}x_i)\mathfrak{Y}_{J_{b'}L1}(\Omega_{\mathbf{x}_i})\cdot\boldsymbol{\sigma}(i)\left\|0^+,\,T=0\right)\right|^2.
\end{aligned}
\tag{5}
$$

The most important contribution of these matrix elements is the first forbidden or dipole. The allowed terms are small. They vanish in the unretarded limit for O^{16} which has doubly closed shells in the ground state. For C^{12} Foldy and Walecka[3] estimate the allowed term contributes 20% of the capture rate. The contribution of the individual levels to the dipole matrix elements is found to be

$$
(M_V{}^2)_D{}^{(i)}=4\pi\frac{y^{(i)}}{y_\mu}\left|\sum_{n'l'j';nlj}\alpha_{(i)}{}^{(n'l'j')(nlj)^{-1}}\left[\frac{3}{4\pi}(2l+1)(2l'+1)(2j+1)(2j'+1)\right]^{1/2}\times\begin{pmatrix}l'&1&l\\0&0&0\end{pmatrix}(n'l'|j_1(\nu_{ab}{}^{(i)}r)|nl)(-1)^{j-1/2}\begin{Bmatrix}l'&j'&\frac{1}{2}\\j&l&1\end{Bmatrix}\right|^2,
\tag{6}
$$

$$(M_A{}^2)_{DJ}{}^{(i)}=\tfrac{1}{3}W_{DJ}{}^{(i)},$$

$$(M_P{}^2)_{DJ}{}^{(i)}=(\delta_{J,0}+\tfrac{2}{5}\delta_{J,2})W_{DJ}{}^{(i)},$$

where

$$
W_{DJ}{}^{(i)}=4\pi\frac{y^{(i)}}{y_\mu}\left|\sum_{n'l'j';nlj}\alpha_{(i)}{}^{(n'l'j')(nlj)^{-1}}\left[\frac{3}{4\pi}(2l+1)(2l'+2)(2j+1)(2j'+1)\right]^{1/2}\times\begin{pmatrix}l'&1&l\\0&0&0\end{pmatrix}(n'l'|j_1(\nu_{ab}{}^{(i)}r)|nl)\sqrt{6}(-1)^{l'}(2J+1)^{1/2}\begin{Bmatrix}l'&l&1\\\frac{1}{2}&\frac{1}{2}&1\\j'&j&J\end{Bmatrix}\right|^2,
$$

and

$$
(n'l'|j_1(\nu r)|nl)\equiv\int_0^\infty R_{n'l'}{}^*(r)j_1(\nu r)R_{nl}(r)r^2dr,
\tag{7}
$$

where R_{nl} is the radial harmonic oscillator wave function defined in Ref. 6, and

$$y^{(i)}\equiv(\tfrac{1}{2}\nu_{ab}{}^{(i)}b)^2\quad\text{and}\quad y_\mu\equiv(\tfrac{1}{2}\nu_\mu b)^2,$$

where b is the oscillator parameter used in Sec. II. The $\alpha_{(i)}{}^{(n'l'j')(nlj)^{-1}}$ are the coefficients of the basis wave functions and are given in Tables I and II. To find the total matrix elements we sum over the individual levels; $(M^2)_D=\sum_i(M^2)_D{}^{(i)}$ for $M=M_V$, M_A, and M_P. The contributions of the individual levels are given in Table

[17] We note that $\mathbf{S}|a\rangle=0$ does not necessarily imply $M_A=M_P$ as stated by Foldy and Walecka (Ref. 3).

TABLE IV. Squared matrix elements for the total muon capture rate. The primed results are those of Foldy and Walecka, (Ref. 3) who estimated $(M_V^2)_{UD}$ by integrating over the experimental photoabsorption cross section.

Nucleus	$(M_V^2)_U$	$(M_A^2)_D$	$(M_P^2)_D$	$(M_V^2)_{UD}$	$(M_A^2)_{UD}$	$(M_P^2)_{UD}$	$(M_V^2)_D'$	$(M_V^2)_{UD}'$
C^{12}	0.567	0.608	0.611	0.795	0.857	0.861	0.324	0.45
C^{12} (RPA)	0.506	0.568	0.570	0.711	0.798	0.803	0.324	0.45
O^{16}	0.857	0.938	0.938	1.259	1.414	1.415	0.485	0.77

III, and the total matrix elements are shown in Table IV. The results of Foldy and Walecka for $(M_V^2)_D$ and $(M_V^2)_{UD}$ are also included in Table IV for comparison. In Table V the ratios $(M_V^2)_D/(M_V^2)_{UD}$ for $M=M_V$, M_A and M_P are compared with $|F_{\rm el}(\nu_{\rm res})|^2$.[18]

TABLE V. Square of the elastic form factor and ratios of the retarded to unretarded squared matrix elements.

