APPENDIX

We wish to give here an argument to show that the inclusion of vector mesons does not affect the above conclusions. For simplicity, we shall consider the πN case, although the argument can be trivially generalized to the $SU(3)$ case.

Let $\Delta B_{IJ}^{(\rho)}$ be the contribution of ρ-meson exchange to the force term B_{IJ}. Then the contribution to N_{IJ} is

$$\Delta N_{IJ}^{(\rho)}(\omega)=\frac{1}{\pi}\int_L \frac{\mathrm{Im}\Delta B_{IJ}^{(\rho)}(\omega')}{\omega'-\omega}D_{IJ}(\omega').$$

With the linear D approximation (Eq. 7) this becomes

$$\Delta N_{IJ}^{(\rho)}(\omega) =\Delta B_{IJ}^{(\rho)}(\omega)D_{IJ}(\omega)+\frac{1}{\omega_{IJ}-\omega_0}\lim_{\omega\to\infty}(\omega\Delta B_{IJ}^{(\rho)}(\omega)).$$

Then

$$\Delta g_{IJ}^{(\rho)}(\omega)=\Delta B_{IJ}^{(\rho)}(\omega)+\frac{1}{\omega_{IJ}-\omega}\lim_{\omega\to\infty}(\omega\Delta B_{IJ}^{(\rho)}(\omega)).$$

This gives for the change in the reduced width γ_{IJ}

$$\Delta\gamma_{IJ}^{(\rho)}=\lim_{\omega\to\infty}(\omega\Delta B_{IJ}^{(\rho)}(\omega)).$$

With the expression for $\Delta B_{IJ}^{(\rho)}(\omega)$ given by Chew,[1] viz.

$$\Delta B_{IJ}^{(\rho)}(\omega)=\frac{C^{(\rho)}}{4k^2}\ln\left(1+\frac{4k^2}{m_\rho^2}\right),$$

we have

$$\lim_{\omega\to\infty}(\omega\Delta B_{IJ}^{(\rho)}(\omega))=0.$$

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Precession of Relativistic Particles of Arbitrary Spin in a Slowly Varying Electromagnetic Field

DANIEL ZWANZIGER*

Centre d'Etudes Nucléaires de Saclay, Gif-sur-Yvette (S et O), Saclay, France

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It is shown that, under accelerator or bubble-chamber conditions, the passage of a particle of arbitrary spin through an electromagnetic field effects a Lorentz transformation on its momentum and polarization, and a linear differential equation determining this transformation is given. We also give explicitly the decay-time dependence of the angular distribution that describes the decay of a particle moving in an electromagnetic field, and thereby obtain a method, explained in detail, of measuring the magnetic moment of an unstable, higher spin particle like the Ω^-. It is noted that the gyromagnetic ratio $g=2$ leads to particularly simple equations of motion for all spins, and not only for spin $\frac{1}{2}$. In an appendix we use a novel covariant algebraic method to solve the equations of motion and obtain the finite Lorentz transformation, in the case of a constant and homogeneous electromagnetic field. The method involves the introduction of an algebra of 4-by-4 matrices that plays the same role for 4-vectors as the Dirac algebra for 4-spinors.

I. RELATIVISTIC LARMOR THEOREM

We wish to describe the time evolution of the polarization matrix, or density matrix in spin space, of a relativistic particle of arbitrary spin in a slowly varying electromagnetic field. This matrix is perhaps most directly observable if the particle decays, for it determines the angular distribution of the decay products, a function, $I(p_1,p_2,\cdots)$, of the 4-momenta p_1, $p_2\cdots$ of the daughter particles. Knowledge of the momentum and polarization matrix at a time $t=0$, and of its subsequent time evolution, allows one to predict the dependence $I(p_1p_2\cdots t)$ of the decay angular distribution on the decay time t. We will obtain this dependence explicitly.

