Compton Scattering of an Intense Photon Beam

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The fully quantum-mechanical description of the Compton scattering of a photon beam is shown to be equivalent under certain conditions to a semiclassical description, thereby confirming the prediction by Brown and Kibble and by Goldman of an intensity-dependent increment to the frequency shift. If the incident beam is in a coherent state, the equivalence is exact under all conditions. If it contains a definite number of photons, the equivalence is approximate, and requires many photons and convergence of the expansion in powers of the photon density. The demonstration is based upon the explicit use of wave packets to introduce the boundary conditions, and the equivalence is shown to hold in the sense that the transition probabilities, but not the transition amplitudes, are the same in the quantum and semiclassical treatments. The apparent failure of energy-momentum conservation implied by the incremental shift is seen to be accounted for by the energy-momentum uncertainty of the ensemble of localized wave packets even in the monochromatic limit. It is proved in the Appendix that the coherent states are the only kind for which the equivalence is exact.

I. INTRODUCTION

THE Compton scattering of an intense photon beam has been studied by several authors who have reached conflicting conclusions. Brown and Kibble[1] and Goldman[2] (BK, G), using a semiclassical approach without radiative reaction effects (the electron and the scattered photon are treated quantum-mechanically, and the incident beam is replaced by a classical electromagnetic field), have predicted an intensity-dependent increment to the frequency shift of the scattered photon. Fried and Eberly[3] (FE), using a quantized electromagnetic field but also neglecting radiative corrections, do not find the additional shift. Stehle and DeBaryshe,[4] on the basis of certain properties of the electron propagator (as defined in the presence of the photon beam), have been led to a different interpretation of the energy-momentum variable that appears in the semiclassical calculations. As a result, they conclude that there is no shift. The issue is resolved below with a proof that the semiclassical treatment is the proper one to use in that it is equivalent to a correct fully quantum-mechanical treatment; furthermore, it is shown by explicitly using wave packets to define the momentum of the free electron, that the interpretation which leads to the prediction of the incremental frequency shift is correct.

The difference between the BK, G, and FE treatments can be stated in terms of their methods of handling certain types of Feynman diagrams.[5] FE observe that, since they use a monochromatic beam, there are diagrams in which the electron returns one or more times to the mass shell in intermediate states. These diagrams are intensity-dependent analogs of the ordinary self-energy corrections to the free-electron line, and their treatment by FE is analogous to ordinary wave-function renormalization. They are discarded, and then corrected for by the introduction of an intensity-dependent wave-function normalization constant. In the approach of BK, G, on the other hand, these diagrams are retained. In the latter case, the incident beam must be described by a spread of frequencies so that the pole on the mass shell can be defined by an integration over the electron energy. The use of a frequency spread also allows the incident beam to be localized spatially, and therefore permits an introduction of the boundary conditions, viz., that the beam and the electron are isolated from one another in the remote past and future. FE, since their renormalization procedure requires quanta of a unique frequency, rely instead on the usual adiabatic switching of the coupling constant to introduce the boundary conditions.

The following comparison of the two treatments might now be made. The BK, G method takes proper account of the boundary conditions, but is not manifestly fully quantum mechanical. The FE method is fully quantum mechanical, but uses an artificial, and hence questionable device to introduce the boundary conditions.

It is the principal purpose of this paper to demonstrate that a completely quantum-mechanical description in which the correct boundary conditions are guaranteed by the explicit use of photon wave packets is equivalent to a semiclassical description, and therefore, if radiative corrections are ignored, to that of BK, G. Furthermore, it will be seen that in the original quantum-mechanical picture the energy-momentum variables of the free electron can be identified unambiguously. The identification can then be easily carried through to the semiclassical version, where it agrees with the interpretation of BK, G. It then follows that the intensity-dependent frequency shift is, in fact, predicted by quantum electrodynamics.

* Supported by the U. S. Army Research Office-Durham.
[1] L. S. Brown and T. W. B. Kibble, Phys. Rev. **133**, A705 (1964).
[2] I. I. Goldman, Phys. Letters **8**, 103 (1964).
[3] Z. Fried and J. H. Eberly, Phys. Rev. **136**, B871 (1964).
[4] P. Stehle and P. G. DeBaryshe, University of Pittsburgh, Pittsburgh, Pennsylvania (unpublished).
[5] P. J. Redmond, Conference on Quantum Electrodynamics of High Intensity Photon Beams, Durham, 1964 (unpublished).