Nucleus	$\frac{(M_V^2)_D}{(M_V^2)_{UD}}$	$\frac{(M_A^2)_D}{(M_A^2)_{UD}}$	$\frac{(M_P^2)_D}{(M_P^2)_{UD}}$	$\|F_{\rm el}(\nu_{\rm res})\|^2$
C^{12}	0.713	0.713	0.710	0.719
C^{12} (RPA)	0.711	0.711	0.711	0.719
O^{16}	0.681	0.664	0.663	0.676

If we neglect ground-state correlations we see that the assumption

$$(M_V^2)_D=(M_A^2)_D=(M_P^2)_D \tag{8}$$

is good to within 8% for C^{12} and 12% for O^{16}. Letting the energies of the excited states become degenerate, we find the assumption holds exactly. To see this we first make a unitary transformation to the basic particle-hole states. For M_A^2 and M_P^2 we must then evaluate an expression of the form

$$S_z\equiv\sum_{ph}|(p|\sigma_z(i)\tau^{(3)}(i)\exp[-i\mathbf{v}_{ab}\cdot\mathbf{x}(i)]|h)|^2, \tag{9}$$

where p and h are the particle and hole wave functions for the ith nucleon. We have arbitrarily chosen the z component of $\boldsymbol{\sigma}$, since all components give the same result. For the cases being considered the sum may be broken up into terms which contain particles with both $j=l+\frac{1}{2}$ and $j=l-\frac{1}{2}$ for a given hole or vice versa. Thus in every term we may change either the particles or the holes from $|nljm_j)$ to $|nlm_lm_s)$ states. By quantizing these states in the z direction we see that

$$S_z=\sum_{ph}|(p|\tau^{(3)}(i)\exp[-i\mathbf{v}_{ab}\cdot\mathbf{x}(i)]|h)|^2. \tag{10}$$

Since $S_x=S_y=S_z=S_{\hat{v}}$ we find $M_V^2=M_A^2=M_P^2$. Foldy and Walecka have shown that if the nucleon-nucleon force is spin-independent, one can apply Wigner's supermultiplet theory[19] with the result that $M_V^2=M_A^2=M_P^2$. In the present calculation we find that this equality is no longer exact when spin-dependent forces are considered. We have included spin dependence in two ways. The correct singlet-triplet ratio has been used for the internucleon force, and the configuration energies have been taken from the empirically observed levels of the neighboring nuclei. We have not included a tensor force in the internucleon interaction; what effect it would have is not known.

The assumption

$$(M^2)_D=(M^2)_{UD}|F_{\rm el}(\nu_{\rm res})|^2 \tag{11}$$

is satisfied to about 1% in all cases [including C^{12}(RPA)] in our calculation. Again we note that if we let the energies become degenerate, the ν dependences of, for example, $(M_V^2)_D$ and $|F_{\rm el}|^2$ become almost identical (to about 1% at $\nu_{ab}=\nu_{\rm res}$). Goldemberg *et al.*[20] have shown this relation holds whenever the neutrons and protons oscillate with small amplitude against each other while maintaining their ground-state spatial distributions. The Goldhaber-Teller[21] model of the giant-dipole resonance satisfies these conditions. Another example is the giant-dipole resonance for the case of a harmonic oscillator with an interparticle harmonic force.[22] Furthermore, Bishop and Isabelle[23] find this relation is consistent with experiment for the giant resonance in O^{16}.

The inclusion of ground-state correlations in the C^{12} calculation reduces the squared matrix elements by about 10%. Equation (8) holds to within 13% instead of 8%, and Eq. (11) is still found to be accurate to 1%.

Foldy and Walecka have obtained a value for $(M_V^2)_{UD}$ by integrating the empirical photoabsorption cross section weighted by an energy-dependent factor. We find that our value of $(M_V^2)_{UD}$ is about twice what they obtain. Since their results for the total muon capture rates are in good agreement with experiment, we believe our values for all the matrix elements are too high. This will be discussed further in Sec. V.

IV. ELECTRON SCATTERING

The cross section for exciting a nucleus from a 0^+ ground state to a 2^- state by inelastic electron scattering is proportional to the square of the matrix element of the magnetic quadrupole operator. In Born approxima-

[18] $F_{\rm el}(\nu)=(1-4y/9)e^{-y}$ for C^{12} and $(1-\frac{1}{2}y)e^{-y}$ for O^{16} where $y=(\frac{1}{2}b\nu)^2$. As in Ref. 3 we have used $\hbar\nu_{\rm res}=E_m-E_{\rm res}=83$ MeV for C^{12} and 85 MeV for O^{16}.