The equation of motion of the dipole polarization, corresponding to spherical harmonics of order 1 in the decay angular distribution, has been described in the literature,[1] and is known most familiarly in covariant form as the Bargmann-Michel-Teledgi (BMT) equation.[2] However, particles of spin $j>\frac{1}{2}$ also have higher multipole polarization, corresponding to harmonics of all orders up to $2j$ in the angular distribution. The new content of the description given here is that it is applied to these higher moments as well. It takes the form of a simple generalization of Larmor's theorem which, however, when stated relativistically is found to apply to the momentum as well as to the polarization.

* Present address: Courant Institute of Mathematical Sciences, New York University, New York, New York.

[1] H. Bacry, Nuovo Cimento 3, 1164 (1962). This article contains many references to earlier work on the subject.

[2] V. Bargmann, L. Michel, and V. Teledgi, Phys. Rev. Letters 2, 435 (1959).

The result may be put geometrically in the form of the following theorem.

Theorem: In a slowly varying electromagnetic field F, the polarization matrix and the unit velocity 4-vector $u=p/m$ of a particle of arbitrary spin j, mass m, charge e, and gyromagnetic[3] ratio g, undergo a continuous Lorentz transformation whose infinitesimal generator $\Omega d\tau$, for an increment in the particle's proper time $d\tau$, is given by

$$\Omega=(e/m)[F+\tfrac{1}{2}(g-2)(I-uu)F(I-uu)]. \quad (1)$$

We are using a matrix notation where F is the matrix with components $F_\nu{}^\mu$, and where uu is the matrix with components $u^\mu u_\nu$. It is understood that e may be zero, with finite eg.

Ω is an antisymmetric tensor like F. "Slowly varying" means that the force due to the electromagnetic multipole moments $l=1, 2, \cdots, 2j$, and the torque due to the multipole moments $l=2, 3, \cdots, 2j$ is negligible. For an elementary particle this requires that the field variation in space (time) is negligible over a Compton wavelength ($c^{-1}\times$Compton wavelength), which certainly obtains under accelerator and bubble-chamber conditions.

The theorem is easily established by a direct application of the argument of BMT.[2] Let us first restate their result and review their reasoning. The Lorentz force on the velocity 4-vector u, where $u^2=1$, and the BMT equation for a classical spin 4-vector s, where $s\cdot u=0$, are

$$du/d\tau=(e/m)Fu, \quad (2)$$

$$ds/d\tau=(e/m)[Fs+\tfrac{1}{2}(g-2)(I-uu)Fs]. \quad (3)$$

Since $u^2=1$ and $u\cdot s=0$, we may rewrite these equations as

$$du/d\tau=\Omega u, \quad (4)$$

$$ds/d\tau=\Omega s, \quad (5)$$

with Ω given by Eq. (1). We note incidentally that the familiar "precession of spin" with respect to the momentum is simply an illustration of the elementary fact that the same Lorentz transformation affects in a different way the 3-vector parts of different 4-vectors.

As pointed out by BMT, Eq. (3), and hence also Eq. (5), follow upon expressing in covariant fashion the Larmor precession of s in the rest frame, given the condition $u\cdot s=0$.

It has recently been shown[4] that the polarization matrix of a particle of spin j and 4-velocity u may be specified by a real, traceless symmetric tensor of rank $2j$, $D_{\mu_1,\mu_2,\cdots,\mu_{2j}}$, or by a set of real traceless, symmetric tensors orthogonal to u, of rank $r=1, 2, \cdots, 2j$, $M_{\mu_1,\cdots,\mu_r}$. These quantities are of course c numbers. They are parameters that determine the density matrix. The dipole polarization 4-vector M_μ coincides with the classical 4-vector s_μ of BMT.

We now assume the validity of a classical relativistic limit whereby the particle is characterized by a well-defined trajectory and velocity and by these polarization tensors, all of which may be regarded as functions of the proper time τ of the particle. This is exactly the situation which is familiar in the theory of light where one may understand polarization phenomena, such as the rotation of polarization produced by optically active substances, in the ray limit of optics.