Kibble,[6] in a subsequent paper, has also studied the connection between the semiclassical and the fully quantum-mechanical pictures. He makes use of the coherent states, discussed by Glauber and others,[7–10]

$$|C\rangle = e^{-N/2} \exp\left[N^{1/2} \int d\mathbf{k}\, C^{\mu}(\mathbf{k}) a_{\mu}^{\dagger}(\mathbf{k}) \right] |\text{vac}\rangle, \quad (1.1a)$$

$$\int d\mathbf{k}\, C_{\mu}^{*}(\mathbf{k}) C^{\mu}(\mathbf{k}) = 1, \quad (1.1b)$$

which are normalized eigenstates of the photon annihilation operators $a_{\mu}(\mathbf{k})$,

$$a_{\mu}(\mathbf{k})|C\rangle = |C\rangle N^{1/2} C^{\mu}(\mathbf{k}), \quad (1.2)$$

and therefore also of the positive-frequency part of the vector-potential field operator,

$$A_{\mu}^{(+)}(x) = (2\pi)^{-3/2} \int d\mathbf{k} (2\omega_k)^{-1/2} e^{-ik\cdot x} a_{\mu}(\mathbf{k}). \quad (1.3)$$

Briefly, he observes that if one calculates the S-matrix element $\langle f|S|i\rangle$ with the initial state $|i\rangle$ taken to be a coherent photon state plus an electron of momentum $\mathbf{p}$ and spin state λ,

$$|i\rangle = |\mathbf{p}\lambda, C\rangle, \quad (1.4)$$

and the final state $|f\rangle$ taken to be the same coherent photon state plus an electron of momentum $\mathbf{p}'$ and spin state λ' and a scattered photon of momentum $\mathbf{k}$ and polarization ϵ,

$$|f\rangle = |\mathbf{p}'\lambda', \mathbf{k}\epsilon, C\rangle, \quad (1.5)$$

then after the Wick expansion of the field operators into normal products[11] the property (1.2) of the coherent states permits the replacement of the vector potential field operator by the classical vector potential,

$$\tilde{A}_{\mu}(x) = (2\pi)^{-3/2} N^{1/2} \int d\mathbf{k} (2\omega_k)^{-1/2} \times [e^{-ik\cdot x} C_{\mu}(\mathbf{k}) + e^{ik\cdot x} C_{\mu}^{*}(\mathbf{k})], \quad (1.6)$$

at all but one of the external photon vertices. The result is

$$\langle \mathbf{p}'\lambda', \mathbf{k}\epsilon, C|S|\mathbf{p}\lambda, C\rangle = \langle \mathbf{p}'\lambda', \mathbf{k}\epsilon|S(\tilde{A})|\mathbf{p}\lambda\rangle, \quad (1.7)$$

where $S(\tilde{A})$ is the S matrix in the presence of the external classical field $\tilde{A}$. The right-hand side of (1.7) is the desired semiclassical expression, and the use of the photon wave packets $\int d\mathbf{k}\, C^{\mu}(\mathbf{k}) a_{\mu}^{\dagger}(\mathbf{k})|\text{vac}\rangle$ guarantees the initial and final isolation of the photon beam from the electron in the fully quantum-mechanical expression on the left-hand side.

However, as a justification of the semiclassical description, the use of Eq. (1.7) is subject to the criticism that while $\langle f|S|i\rangle$ is well defined mathematically, its physical interpretation is not at all clear. The final state $|f\rangle$ is composed of wave packets, instead of being an eigenstate of the momentum. Therefore, $\langle f|S|i\rangle$ is the *overlap* of the state vector in the remote future with $|f\rangle$, rather than the *projection* of the state vector onto an eigenstate of some set of commuting Hermitian operators that correspond to observables. The projection onto an eigenstate has a well-known interpretation, according to the postulates of quantum mechanics, as the probability amplitude that a measurement of the observables will result in the corresponding eigenvalues.[12] The overlap integral, however, has no such clearly defined meaning.

In the present treatment the final state is always taken to be a momentum eigenstate, although the initial one is a "prepared" state composed of localized photon wave packets. It will be shown that it is nevertheless correct, under certain conditions, to use the semiclassical expression $\langle \mathbf{p}'\lambda', \mathbf{k}\epsilon|S(\tilde{A})|\mathbf{p}\lambda\rangle$. The conditions referred to depend on the specific nature of the incident beam, and we shall treat two cases of particular interest. In the first case, the beam contains an arbitrary but precise number N of quanta, all having wave packets of the same functional form $C_{\mu}(\mathbf{k})$. We shall show that if N is sufficiently large compared to unity, and if the local photon density [determined by N and the shape of the packet $C_{\mu}(\mathbf{k})$] is small enough to permit a sufficiently rapid convergence of the series expansion in $\tilde{A}$, then the transition *probability* obtained from $\langle \mathbf{p}'\lambda', \mathbf{k}\epsilon|S(\tilde{A})|\mathbf{p}\lambda\rangle$ is a satisfactory approximation to the exact transition probability. The approximation can be made arbitrarily accurate by increasing N while adjusting $C_{\mu}(\mathbf{k})$ to keep the local photon density fixed, i.e., by approaching the monochromatic limit at fixed intensity. It is of considerable interest to note that it is not the quantum mechanical and semiclassical transition *amplitudes* which become equated, but only the transition *probabilities*.