[19] E. Wigner, Phys. Rev. **51**, 106 (1937).

[20] J. Goldemberg, Y. Torizaka, W. C. Barber, and J. D. Walecka, Nucl. Phys. **43**, 242 (1963).

[21] M. Goldhaber and E. Teller, Phys. Rev. **74**, 1046 (1948).

[22] D. M. Brink, Nucl. Phys. **4**, 215 (1960).

[23] G. R. Bishop and D. B. Isabelle, Nucl. Phys. **45**, 209 (1963).

tion, with neglect of nuclear recoil and the electron mass with respect to its energy,

$$
\begin{aligned}
\frac{d\sigma}{d\Omega}(2^- \leftarrow 0^+) &= \frac{k_2}{k_1}\frac{8\pi\alpha^2}{\Delta^4} V_T(\theta)\,|(2^-\|T_2^{\mathrm{mag}}(q)\|0^+)|^2 ,\\
V_T(\theta) &= \frac{2k_1k_2}{q^2}\sin^2(\theta/2)[(k_1+k_2)^2 - 2k_1k_2\cos^2(\theta/2)], \qquad (12)\\
T_{2M}^{\mathrm{mag}}(q) &= \int d\mathbf{x}[\boldsymbol{\mu}_N(\mathbf{x})\cdot(\nabla\times j_2(qx)\mathfrak{Y}_{221}{}^M(\Omega_x)) + j_2(qx)\mathfrak{Y}_{221}{}^M(\Omega_x)\cdot\mathbf{j}_N(\mathbf{x})];
\end{aligned}
$$

q and Δ are the three- and four-momentum transfers, and k_1 and k_2 are the initial and final electron energies.[6] For the case of a $T=0$ ground state and a $T=1$ excited state only the isovector part of T_{2M}^{mag} has a nonvanishing matrix element. Inserting single-particle operators, we have

$$
|(\Phi_{J=2;T=1;M_T=0}\|T_2^{\mathrm{mag}}(q)\|\Phi_0)|^2 \equiv F^2(q) = \tfrac{1}{2}\Big|\sum_{n'l'j';nlj}\alpha_{J=2,T=1}{}^{(n'l'j')(nlj)^{-1}}(n'(l'\tfrac{1}{2})j'\|t_{2,v}{}^{\mathrm{mag}}\|n(l\tfrac{1}{2})j)\Big|^2, \qquad (13)
$$

where

$$
t_{2M,v}{}^{\mathrm{mag}} = \nabla\times j_2(qr)\mathfrak{Y}_{221}{}^M(\theta,\phi)\cdot\frac{\hbar}{2Mc}(\lambda_p-\lambda_n)\boldsymbol{\sigma} + j_2(qr)\mathfrak{Y}_{221}{}^M(\theta,\phi)\cdot\frac{\hbar}{Mc}\frac{1}{i}\nabla .
$$

The matrix element between single-particle states may be evaluated in standard fashion[24] yielding

$$
\begin{aligned}
(n'(l'\tfrac{1}{2})j'\|t_{2,v}{}^{\mathrm{mag}}(q)\|n(l\tfrac{1}{2})j) &= i(\hbar/mc)\left(\frac{5}{4\pi}(2l+1)(2l'+1)(2j+1)(2j'+1)\right)^{1/2}\Bigg\{q(\lambda_p-\lambda_n)(-1)^{l'}\\
&\times\Bigg[\left(\frac{27}{10}\right)^{1/2}\begin{Bmatrix} l' & l & 1\\ \tfrac{1}{2} & \tfrac{1}{2} & 1\\ j' & j & 2\end{Bmatrix}\begin{pmatrix} l' & 1 & l\\ 0 & 0 & 0\end{pmatrix}(n'l'|j_1(qr)|nl) - \left(\frac{21}{5}\right)^{1/2}\begin{Bmatrix} l' & l & 3\\ \tfrac{1}{2} & \tfrac{1}{2} & 1\\ j' & j & 2\end{Bmatrix}\begin{pmatrix} l' & 3 & l\\ 0 & 0 & 0\end{pmatrix}(n'l'|j_3(qr)|nl)\Bigg]\\
&+(-1)^{l'+j+1/2}\sqrt{5}\begin{Bmatrix} l' & j' & \tfrac{1}{2}\\ j & l & 2\end{Bmatrix}\Bigg[\begin{Bmatrix} 2 & 1 & 2\\ l & l' & l+1\end{Bmatrix}\frac{\begin{pmatrix} l' & 2 & l+1\\ 0 & 0 & 0\end{pmatrix}}{\begin{pmatrix} l+1 & 1 & l\\ 0 & 0 & 0\end{pmatrix}}\frac{l+1}{2l+1}\left(n',l'\left|j_2(qr)\left(\frac{d}{dr}-\frac{l}{r}\right)\right|n,l\right)\\
&+\begin{Bmatrix} 2 & 1 & 2\\ l & l' & l-1\end{Bmatrix}\frac{\begin{pmatrix} l' & 2 & l-1\\ 0 & 0 & 0\end{pmatrix}}{\begin{pmatrix} l-1 & 1 & l\\ 0 & 0 & 0\end{pmatrix}}\frac{l}{2l+1}\left(n',l'\left|j_2(qr)\left(\frac{d}{dr}+\frac{l+1}{r}\right)\right|n,l\right)\Bigg]\Bigg\}. \qquad (14)
\end{aligned}
$$