In the rest frame of the particle, the tensors M, orthogonal to u, have only nonvanishing space components and are the familiar nonrelativistic multipole-moment tensors. In this frame they undergo Larmor precession. The argument of BMT applies to them, giving the tensor form of Eq. (5),

$$\frac{d}{d\tau}M^{\mu_1\mu_2\cdots\mu_l}=\Omega^{\mu_1}{}_\nu M^{\nu\mu_2\cdots\mu_l}+\Omega^{\mu_2}{}_\nu M^{\mu_1\nu\cdots\mu_l} +\Omega^{\mu_l}{}_\nu M^{\mu_1\mu_2\cdots\nu}. \quad (6)$$

But any antisymmetric tensor Ω is the infinitesimal generator of a Lorentz transformation, and the last equation is the Lorentz transformation law of a tensor $M^{\mu_1\mu_2\cdots\mu_l}$.

Consequently, the 4-vector u and the multipole-moment tensors M, which specify the momentum and the polarization matrix, undergo the same instantaneous Lorentz transformation. Because the Lorentz transformations form a Lie group, it follows that the net effect on the momentum and polarization of the passage of a particle through a slowly varying electromagnetic field is a certain finite Lorentz transformation Λ. This Lorentz transformation, regarded as a function of the proper time τ of the particle, is the solution of the differential equation

$$\frac{d}{d\tau}\Lambda(\tau)=\Omega(\tau)\Lambda(\tau). \quad (8)$$

The trajectory is obtained by solving

$$\frac{d}{d\tau}x(\tau)=u(\tau), \quad (9)$$

where $x^\mu(\tau)=(t(\tau),\mathbf{x}(\tau))$, which gives t as a function of τ or inversely. The 4-velocity $u(\tau)$, which appears here and in the definition of Ω, Eq. (1), is the solution of Eq. (2). In the Appendix we solve Eqs. (2), (8), and (9) for a constant homogeneous electromagnetic field.

We observe that the particular significance of a gyromagnetic ratio $g=2$ does not seem limited to spin $\frac{1}{2}$, but leads to the simplest equations of motion for all spin. This lends support to a recent speculation of the author

[3] If, in violation of parity conservation and time-reversal invariance, the particle also has an electric-dipole moment, this may be accounted for by adding, on the right-hand side of Eq. (1), the term $(e/2m)p(I-uu)F^d(I-uu)$, where F^d is the dual to F.

[4] D. Zwanziger, Phys. Rev. 137, B1535 (1965).

on minimal electromagnetic coupling[5] which is in contradiction with various other suggestions.[6]

Let us now recall the decay angular distribution mentioned at the beginning, $I(p_1p_2\cdots\tau)$. We adopt a covariant normalization so that the probability P_Ω of finding p_1, $p_2\cdots$ in a volume Ω of phase space is

$$P_\Omega=\int_\Omega \delta^4(p(\tau)-p_1-p_2\cdots) \times I(p_1,p_2\cdots\tau)\frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2}\cdots. \quad (10)$$

Then the time dependence of the angular distribution I is simply

$$I(p_1,p_2,\cdots,\tau)=I(\Lambda^{-1}(\tau)p_1,\Lambda^{-1}(\tau)p_2,\cdots,0), \quad (11)$$

where $\Lambda(\tau)$ is the solution of Eq. (8), determined by the initial condition $\Lambda(0)=1$. This follows from the fact[4] that I is an invariant function of the final momenta, and the initial momentum and polarization tensor.

II. A METHOD OF MEASURING DIPOLE MOMENTS

This work was undertaken in preparation for the day when the magnetic moment of the Ω^-, which is believed to have spin $\frac{3}{2}$, or of some other unstable particle of spin $>\frac{1}{2}$, is measured. This will presumably be done by observing the precession of the decay angular distribution,[7] as is done at present for μ, Λ, and Σ. We will now describe a way of obtaining the magnetic- and electric-dipole moments from observation of this precession. It requires solving Eq. (8) in a particular way.