In the second case considered, the incident beam is described by a coherent state, such as (1.1). Again we find that the quantum-mechanical and semiclassical probabilities transition probabilities, but not the amplitudes, are to be equated. However, this case has the remarkable feature that the semiclassical description is *exactly* correct, regardless of the value of N, the mean number of quanta in the beam.[13] In the Appendix we

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[6] T. W. B. Kibble, Phys. Rev. **138**, B740 (1965).
[7] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).
[8] E. C. G. Sudershan, Phys. Rev. Letters **10**, 277 (1963).
[9] S. S. Schweber, J. Math. Phys. **3**, 831 (1962).
[10] $\int d\mathbf{k}$ is a volume integral in three-space, $\int dx$ is a volume integral in four-space, and our notation for the four-vector inner product is illustrated by $k\cdot x = k^{\mu}x_{\mu} = k^0x^0 - \mathbf{k}\cdot\mathbf{x}$. Also, we use units in which $\hbar = c = 1$.
[11] G. C. Wick, Phys. Rev. **80**, 268 (1950).
[12] J. Von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1955), Chap. III.
[13] See Refs. 7 and 8 for further discussions of the connection between quantum-mechanical coherent states and classical electrodynamics.

show that the coherent states are the *only* kind for which the semiclassical description is exactly correct.

II. DEFINITE NUMBER OF PHOTONS

In this section we shall consider the Compton scattering of a beam of precisely N photons, all of which have identical wave packets described by the normalized wave functions $C_\beta(\mathbf{l})$,

$$\int d\mathbf{l}\, C_\beta^*(\mathbf{l})C^\beta(\mathbf{l})=1, \tag{2.1}$$

where $\mathbf{l}$ is the wave vector and β is the polarization variable. The probability amplitude $\psi(K_M)$ for finding the system in the state K_M in the remote future, is in terms of the S matrix,

$$\psi(K_M)=\int d\mathbf{l}_1\cdots d\mathbf{l}_N \sum_{\beta_1}\cdots\sum_{\beta_N}\langle K_M|S|L_N\rangle \times C_{\beta_1}(\mathbf{l}_1)\cdots C_{\beta_N}(\mathbf{l}_N)(N!)^{-1/2}, \tag{2.2}$$

where K_M and L_N are momentum eigenstates of the system for M and N photons, respectively,

$$\begin{aligned}|K_M\rangle&=|\mathbf{p}'\lambda',\mathbf{k}_1\alpha_1,\cdots\mathbf{k}_M\alpha_M\rangle\\&=a_{\alpha_1}^\dagger(\mathbf{k}_1)\cdots a_{\alpha_M}^\dagger(\mathbf{k}_M)|\mathbf{p}'\lambda'\rangle,\\|L_N\rangle&=|\mathbf{p}\lambda,\mathbf{l}_1\beta_1,\cdots\mathbf{l}_N\beta_N\rangle\\&=a_{\beta_1}^\dagger(\mathbf{l}_1)\cdots a_{\beta_N}^\dagger(\mathbf{l}_N)|\mathbf{p}\lambda\rangle,\end{aligned} \tag{2.3}$$

and $\mathbf{p}\lambda$ and $\mathbf{p}'\lambda'$ are the initial and final momenta and spin states of the electron. Strictly speaking, the initial state of the electron should also be described by a wave packet, i.e., the replacement

$$|L_N\rangle\rightarrow\sum_\lambda\int d\mathbf{p}|\mathbf{p}\lambda,\mathbf{l}_1\beta_1,\cdots\mathbf{l}_N\beta_N\rangle C'(\mathbf{p}\lambda), \tag{2.4}$$

should be made. However, we shall not complicate our expressions by doing so explicitly, since the electron variables are not manipulated below, and (2.4) may be invoked, if desired, at the very end.

The S matrix is given by

$$S=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{\infty}dx_1\cdots dx_n P[H(x_1)\cdots H(x_n)], \tag{2.5}$$

where the Hamiltonian density is

$$H(x)=j(x)\cdot A(x). \tag{2.6}$$

$A(x)$ is the vector-potential field operator, whose positive and negative frequency parts are

$$A_\mu^{(+)}(x)=\int d\mathbf{k}B(\omega_k)e^{-ik\cdot x}a_\mu(\mathbf{k}), \tag{2.7a}$$

$$A_\mu^{(-)}(x)=\int d\mathbf{k}B(\omega_k)e^{ik\cdot x}a_\mu^\dagger(\mathbf{k}), \tag{2.7b}$$

$$B(\omega_k)\equiv(2\pi)^{-3/2}(2\omega_k)^{-1/2},\quad k^0\equiv\omega_k, \tag{2.7c}$$

with the creation and annihilation operators satisfying

$$[a_\mu(\mathbf{k}),a_\nu^\dagger(\mathbf{k}')]=\delta^{(3)}(\mathbf{k}-\mathbf{k}')\delta_{\mu\nu}. \tag{2.8}$$