We shall call $F(q)$ the (transverse) form factor for the state. The corresponding (transverse) form factors for excitation of the 1^- states have been calculated by Lewis and Walecka[6] for C^{12} and by Lewis[7] for O^{16}. All the calculations are for a scattering angle of 180°. In this case the longitudinal form factors do not contribute, leaving only the transverse ones mentioned above. These are the same transverse form factors that appear in processes involving real photons. However, such processes determine the form factor for only one momentum transfer since the photon must be on the mass shell.

We note that our results using a residual interparticle that fits low-energy nucleon-nucleon scattering give energy levels which are approximately 1 or 2 MeV higher than those using a δ-function ordinary force with a

[24] For example see A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957). We use Edmonds' notation.

strength chosen to fit the known 1^- levels.[6,7,9,10] We therefore compare our form factors with those of experimental levels of slightly lower energy.

Figure 1 shows the calculated spectrum of the 1^- and 2^-, $T=1$ states in C^{12} for various values of q. The contribution of the 2^- states is small for low q, but becomes quite sizeable at about $q=100$ MeV/c. In particular the state at 20.76 MeV carries most of the 2^- strength at this q and should be clearly noticeable in experiments.[25] The experimental spectrum for 65-MeV incident electrons shown in Fig. 2, indicates a very strong state at

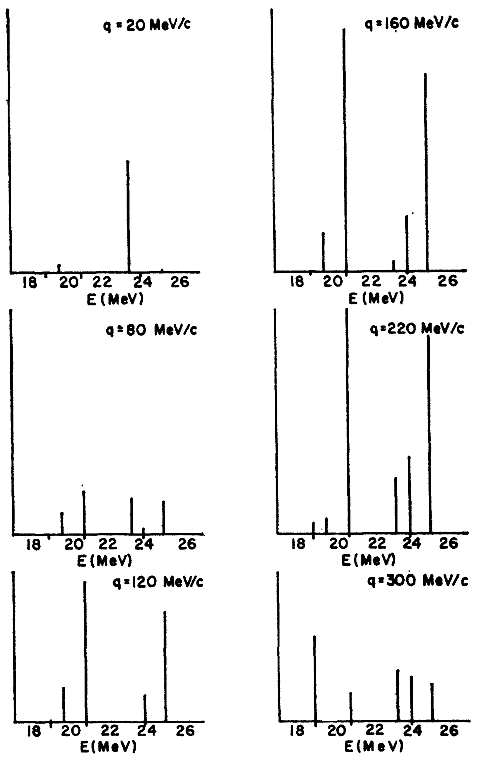

FIG. 1. Spectrum of the 1^- and 2^-, $T=1$ states in C^{12}. The length of the line is proportional to the square of the transverse factor for the state. For the 2^- states the line is extended slightly below the base line. An additional 1^- state at 35.80 MeV is not shown.

19.2 MeV.[12] The squared form factors for these states are compared in Fig. 3. The calculated curve is too large by about a factor of 2. We note, however, that the form factor appears to be rising rapidly with q as predicted by the theory. Sanderson[26] has calculated inelastic proton scattering from the 2^-, $T=1$ state at 19.3 MeV calculated by Vinh-Mau and Brown[10] (which corresponds to our 20.76 MeV state) and from other states in this region and has shown the results are consistent with the

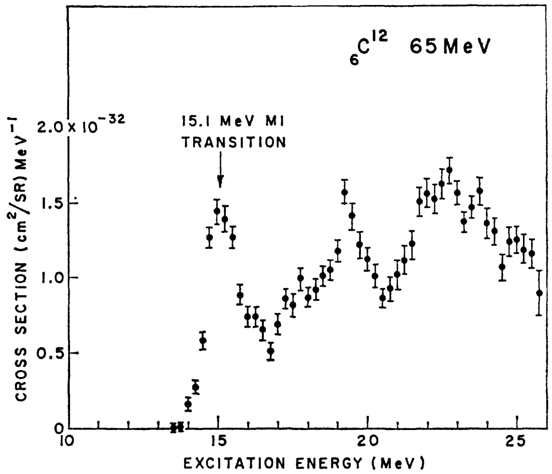

FIG. 2. The experimental energy spectrum of 65-MeV electrons scattered through 180° from C^{12}. (Ref. 12.)