Set

$$\Omega_0=(e/m)F, \quad (12)$$

and let $\Lambda_0(\tau)$ be the solution of

$$\dot{\Lambda}_0(\tau)=\Omega_0\Lambda_0(\tau), \quad \Lambda_0(0)=1, \quad (13)$$

where the dot means differentiation with respect to τ. By virtue of Eq. (2), we have immediately

$$u(\tau)=\Lambda_0(\tau)u(0). \quad (14)$$

We now break up Eq. (8) into two parts. Set

$$\Lambda(\tau)=\Lambda_0(\tau)\Lambda_1(\tau), \quad (15)$$

and substitute this into Eq. (8) to obtain an equation for Λ_1:

$$\dot{\Lambda}_1(\tau)=\Omega_1(\tau)\Lambda_1(\tau), \quad \Lambda_1(0)=1, \quad (16)$$

with

$$\Omega_1(\tau)=\Lambda_0^{-1}(\tau)[1-u(\tau)u(\tau)] \times[\tfrac{1}{2}(g-2)\Omega_0(\tau)+\tfrac{1}{2}p\Omega_0^d(\tau)] \times[1-u(\tau)u(\tau)]\Lambda_0(\tau), \quad (17)$$

where we have included the effect of a possible electric dipole moment, in accordance with footnote 3. Now in virtue of Eq. (14) for $u(\tau)$, we find

$$\Omega_1(\tau)=[1-u(0)u(0)]\{\Lambda_0^{-1}(\tau)[\tfrac{1}{2}(g-2)\Omega_0(\tau) +\tfrac{1}{2}p\Omega_0^d(\tau)]\Lambda_0(\tau)\}\times[1-u(0)u(0)]. \quad (18)$$

From this expression for $\Omega_1(\tau)$, we see immediately that it annihilates $u(0)$,

$$\Omega_1(\tau)u(0)=0,$$

so that $\Omega_1(\tau)$ is the generator of a rotation in the rest system of $u(0)$. By virtue of Eq. (16), $\Lambda_1(\tau)$ *is a pure rotation in the rest system of* $u(0)$,

$$\Lambda_1(\tau)u(0)=u(0). \quad (19)$$

Furthermore the entire dependence of Λ on the value of the magnetic- and electric-dipole moments is contained in the rotation Λ_1, because Λ_0, defined by Eqs. (12), (13), and (15), is independent of g and p. Consequently, it proves very convenient to refer all quantities to the rest frame R_0 of the particle at $\tau=t=0$ rather than the laboratory frame L, or the instantaneous rest frame R_t.

A particularly simple way of interpreting the observations is as follows. All quantities (time t, position $\mathbf{x}$, observed momenta of daughter particles p_1p_2, field strength $\mathbf{E}$, $\mathbf{B}$) are understood to refer to the R_0 frame and are obtained from the corresponding quantities, as measured in the laboratory, by the Lorentz transformation which brings $u(0)$, as measured in the laboratory, to rest. We assume that Eqs. (9), (13), and (16) have been solved, so τ and $\Lambda_0(\tau)$ are known for each disintegration and $\Lambda_1(\tau)$ is a known function of the magnetic- and electric-dipole moments. (In the Appendix they are found explicitly for constant homogeneous electromagnetic fields, which is perhaps the most important practical case.) Then for each momentum p_1, $p_2\cdots$, resulting from a decay observed to occur at proper time τ, calculate $\bar{p}_1=\Lambda_0^{-1}(\tau)p_1$, $\bar{p}_2=\Lambda_0^{-1}(\tau)p_2\cdots$, and plot the observed angular distribution for the decays occurring at proper time τ as a function of $\bar{p}_1$, $\bar{p}_2\cdots$. This results in an angular distribution

$$\bar{I}(\bar{p}_1,\bar{p}_2\cdots,\tau)=I(p_1,p_2,\cdots,\tau)$$

[5] D. Zwanziger in *Proceedings of the Symposium on the Lorentz Group, Boulder, Colorado, 1964* (University of Colorado Press, Boulder, Colorado, to be published).