When Wick's theorem[11] is applied to the expansion of S in $\langle K_M|S|L_N\rangle$, only terms with certain types of normal products will survive, viz., those with no more than N photon destruction operators $A_\mu^{(+)}(x)$, and no more than M photon creation operators $A_\mu^{(-)}(x)$. We consider here only cases for which $N\geq M$, i.e., only cases for which the number of outgoing photons equals the incident number, or for which frequency harmonics are generated by the absorption of more photons than are re-emitted. If the number of creation and annihilation operators in a given term is m and n, respectively, then the operators applied to $|L_N\rangle$ create a state with $N+m-n$ photons, so we have the relation

$$n=N-M+m. \tag{2.9}$$

In the Wick expansion of the operators comprising S into sums of products of contractions and normal products there is complete symmetry with respect to the integration four-variables in each order. In other words, if the vertices in each diagram are labeled by the x's that appear in the corresponding operators, then for any diagram having, for example, x_i and x_j appearing at a given pair of vertices, there is another diagram with the two labels interchanged. Since the x's are dummy integration variables, all of the topologically identical diagrams give identical mathematical contributions, so that they can all be grouped together and a unique labeling of the vertices assigned to their total contribution. Furthermore, all of the contributions from diagrams having the same number n of external absorption vertices and the same number m of external emission vertices can be grouped together and written in the form,

$$\int dx_1\cdots dx_m dy_1\cdots dy_n\sum_{\mu\nu}f(x_1\mu_1\cdots x_m\mu_m|y_1\nu_1\cdots y_n\nu_n) \times A_{\mu_1}{}^{(-)}(x_1)\cdots A_{\mu_m}{}^{(-)}(x_m)A_{\nu_1}{}^{(+)}(y_1)\cdots A_{\nu_n}{}^{(+)}(y_n), \tag{2.10}$$

where the external absorption vertices are labeled by the dummy variables $y\nu$, and the external emission vertices are labeled by the $x\mu$. The ordering of the $x\mu$ in the function f corresponds to some unique relative ordering of the external emission vertices in each diagram, and similarly for the $y\nu$ and the absorption vertices. For example, if Fig. 1 corresponds to $f(x_1\mu_1,x_2\mu_2|y_1\nu_1)$, then Fig. 2 corresponds to $f(x_2\mu_2,x_1\mu_1|y_1\nu_1)$. Vertices involving only virtual particles are not explicitly indicated, and the integrations over their variables are implicitly contained in f.

The expression for the matrix element $\langle K_M|S|L_N\rangle$

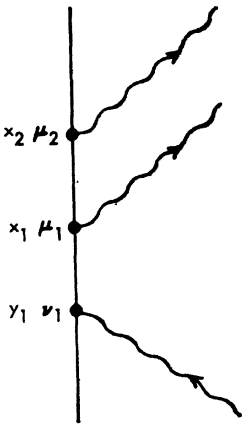

FIG. 1. Feynman diagram in which the relative ordering of the external photon emission vertices corresponds to the ordering of $x_1\mu_1$ and $x_2\mu_2$ in the function $f(x_1\mu_1,x_2\mu_2|y_1\nu_1)$.

appearing in (2.2) can now be written as[14]

$$\langle K_M|S|L_N\rangle=\sum_{m=0}^{M}\int dx_1\cdots dx_m dy_1\cdots dy_n$$
$$\times\langle K_M|f(x_1\cdots x_m|y_1\cdots y_n)$$
$$\times A^{(-)}(x_1)\cdots A^{(-)}(x_m)$$
$$\times A^{(+)}(y_1)\cdots A^{(+)}(y_n)|L_N\rangle, \quad (2.11)$$

with m and n related by (2.9). It then follows from a straightforward evaluation of the now normally ordered operators between $|K_M\rangle$ and $|L_N\rangle$ that the transition amplitude $\psi(K_M)$ can be reduced to

$$\psi(K_M)=\sum_{m=0}^{M}\int dx_1\cdots dy_n$$
$$\times\sum_{P(k)}\langle\mathbf{p}'|f(x_1\cdots x_m|y_1\cdots y_n)B(\omega_{k_1})e^{ik_1\cdot x_1}\cdots$$
$$\times B(\omega_{k_m})e^{ik_m\cdot x_m}\Omega^{(+)}(y_1)\cdots\Omega^{(+)}(y_n)|\mathbf{p}\rangle$$
$$\times C(\mathbf{k}_{m+1})\cdots C(\mathbf{k}_M)[(N!)^{1/2}/(N-n)!], \quad (2.12a)$$
$$n=N-M+m, \quad (2.12b)$$

where $\sum_{P(k)}$ means sum over all permutations of k_1 to k_M. We have defined the positive frequency part $\Omega^{(+)}(y)$ of a "classical vector potential" $\Omega(y)$ by

$$\Omega^{(+)}(y)=\int d\mathbf{k}\, B(\omega_k)e^{-ik\cdot y}C(\mathbf{k}). \quad (2.13a)$$

Later we shall also need the negative frequency part

$$\Omega^{(-)}(y)=\int d\mathbf{k}\, B(\omega_k)e^{ik\cdot y}C^*(\mathbf{k}). \quad (2.13b)$$

[14] In the remaining discussion, the polarization and spin variables are suppressed, so as to make the notation less cumbersome. This abbreviation does not alter the essentials of the argument.