inelastic proton-scattering experiments of Tyren and Maris.[27]

Hanna and his co-workers[28] have also seen a state at

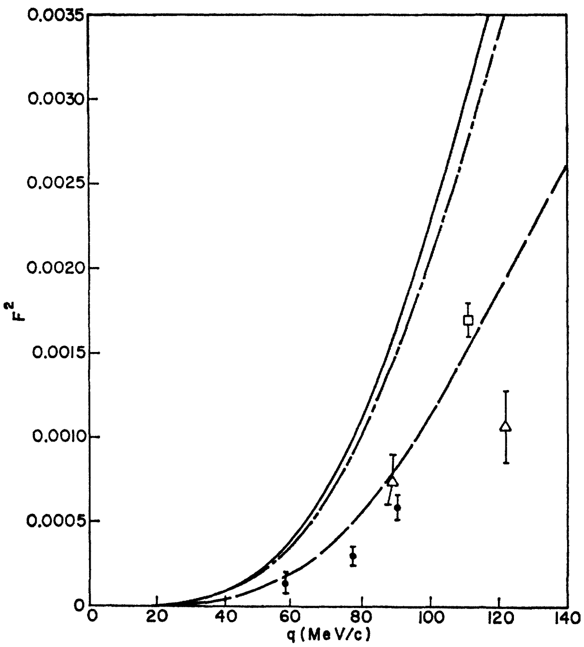

FIG. 3. Squared form factor versus momentum transfer for the 2^-, $T=1$ state at 20.76 MeV in C^{12}. The dashed curve is the same but with half the strength, while the dot-dash curve is the result of a random-phase approximation calculation. The triangles are from Stanford 180° scattering experiments by Goldemberg and Barber (Ref. 11). The circles are from 152° scattering experiments at Darmstadt (Ref. 11), and the data have been treated as though the state had no longitudinal component. The square is from recent Stanford 180° data (Ref. 12). *Note added in proof.* Recent experiments by Vanpraet (private communication) provide two new points in Fig. 3: $F^2=0.0006$ at $q=81$ MeV/c and $F^2=0.0018$ at $q=121$ MeV/c with an error of about ±0.0001 in each case.

[25] The importance of this state for $M2$ transitions has been noted previously by Vinh-Mau and Brown (Ref. 10).

[26] E. A. Sanderson, Nucl. Phys. **35**, 557 (1962).

[27] H. Tyren and Th. A. J. Maris, Nucl. Phys. **3**, 52 (1957); **4**, 637 (1957).

[28] R. E. Segel, S. S. Hanna, and R. C. Allas, Phys. Rev. (to be published).

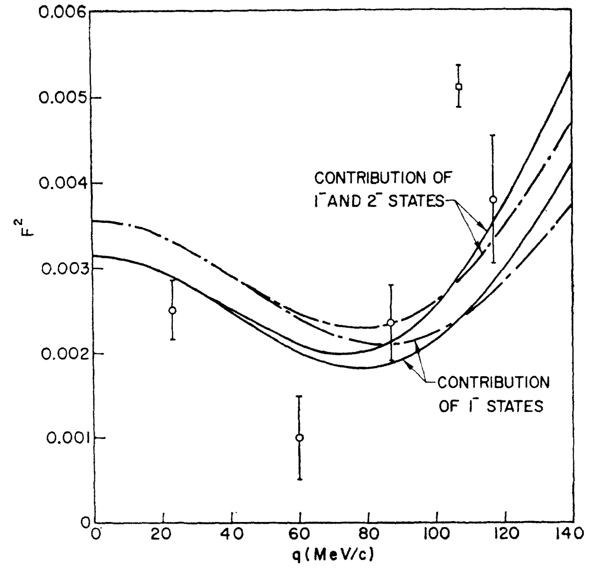

FIG. 4. Squared form factor for the giant resonance in C^{12}. The lower curve includes only the contribution of the 1^-, $T=1$ states while the upper curve also includes that of the 2^- states. The dot-dash curves are the result of a random-phase approximation calculation. The point at 23 MeV/c is from photon experiments (Ref. 6). The three other circles are from Stanford 180° scattering data (Ref. 11), and the square is more recent Stanford data (Ref. 13). The experimental points were obtained by integrating the cross section between 20 and 26 MeV.