[6] Yukawa, Sakata, Taketani, Proc. Phys.-Math. Soc. Japan **20**, 319 (1938); F. J. Belinfante, Phys. Rev. **92**, 997 (1953); P. A. Moldauer and K. M. Case, Phys. Rev. **102**, 279 (1956); V. S. Tumanov, Zh. Eksperim. i Teor. Fiz. **46**, 1755 (1964) [English transl.: Soviet Phys.—JETP **19**, 1182 (1964)]; T. D. Lee and C. N. Yang, Phys. Rev. **128**, 885 (1962); T. D. Lee, Phys. Rev. **128**, 899 (1962); R. C. Brunet, Nuovo Cimento **30**, 1317 (1963) and **34**, 599 (1964); L. M. Nath, Nucl. Phys. **57**, 611 (1964).

[7] For spin $>\frac{1}{2}$ it is not necessary that the decay violate parity for this precession to be observable in principle, although the shorter lifetime usually associated with parity-conserving decays seems to preclude this in practice.

whose time dependence is

$$\bar{I}(\bar{p}_1,\bar{p}_2,\cdots,\tau)=\bar{I}(R^{-1}(\tau)\bar{p}_1,R^{-1}(\tau)\bar{p}_2,\cdots,0)\,, \quad (20)$$

where we have written $R(\tau)\equiv\Lambda_1(\tau)$ to indicate that it is a pure rotation. From the observed rotation of the angular distribution $\bar{I}$ as a function of time, $R(\tau)$ may be deduced. The magnetic- and electric-dipole moments may then be determined because the dependence of of $R(\tau)$ on them is known. In the case of constant and homogeneous electromagnetic fields **E** and **B**, R is a rotation about the direction $(g-2)\mathbf{B}+p\mathbf{E}$, through an angle

$$\varphi=(e/2m)\,|(g-2)\mathbf{B}+p\mathbf{E}|\,\tau\,. \quad (21)$$

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APPENDIX: SOLUTION FOR A CONSTANT FIELD, AND THE ALGEBRA OF LORENTZ TENSORS

We will find the solution to Eq. (8) in the case of a constant, homogeneous electromagnetic field, by the method of Sec. II. We adopt the metric $(+,-,-,-)$ and a matrix notation for Lorentz tensors, whereby the first (row) index is upper (contravariant) and the second (column) is lower (covariant), so that, for example, F is the matrix with components

$$F=F^{\mu}{}_{\nu}=\begin{bmatrix}0 & E_1 & E_2 & E_3\\ E_1 & 0 & B_3 & -B_2\\ E_2 & -B_3 & 0 & B_1\\ E_3 & B_2 & -B_1 & 0\end{bmatrix}. \quad (A1)$$

The solution to Eq. (13), for constant, homogeneous F is obviously

$$\Lambda_0(\tau)=\exp(\Omega_0\tau)\,, \quad (A2)$$

with Ω_0 given by Eq. (12). The velocity 4-vector $u(\tau)$ is given by Eq. (14), which may be integrated to give

$$x(\tau)=\tau\{(\Omega_0\tau)^{-1}[\exp(\Omega_0\tau)-1]\}u(0)+x(0)\,, \quad (A3)$$

which gives the dependence of the time t on τ and the equation of the orbit. The function of $\Omega_0\tau$ in the braces is the entire function represented by the corresponding power series, even though $(\Omega_0\tau)^{-1}$ may not be defined. We will shortly evaluate explicitly these functions of antisymmetric tensors.