Ω is not the classical vector potential $\tilde{A}$ that we are seeking. Because of the normalization,

$$\int d\mathbf{k}|C(\mathbf{k})|^2=1,$$

Ω describes an electromagnetic field whose total energy corresponds to that of only one of the incident quanta. It differs from $\tilde{A}$, according to Eq. (1.6), by an "intensity" factor $N^{1/2}$,

$$\tilde{A}=N^{1/2}\Omega. \quad (2.14)$$

Later it will be seen that the intensity factor $N^{1/2}$ emerges from the numerical terms like $(N!)^{1/2}/(N-n)!$ in (2.12), which express the indistinguishability of the photons. That is, these terms are consequences of the fact that there is an identical contribution to $\psi(K_M)$ from the amplitude for absorbing each of the N photons at any given external absorption vertex.

$\psi(K_M)$ has the appearance of an amplitude for n absorptions from the "classical field," m emissions of photons, and the direct transmission without scattering of the remaining $M-m=N-n$ incident photons. The field Ω has appeared at the absorption vertices because the corresponding vertex functions are weighted by the wave functions $C(\mathbf{k})$ of the incident photon wave packets. No such weight factors occur at the emission vertices, because the final state K_M is a single momentum eigenstate, not a weighted superposition of momentum eigenstates. Thus the quantum-mechanical amplitude is *not* equal to an equivalent semiclassical amplitude. However, the emission vertex operators will acquire weight factors $C^*(\mathbf{k})$ when the transition probability is formed. We shall see below that $\Omega^{(-)}$ appears as a consequence of quantum-mechanical interferences, i.e., because of the interference between (a) amplitudes in which one final-state photon has been emitted *into the beam*, while a second has been in the beam from the first and never interacted, and (b) amplitudes obtained by interchanging the variables of the two photons. The interference terms thus occur because of the indis-

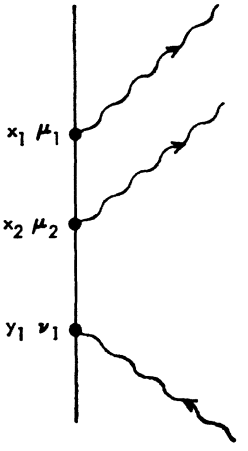

FIG. 2. Feynman diagram in which the labels of the external emission vertices have been interchanged relative to those of Fig. 1, so as to correspond to the ordering in $f(x_2\mu_2,x_1\mu_1|y_1\nu_1)$.

tinguishability of the unscattered photons from those scattered back into the beam. This indistinguishability will also be seen to result in a summation over many identical integrals, and therefore to produce a dependence on the number of incident photons, and, ultimately, the desired intensity factor $N^{1/2}$.

Now, let us evaluate the probability that precisely one photon scatters out of the beam. We first specify that the wave vectors of the incident packets occupy some volume Δ_B of wave vector space, i.e., $C(\mathbf{k})$ is nonzero if $\mathbf{k}$ is in the volume Δ_B, and is zero if $\mathbf{k}$ is outside of Δ_B. Note that if a photon's final-state wave vector is inside Δ_B, one cannot tell in principle whether it scattered back into the beam or did not scatter at all. Next, we specify a larger region Δ_F, which *includes* Δ_B. This is the region of small-angle or *laboratory* forward scattering, where we cannot distinguish an unscattered from a forward scattered photon because of practical limitations. Finally, we specify a region Δ_S outside of Δ_F, where the final-state wave vector of an observably scattered photon may be found.

We denote by $P_r(N)$ the probability that of the N incident photons one will be found in the scattered region Δ_S, and $N-1-r$ will be found in the forward region Δ_F. Therefore we have

$$r=N-M\,, \tag{2.15}$$

and r is the order of the approximate frequency harmonic generated in a process in which r fewer photons come out than go in. The probability is given by[15]

$$P_r(N)=\frac{1}{(M-1)!}\int_{\Delta_S}d\mathbf{k}_1\int_{\Delta_F}d\mathbf{k}_2\cdots\int_{\Delta_F}d\mathbf{k}_M|\psi(K_M)|^2. \tag{2.16}$$

The factor $1/(M-1)!$ has the following origin. Since the photons are indistinguishable, it does not matter which wave vector is integrated over Δ_S and which ones are integrated over Δ_F. We merely ask for the probability of a state in which *some* wave vector is in Δ_S and the rest are in Δ_F. However, by the same token, in integrating $\mathbf{k}_2$ to $\mathbf{k}_M$ all over the same region Δ_F, we are counting each state $(M-1)!$ times.