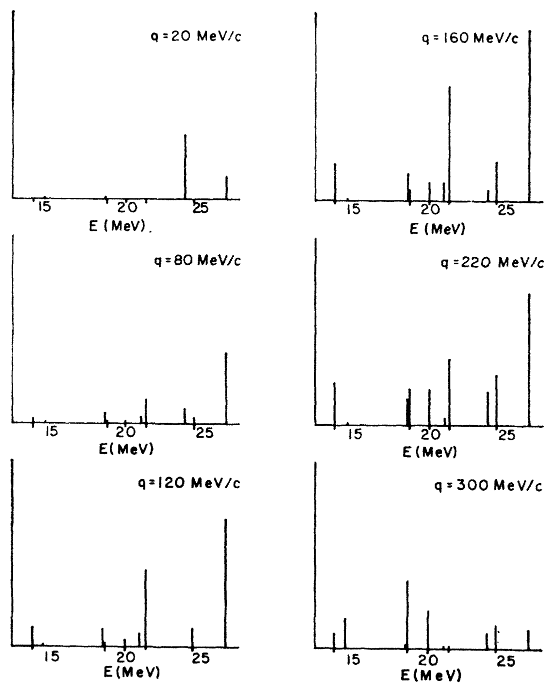

FIG. 5. Spectrum of 1^- and 2^-, $T=1$ states in O^{16}. The length of the line is proportional to the square of the transverse form factor for the state. For the 2^- states the line is extended slightly below the base line.

19.2 MeV in a $B^{11}(p,\gamma)C^{12}$ experiment with a width of about 25 eV. Our calculations indicate the state at 20.76 MeV should have a partial width for decay to the ground state of only 4 eV. We have also calculated the partial width for this state to decay to the 2^+, $T=0$ state at 4.43 MeV. The wave functions of Goswami and Pal[29] have been used for the latter state. As a result of almost complete cancellation of the matrix elements we find a width of 1 eV. This decay has not been observed, which is consistent with the predicted width. Since the observed state at 19.2 MeV has a much larger width for ground-state decay than predicted by the theory, we believe some other state has been seen in the experiment.

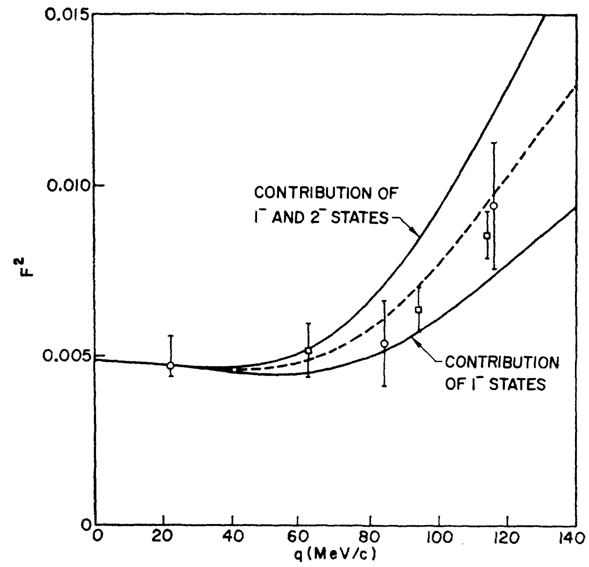

FIG. 6. The squared form factor for the giant resonance in O^{16}. The lower line includes only the 1^- states while the upper line also includes the contribution of the 2^- states. If only half of the 2^- contribution is included the dashed lines result. The point at 22 MeV/c comes from photon experiments (Ref. 7). The other circles are Stanford 180° data (Ref. 11), while the squares are more recent Stanford data (Ref. 13). Owing to the different spectra obtained in the two Stanford experiments, the experimental points were obtained by integrating the cross section in the region 20–27 MeV in Ref. 11, and 19–27 MeV in Ref. 13.

The 2^- state at 23.94 MeV falls in the giant resonance region in C^{12}. In Fig. 4 we show the correction this state makes to the form factor of the giant resonance using only the 1^-, $T=1$ states as calculated by Lewis and Walecka. The modification is quite small, but depends critically on the mixing of the unperturbed states. Most of the unperturbed strength is in the $(1d_{5/2})(1p_{3/2})^{-1}$ configuration; thus only a slight admixture of this state can greatly increase the form factor of one of the other states. About $\frac{1}{2}$ of the strength of the 23.94-MeV state comes from this mixing although it is quite small (0.131).

Finally we note that the inclusion of ground-state

[29] A. Goswami and M. K. Pal, Nucl. Phys. 35, 544 (1962).

correlations in the calculation does not substantially alter the results.