Having obtained $\Lambda_0(\tau)$ we now require $\Lambda_1(\tau)$, defined by the differential equation (16) in terms of $\Omega_1(\tau)$, which is given in Eq. (18). In the present case of constant, homogeneous electromagnetic field, Ω_0 is independent of τ, and Eq. (18) reduces to

$$\Omega_1=[1-u(0)u(0)][\tfrac{1}{2}(g-2)\Omega_0+\tfrac{1}{2}p\Omega_0{}^d]\times[1-u(0)u(0)]\,, \quad (A4)$$

which follows from Eq. (A2) for $\Lambda_0(\tau)$, and the fact, verified below Eq. (A20), that Ω_0 and $\Omega_0{}^d$ commute. We observe that Ω_1 is time-independent, so Eq. (16) may be immediately integrated, with solution

$$\Lambda_1(\tau)=\exp(\Omega_1\tau)\,. \quad (A5)$$

Equations (A2), (A3), and (A5) are the desired formal solutions to Eqs. (9), (13), and (16), and we have for total Lorentz transformation

$$\Lambda(\tau)=\Lambda_0(\tau)\Lambda_1(\tau)=\exp(\Omega_0\tau)\exp(\Omega_1\tau)\,.$$

For these formulas to be useful we must be able to evaluate these functions of antisymmetric tensors. Let the generic antisymmetric Lorentz tensor be represented by Eq. (A1). We introduce the complex 3-vectors

$$\mathbf{F}^+=(\mathbf{E}+i\mathbf{B})\,,\quad \mathbf{F}^-=(-\mathbf{E}+i\mathbf{B})=-(\mathbf{F}^+)^*. \quad (A6)$$

Then Eq. (A1) may be trivially rewritten

$$F=F^{\mu}{}_{\nu}=\tfrac{1}{2}\begin{bmatrix}0 & F_1{}^+ & F_2{}^+ & +F_3{}^+\\ F_1{}^+ & 0 & -iF_3{}^+ & iF_2{}^+\\ F_2{}^+ & iF_3{}^+ & 0 & -iF_1{}^+\\ F_3{}^+ & -iF_2{}^+ & iF_1{}^+ & 0\end{bmatrix}+\tfrac{1}{2}\begin{bmatrix}0 & -F_1{}^- & -F_2{}^- & -F_3{}^-\\ -F_1{}^- & 0 & -iF_3{}^- & iF_2{}^-\\ -F_2{}^- & iF_3{}^- & 0 & -iF_1{}^-\\ -F_3{}^- & -iF_2{}^- & iF_1{}^- & 0\end{bmatrix}. \quad (A7)$$

We now define the matrices $\boldsymbol{\sigma}^+$ and $\boldsymbol{\sigma}^-=-(\boldsymbol{\sigma}^+)^*$ by

$$\sigma_1{}^+=\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&-i\\0&0&i&0\end{bmatrix},\quad \sigma_1{}^-=\begin{bmatrix}0&-1&0&0\\-1&0&0&0\\0&0&0&-i\\0&0&i&0\end{bmatrix},$$

$$\sigma_2{}^+=\begin{bmatrix}0&0&1&0\\0&0&0&i\\1&0&0&0\\0&-i&0&0\end{bmatrix},\quad \sigma_2{}^-=\begin{bmatrix}0&0&-1&0\\0&0&0&i\\-1&0&0&0\\0&-i&0&0\end{bmatrix},$$

$$\sigma_3{}^+=\begin{bmatrix}0&0&0&1\\0&0&-i&0\\0&i&0&0\\1&0&0&0\end{bmatrix},\quad \sigma_3{}^-=\begin{bmatrix}0&0&0&-1\\0&0&-i&0\\0&i&0&0\\-1&0&0&0\end{bmatrix},$$

so that Eq. (A7) takes the form

$$F=F^{\mu}{}_{\nu}=\tfrac{1}{2}\boldsymbol{\sigma}^+\cdot\mathbf{F}^++\tfrac{1}{2}\boldsymbol{\sigma}^-\cdot\mathbf{F}^-, \quad (A9)$$

and the dual tensor is given by

$$F^d=i(\tfrac{1}{2}\boldsymbol{\sigma}^+\cdot\mathbf{F}^+-\tfrac{1}{2}\boldsymbol{\sigma}^-\cdot\mathbf{F}^-)\,. \quad (A10)$$