In performing the integrations indicated in (2.16), we are concerned only with the final-state variables k_1 to k_M, so we shall abbreviate the notation and simplify the accounting by defining

$$W(k_1\cdots k_m)=\sum_{P(k)}\int dx_1\cdots dy_n\langle\mathbf{p}'|f(x_1\cdots x_m|y_1\cdots y_n)B(\omega_{k_1})e^{ik_1\cdot x_1}\cdots B(\omega_{k_m})e^{ik_m\cdot x_m}\Omega^{(+)}(y_1)\cdots\Omega^{(+)}(y_n)|\mathbf{p}\rangle. \tag{2.17}$$

This definition permits $P_r(N)$ to be written as

$$P_r(N)=\frac{N!}{(M-1)!}\int_{\Delta_S}d\mathbf{k}_1\int_{\Delta_F}d\mathbf{k}_2\cdots\int_{\Delta_F}d\mathbf{k}_M|\sum_{m=0}^{M}\sum_{G(k)}W(k_1\cdots k_m)C(\mathbf{k}_{m+1})\cdots C(\mathbf{k}_M)|^2, \tag{2.18}$$

where $\sum_{G(k)}$ means sum over all groupings (combinations) of the k's into "scattered" and "unscattered" variables, a scattered variable being one that appears in a W function and an unscattered variable being one of the complementary set that appears in a C function. Note that of these groupings only those for which k_1 is a scattered variable will contribute, since $C(\mathbf{k}_1)$ vanishes in Δ_S.

Now, the acquisition of the weight factors $C^*(\mathbf{k})$ by the emission vertex functions occurs in the following step. Consider any product term resulting from the squaring operation in (2.18). It will contain one W function and one W^* function, as well as a number of wave functions C and complex conjugate wave functions C^*. If any variable is common to W and W^*, there is no wave function containing it in the product term. If any variable does not appear in either W or W^*, it appears in a factor $|C|^2$ in the product term. Furthermore, if a variable appears in W^* but not W, there is a factor C containing it, and similarly if it appears in W but not W^*, there is a factor C^* containing it. A typical product term contributing to (2.18) with j variables common to W and W^* is then

$$\begin{aligned}\frac{N!}{(M-1)!}&\int_{\Delta_S}d\mathbf{k}_1\int_{\Delta_F}d\mathbf{k}_2\cdots\int_{\Delta_F}d\mathbf{k}_M W(k_1\cdots k_j,k_{j+1}\cdots k_{j+a})C(\mathbf{k}_{j+a+1})\cdots\\ &\times C(\mathbf{k}_{j+a+b})W^*(k_1\cdots k_j,k_{j+a+1}\cdots k_{j+a+b})C^*(\mathbf{k}_{j+1})\cdots C^*(\mathbf{k}_{j+a})|C(\mathbf{k}_{j+a+b+1})|^2\cdots|C(\mathbf{k}_M)|^2\\ &=\frac{N!}{(M-1)!}\int_{\Delta_S}d\mathbf{k}_1\int_{\Delta_F}d\mathbf{k}_2\cdots\int_{\Delta_F}d\mathbf{k}_j(a!)U_a(k_1\cdots k_j)(b!)U_b^*(k_1\cdots k_j),\end{aligned} \tag{2.19}$$

[15] There is also, of course, an integration over the final electron momentum; for simplicity we shall not make it explicit.

where we have defined

$$U_a(k_1\cdots k_j)=\left(\frac{1}{a!}\right)\int_{\Delta_F} d\mathbf{k}_{j+1}\cdots\int_{\Delta_F} d\mathbf{k}_{j+a}C^*(\mathbf{k}_{j+1})\cdots C^*(\mathbf{k}_{j+a})W(k_1\cdots k_{j+a}). \tag{2.20}$$

In a moment we shall interpret $U_a(k_1\cdots k_j)$, but first we note that because of the symmetry of the quantities in the integrand above, and because the regions of integration for k_2 to k_M are identical, there are many identical multiple integrals of the form (2.19) contributing to $P_r(N)$. The number of such identical terms is equal to the number of combinations of $M-1$ variables taken in a common "scattered" group of $j-1$, two different "scattered" groups of a and b, respectively, and a common "unscattered" group of $N-j-a-b$, viz.,

$$(M-1)!/[(M-j-a-b)!(j-1)!a!b!].$$

As a result we have

$$P_r(N)=\sum_{j=1}\frac{1}{(j-1)!}\sum_{a=0}\sum_{b=0}\frac{N!}{(N-r-j-a-b)!}\int_{\Delta_S} d\mathbf{k}_1\int_{\Delta_F} d\mathbf{k}_2\cdots\int_{\Delta_F} d\mathbf{k}_j U_a(k_1\cdots k_j)U_b^*(k_1\cdots k_j). \tag{2.21}$$

Now let us look at $U_a(k_1\cdots k_j)$. In its present form it is not readily interpretable, so we first note that by using the definition (2.17) of W, one can manipulate the expression (2.20) into the form

$$U_a(k_1\cdots k_j)=\int dx_1\cdots dy_n\langle\mathbf{p}',\mathbf{k}_1\cdots\mathbf{k}_j|\,f_j(x_1\cdots x_{j+a}|y_1\cdots y_n)$$
$$\times A^{(-)}(x_1)\cdots A^{(-)}(x_j)\Omega^{(-)}(x_{j+1})\cdots\Omega^{(-)}(x_{j+a})\Omega^{(+)}(y_1)\cdots\Omega^{(+)}(y_n)|\,\mathbf{p}\rangle, \tag{2.22}$$
$$n=N-M+j+a=r+j+a.$$

The function f_j is simply a summation of f functions with the ordering of the variables specified in a particular way,

$$f_j(x_1\cdots x_{j+a}|y_1\cdots y_n)=\sum_{A(j)} f(x_1\cdots x_{j+a}|y_1\cdots y_n), \tag{2.23}$$

where $\sum_{A(j)}$ means sum over all arrangements of x_1 to x_j among the emission vertices, but maintain the sequential ordering 1 to j, and also maintain the sequential ordering $j+1$ to $j+a$ among the variables x_{j+1} to x_{j+a}.