The spectrum of the 1^- and 2^-, $T=1$ states for O^{16} for various values of q is shown in Fig. 5. The 2^- state with the largest form factor at low q occurs at 21.34 MeV, which along with the 24.52-MeV state is in the giant resonance region. In Fig. 6 we plot the correction these states make to the giant resonance form factor calculated by Lewis using the 21.01-, 23.89-, and 26.63-MeV 1^- states. This produces a greater modification than in C^{12} since the 2^- state with the largest form factor is now in the giant resonance region. Corresponding to the predicted level at 21.34 MeV, the experiments of Vanpraet[13] show a peak in the giant resonance at 20.2 MeV. The form factor for this state is shown in Fig. 7. We note that the q dependence of the calculated form factor is in good agreement with experiment. As was the case for the 20.76-MeV state in C^{12} our results for the square of the form factor are larger than experiment by a factor of 2. The recent O^{16} experiments[13] show considerably more structure in the giant resonance than predicted by the particle-hole theory of Brown, and the identification of individual levels is quite difficult (except for the 2^- at 21.34 MeV mentioned above). For this reason only the integrated cross section of the giant resonance has been compared with experiment. Although comparison with individual levels may not be possible, we should be able to predict the shape of the giant resonance. In particular

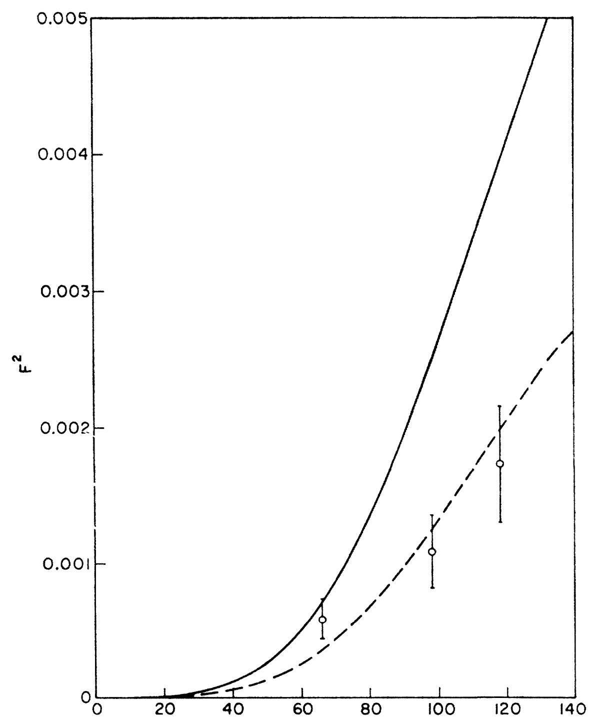

FIG. 7. Squared form factor for the 21.34-MeV 2^- state in O^{16}. The dashed curve is one-half of this result. The experimental results are from Ref. 13.

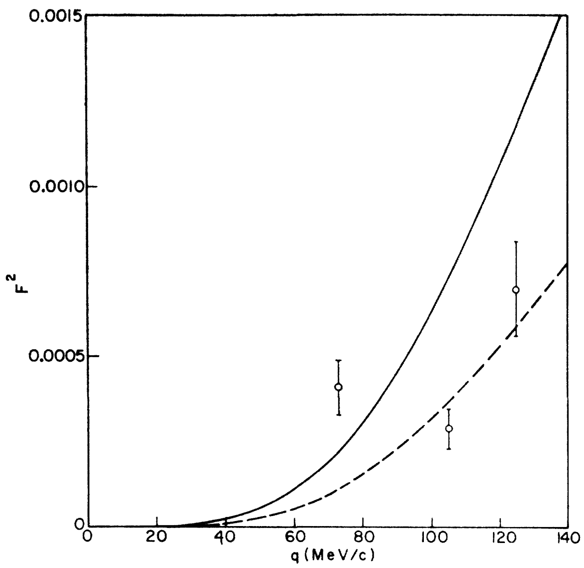

FIG. 8. Squared form factor for the 13.85-MeV 2^- state in O^{16}. Half of this result is indicated by the dashed curve. The points are from Stanford 180° experiments. (Ref. 13).

the 1^- level at 26.63 MeV should be very large at high q (see Fig. 5). Thus as the momentum transfer is increased we expect to see the peak of the giant resonance shift up by about 2 MeV. The original experiments[11] confirmed this shift; however the recent experiments of Vanpraet[13] indicate that such a pronounced shift does not take place. Although there is some shift of strength from the low-energy part to the high-energy part of the giant resonance, it is much smaller than predicted by the theory.[30] Since the agreement of the integrated strength of the giant resonance with experiment is fairly good, we believe that some of the strength predicted for the 1^- state at 26.63 MeV appears instead in additional states at a lower energy.