These matrices satisfy the commutation relations[8]

$$\begin{aligned}\sigma_i{}^+\sigma_j{}^+&=\delta_{ij}+i\epsilon_{ijk}\sigma_k{}^+\,,\\ \sigma_i{}^-\sigma_j{}^-&=\delta_{ij}+i\epsilon_{ijk}\sigma_k{}^-\,,\\ \sigma_i{}^+\sigma_j{}^-&=\sigma_j{}^-\sigma_i{}^+\,,\end{aligned} \quad (A11)$$

[8] The algebra of these matrices plays the same role for 4-vectors as the Dirac algebra for Dirac 4-spinors. In particular the 16 linearly independent matrices $\Sigma_\alpha=1,\ \sigma_i{}^+,\ \sigma_j{}^-,\ \sigma_i{}^+\sigma_j{}^-(i,j=1,2,3)$ each have unit square $\Sigma_\alpha{}^2=1$, and are pairwise orthogonal $\mathrm{tr}\Sigma_\alpha\Sigma_\beta=4\delta_{\alpha\beta}$.

which are those of two independent sets of Pauli matrices, each set commuting with the other.

The problem of evaluating functions of antisymmetric Lorentz tensors is thus reduced to the familiar problem of evaluating functions of Pauli matrices. In particular, for the exponential function $\exp F$, where F again represents the generic antisymmetric tensor, we have

$$\begin{aligned}\exp F &= \exp(\tfrac{1}{2}\boldsymbol{\sigma}^+\cdot\mathbf{F}^+ + \tfrac{1}{2}\boldsymbol{\sigma}^-\cdot\mathbf{F}^-)\\ &= \exp(\tfrac{1}{2}\boldsymbol{\sigma}^+\cdot\mathbf{F}^+)\exp(\tfrac{1}{2}\boldsymbol{\sigma}^-\cdot\mathbf{F}^-)\\ &= [\cosh(\tfrac{1}{2}F^+) + \boldsymbol{\sigma}^+\cdot\hat{F}^+\sinh(\tfrac{1}{2}F^+)]\\ &\quad\times[\cosh(\tfrac{1}{2}F^-) + \boldsymbol{\sigma}^-\cdot\hat{F}^-\sinh(\tfrac{1}{2}F^-)],\end{aligned} \quad (A12)$$

where

$$\cosh(\tfrac{1}{2}F^\pm) = \sum_{n=0}^{\infty}\frac{[(\frac{1}{2}\mathbf{F}^\pm)^2]^n}{(2n)!}$$

$$\boldsymbol{\sigma}^\pm\cdot\hat{F}^\pm\sinh(\tfrac{1}{2}F^\pm) = \tfrac{1}{2}\boldsymbol{\sigma}^\pm\cdot\mathbf{F}^\pm\sum_{n=0}^{\infty}\frac{[(\frac{1}{2}\mathbf{F}^\pm)^2]^n}{(2n+1)!}, \quad (A13)$$

and

$$(\mathbf{F}^\pm)^2 = \mathbf{E}^2 - \mathbf{B}^2 \pm i2\mathbf{E}\cdot\mathbf{B} = z_1 \pm iz_2. \quad (A14)$$

It is now convenient to introduce two real parameters α and β according to $(\mathbf{F}^\pm)^2 = (\alpha + i\beta)^2$ or

$$\begin{aligned}\alpha^2 - \beta^2 &= \mathbf{E}^2 - \mathbf{B}^2 = z_1,\\ 2\alpha\beta &= 2\mathbf{E}\cdot\mathbf{B} = z_2.\end{aligned} \quad (A15)$$

Then Eq. (A12) takes the form

$$\exp F = \left(\frac{\beta^2\cosh\alpha + \alpha^2\cos\beta}{\alpha^2+\beta^2}\right) + \left(\frac{\alpha\sinh\alpha + \beta\sin\beta}{\alpha^2+\beta^2}\right)F + \left(\frac{\cosh\alpha - \cos\beta}{\alpha^2+\beta^2}\right)F^2 + \left(\frac{-\beta\sinh\alpha + \alpha\sin\beta}{\alpha^2+\beta^2}\right)F^d. \quad (A16)$$

If Eqs. (A15) are solved for α and β in terms of the bilinear invariants z_1 and z_2, then the coefficients in parentheses, when expressed in terms of z_1 and z_2, turn out to be entire analytic functions. The coefficient of F^d is odd in the pseudoscalar z_2, the others are even.