In view of the definition of the f functions associated with expression (2.10), one can now interpret U_a in terms of an S matrix $S(\Omega)$ with an interaction Hamiltonian density $H(x)$ involving the "classical field" $\Omega(x)$ in addition to the quantum field $A(x)$,

$$H(x)=j(x)\cdot[A(x)+\Omega(x)]. \tag{2.24}$$

$U_a(k_1\cdots k_j)$ is the sum of contributions to $\langle\mathbf{p}',\mathbf{k}_1\cdots\mathbf{k}_j|\times S(\Omega)|\,\mathbf{p}\rangle$ from all diagrams with $n=r+j+a$ vertices at which the positive frequency part of the "classical field" interacts, a vertices at which the negative frequency part of the "classical field" interacts, and j vertices at which photons are emitted.

This interpretation, however, is not yet the anticipated one in terms of $S(\tilde{A})$. Furthermore, the expression (2.21) for $P_r(N)$ is not in the form of the magnitude squared of a transition amplitude. Equation (2.21) is exact; to obtain the desired result we must now make some assumptions and approximations. First we assume that the *number* of incident photons N is sufficiently large, and the *density* of incident photons (as determined by the wave packet volume) is sufficiently small that one may neglect those terms in (2.21) for which $r+j+a+b$ is comparable to N. In other words, we assume that the series converges rapidly when the terms are ordered according to the number of vertices involving the "classical field" Ω. This assumption permits us to make the approximation

$$N!/[(N-r-j-a-b)!]\cong N^{r+j+a+b}, \tag{2.25}$$

which in turn suggests defining the quantity

$$V_a(k_1\cdots k_j)=N^{(r+j+2a)/2}U_a(k_1\cdots k_j). \tag{2.26}$$

The number of factors of $N^{1/2}$ in V_a is equal to the number of vertices involving Ω in U_a, so that $V_a(k_1\cdots k_j)$ is defined by (2.22) with $\Omega^{(+)}$ replaced by $\tilde{A}^{(+)}=N^{1/2}\Omega^{(+)}$ and $\Omega^{(-)}$ replaced by $\tilde{A}^{(-)}=N^{1/2}\Omega^{(-)}$.

By virtue of our assumption of "sufficiently" rapid convergence we may make the additional approximation of extending the summation limits in (2.21) to infinity and write

$$P_r(N)=\sum_{j=1}^{\infty}\frac{1}{(j-1)!}\int_{\Delta_S} d\mathbf{k}_1\int_{\Delta_F} d\mathbf{k}_2\cdots$$
$$\times\int_{\Delta_F} d\mathbf{k}_j|\sum_n V_n(k_1\cdots k_j)|^2, \tag{2.27}$$

which has, finally, the anticipated form of the magni-

tude squared of a transition amplitude. The quantity $\sum_n V_n(k_1\cdots k_j)$ is the sum of contributions to $\langle \mathbf{p}',\mathbf{k}_1\cdots\mathbf{k}_j|S(\tilde{A})|\mathbf{p}\rangle$ from all diagrams in which the number of classical absorption (positive frequency) vertices exceeds the number of classical emission (negative frequency) vertices by $j+r$. Let us now make the assumption that the momentum spread Δk_B of the incident beam is small compared to the mean momentum k_B [both of these are defined by $C(\mathbf{k})$], i.e., that the beam is highly monochromatic, and let us take Δ_F only slightly bigger than Δ_B. Then energy-momentum can be conserved for these diagrams only if Δ_S occupies some small range in the vicinity of the rth harmonic of k_B. Furthermore, in this same range, which we denote by $\Delta_S(r)$, no other diagrams can contribute because all others fail to conserve energy-momentum except in the vicinity of some other harmonic. Therefore, for a highly monochromatic beam we may set Δ_S equal to $\Delta_S(r)$, and write[16]

$$P_r(N)=\sum_{j=1}^{\infty}\frac{1}{(j-1)!}\int_{\Delta_S(r)}d\mathbf{k}_1\int_{\Delta_F}d\mathbf{k}_2\cdots \times\int_{\Delta_F}d\mathbf{k}_j|\langle \mathbf{p}',\mathbf{k}_1\cdots\mathbf{k}_j|S(\tilde{A})|\mathbf{p}\rangle|^2. \quad (2.28)$$