Of the remaining 2^- states in O^{16}, the one predicted at 13.85 MeV, which is a member of the same $T=1$ multiplet as the ground state of N^{16}, has the greatest strength. The form factor for this state is compared with the experimental results of the observed level at 13 MeV in Fig. 8. Except for the point at $q=73$ MeV/c we find the experiments give about half the predicted cross section.

V. SUMMARY AND DISCUSSION

Using the particle-hole model of the nucleus, we have computed the dipole and unretarded dipole contributions to the matrix elements M_V, M_A, and M_P for muon capture. To do this we needed the wave functions of the 0^-, 1^-, and 2^-, $T=1$ states. The wave functions of the 1^- states in C^{12} and the 0^-, 1^-, and 2^- in O^{16} have been

[30] We note that with the inclusion of the 2^- states, the predicted shift is much smaller than that predicted if only 1^- states are considered.

calculated by Lewis and Walecka. We have extended their calculations to the 0^- and 2^- states in C^{12}. The calculation has essentially no free parameters, since a potential which fits the low-energy scattering of free nucleons was used. The equality $(M_V{}^2)_D=(M_A{}^2)_D=(M_P{}^2)_D$ was found to hold to within 13% in all cases, while the assumption $(M^2)_D=(M^2)_{UD}|F_{\rm el}(\nu_{\rm res})|^2$ was good to 1%.

We have also used the wave functions for the 2^- states to compute the inelastic electron scattering from these states. For both carbon and oxygen one state carries most of the 2^- strength. These "giant-magnetic-quadrupole states" have been identified with states observed in recent electron-scattering experiments. The momentum-transfer dependence of the calculated states are in good agreement with experiment. However, since the spins and parities of the levels were not determined directly from the experiment, we must consider our results as not being definite.

In the case of C^{12} we have repeated the calculations this time using the random-phase approximation to include ground-state correlations. In both muon capture and electron scattering the results did not differ radically from those where correlations were not considered.

As has been noted, our results for the squared form factors for electron scattering from the 2^-, $T=1$ states tend to be too high by about a factor of 2. Similarly, the matrix elements $(M_V{}^2)_{UD}$ for muon capture exhibit a factor of 2 discrepancy when compared with the results of Foldy and Walecka. Lewis and Walecka[6] have shown that if one uses current conservation to evaluate $T_{1M}{}^{\rm el}$ in the limit $q\to 0$,

$$T_{1M}{}^{\rm el}(q_{fi})=-(\sqrt{2/3})q_{fi}\times\int d\mathbf{x}\,x\rho_N(\mathbf{x})Y_{1M}(\Omega_{\mathbf{x}}),\quad q_{fi}\to 0,\quad (15)$$

the resulting squared matrix element is about twice that obtained by using the particle-hole model to evaluate the matrix element of the current. They find the latter method is in good agreement with photoabsorption experiments. The discrepancy is due to the use of approximate wave functions. Since $(M_V{}^2)_{UD}{}^{(i)}$ is proportional to the square of the matrix element of the same operator that appears in Eq. (15), we expect $(M_V{}^2)_{UD}$ to be about twice the correct value. Furthermore, since we believe the approximate equality of $(M_V{}^2)_{UD}$, $(M_A{}^2)_{UD}$, and $(M_P{}^2)_{UD}$ should hold for all reasonable wave functions, we expect $(M_A{}^2)_{UD}$ and $(M_P{}^2)_{UD}$ should also be too large by a factor of 2. However it may easily be seen that the major contribution to $|(2^-,T=1\|T_2{}^{\rm mag}\|0^+,T=0)|^2$ comes from a term which is proportional to the contribution of this 2^- state to $(M_A{}^2)_{UD}$. This is consistent with the observed discrepancy in the squared form factors for the 2^- states.

In our calculation of the form factors for the giant resonance, we have only considered the contributions of the 1^- and 2^-, $T=1$ states. A 1^+, $T=1$ state would also contribute strongly to the form factor, but calculations indicate the probable absence of such states in the region of the giant resonance, at least for C^{12}.[10] Weisskopf estimates of $E2$, $E3$, and $M3$ transitions for $q=120$ MeV/c and $E_{fi}=20$ MeV are about 1% of the observed cross section for the giant resonance for both C^{12} and O^{16}.[11,13] These estimates are quite rough since the long-wavelength limit is not appropriate for $q=120$ MeV/c; however, we believe they give a reasonable order of magnitude. The magnetic transitions to 2^-, $T=0$ states will be small compared with those to the 2^-, $T=1$ states considered in their paper since $[(\lambda_p+\lambda_n)/(\lambda_p-\lambda_n)]^2=(0.88/4.71)^2=0.035$. Finally we note that collective effects may increase the strength of the neglected states.

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