Equation (A16) shows very clearly the dependence of exp F on the invariants. Some special cases are of interest. We call F "rotation" or "magnetic" type if $\alpha = 0$, which means that in some frame $\mathbf{E} = 0$; "velocity" or "electric" type if $\beta = 0$, which means that in some frame $\mathbf{B} = 0$; and "crossed" or "light" type if $\alpha = \beta = 0$ which means that $\mathbf{E}^2 - \mathbf{B}^2 = 0$, $\mathbf{E}\cdot\mathbf{B} = 0$.

Equation (A16) reduces in these cases, respectively, to

$$\exp F = 1 + \frac{\sin\beta}{\beta}F + \frac{1-\cos\beta}{\beta^2}F^2, \quad (A17)$$

$$\exp F = 1 + \frac{\sinh\alpha}{\alpha}F + \frac{\cosh\alpha - 1}{\alpha^2}F^2, \quad (A18)$$

$$\exp F = 1 + F + \tfrac{1}{2}F^2. \quad (A19)$$

We recall that F here represents the generic antisymmetric tensor and not necessarily an electromagnetic field. In particular, if F is the generator of a Lorentz transformation, then $\exp F$ is a Lorentz transformation and the special cases are, respectively, a pure rotation in some frame, a pure velocity transformation in some frame, and an exceptional type of transformation.

A useful relation, obtained directly from Eqs. (A9) and (A10) is

$$FF^d = F^dF = -\mathbf{E}\cdot\mathbf{B} = -\alpha\beta \quad (A20)$$

or

$$F^{-1} = -(\alpha\beta)^{-1}F^d, \quad (A21)$$

which enables us to calculate from Eq. (A16)

$$F^{-1}(\exp F - 1) = \left[\frac{\beta^3\sinh\alpha + \alpha^3\sin\beta}{\alpha\beta(\alpha^2+\beta^2)}\right] + \left[\frac{\cosh\alpha - \cos\beta}{\alpha^2+\beta^2}\right]F + \left[\frac{\beta\sinh\alpha - \alpha\sin\beta}{\alpha\beta(\alpha^2+\beta^2)}\right]F^2 + \left[\frac{\alpha^2+\beta^2-\beta^2\cosh\alpha-\alpha^2\cos\beta}{\alpha\beta(\alpha^2+\beta^2)}\right]F^d. \quad (A22)$$

The three special cases are, respectively,

$$F^{-1}(\exp F - 1) = 1 + \frac{1-\cos\beta}{\beta^2}F + \frac{\beta - \sin\beta}{\beta^3}F^2, \quad (A23)$$

$$F^{-1}(\exp F - 1) = 1 + \frac{\cosh\alpha - 1}{\alpha^2}F + \frac{\sinh\alpha - \alpha}{\alpha^3}F^2, \quad (A24)$$

$$F^{-1}(\exp F - 1) = 1 + \tfrac{1}{2}F + (1/3!)F^2. \quad (A25)$$

The required solutions of the equations of motion Eqs. (A2) and (A3) are obtained, respectively, from Eqs. (A16) and (A22), or their special cases, by the substitution $F \to (e/m)F\tau$. The anomalous additional precession induced by an anomalous magnetic moment and/or an electric-dipole moment is given by Eq. (A5). It is a "rotation"-type transformation and hence is obtained from Eq. (A17) by the substitution $F \to \Omega_1\tau$.