Now, provided the various $\Delta_S(r)$ do not overlap, which of course they do not in the monochromatic limit, we may take a Δ_S which is a sum over all of the $\Delta_S(r)$, and evaluate the total probability of scattering, regardless of the harmonic generated,

$$P(N)=\sum_r P_r(N), \quad (2.29)$$

which is immediately seen to be

$$P(N)=\sum_{j=1}^{\infty}\frac{1}{(j-1)!}\int_{\Delta_S}d\mathbf{k}_1\int_{\Delta_F}d\mathbf{k}_2\cdots \times\int_{\Delta_F}d\mathbf{k}_j|\langle \mathbf{p}',\mathbf{k}_1\cdots\mathbf{k}_j|S(\tilde{A})|\mathbf{p}\rangle|^2. \quad (2.30)$$

$P(N)$ is the probability that of the N incident photons one will be found in Δ_S and all others will be found in Δ_F, with no restrictions on how many others there are. It has the anticipated form of a semiclassical description. If radiative corrections are neglected, and the summation is limited to $j=1$ (that is, if only the first perturbation term in powers of the electronic charge is retained), it reduces to the description used by Brown and Kibble, and Goldman.[1,2]

III. COHERENT STATE

In this section, instead of describing the incident beam by a state vector with a precise number of photons, we shall describe it by a coherent state $|C\rangle$, as given by (1.1). The state vector $|C\rangle$ is normalized,[17]

$$\langle C|C\rangle=1, \quad (3.1)$$

and corresponds to a *mean* number N of incident photons. That is, according to (1.2) the expectation value of the photon number operator is

$$\langle N_{\rm op}\rangle=\langle C|\int d\mathbf{k}\,a^\dagger(\mathbf{k})a(\mathbf{k})|C\rangle =N\int d\mathbf{k}|C(\mathbf{k})|^2=N. \quad (3.2)$$

In addition to the photon beam, the system contains an electron of initial momentum $\mathbf{p}$, so we take for our initial state

$$|\mathbf{p},C\rangle=e^{-N/2}\exp\left[N^{1/2}\int d\mathbf{k}\,C(\mathbf{k})a^\dagger(\mathbf{k})\right]|\mathbf{p}\rangle. \quad (3.3)$$

By expanding the exponential one can immediately see that $|\mathbf{p},C\rangle$ is a weighted superposition of states with all possible numbers of identical incident photon wave packets,

$$|\mathbf{p},C\rangle=e^{-N/2}\sum_n(1/n!)\int d\mathbf{k}_1\cdots d\mathbf{k}_nN^{1/2}C(\mathbf{k}_1)\cdots \times N^{1/2}C(\mathbf{k}_n)|\mathbf{p},\mathbf{k}_1\cdots\mathbf{k}_n\rangle. \quad (3.4)$$

Each state in the superposition is of the type used for the initial state in the previous section. We take the same final momentum eigenstate as before,

$$|K_M\rangle=|\mathbf{p}',\mathbf{k}_1\cdots\mathbf{k}_M\rangle, \quad (3.5)$$

and again evaluate the probability amplitude $\psi(K_M)$ for finding the system in the state K_M,

$$\psi(K_M)=\langle K_M|S|\mathbf{p},C\rangle. \quad (3.6)$$

As before, this can be expressed in terms of normally ordered products of field operators,

$$\psi(K_M)=\langle K_M|\sum_{mn}\int dx_1\cdots dx_m dy_1\cdots dy_n f(x_1\cdots x_m|y_1\cdots y_n)A^{(-)}(x_1)\cdots A^{(-)}(x_m)A^{(+)}(y_1)\cdots A^{(+)}(y_n)|\mathbf{p},C\rangle, \quad (3.7)$$

[16] For nonzero Δk_B these energy-momentum conservation arguments are true only out to some finite order in the number of vertices involving the classical field. However, the order in which the approximation fails can be made arbitrarily large by taking $\Delta k_B/k_B$ arbitrarily small, i.e., by taking the wave packets to be sufficiently monochromatic. The approximation is then consistent with our previous ones based on the assumptions of large photon number but limited photon density.

[17] This follows from the identity $e^{\mu a}e^{\lambda a^\dagger}=e^{\lambda a^\dagger}e^{\mu a}e^{\lambda\mu}$, which holds if $[a,a^\dagger]=1$.

but this time there is no limit to the number of n of possible absorption vertices. By using the property

$$A^{(+)}(y)|C\rangle=|C\rangle\tilde{A}^{(+)}(y), \tag{3.8}$$

which follows directly from (1.2), one can reduce the expression for $\psi(K_M)$ to

$$\psi(K_M)=\sum_{m=0}^{M}\sum_{n=0}^{\infty}\int dx_1\cdots dy_n \sum_{P(k)}\langle \mathbf{p}'|f(x_1\cdots x_m|y_1\cdots y_n)B(\omega_{k_1})e^{ik_1\cdot x_1}\cdots$$
$$\times B(\omega_{k_m})e^{ik_m\cdot x_m}\tilde{A}^{(+)}(y_1)\cdots\tilde{A}^{(+)}(y_n)|\mathbf{p}\rangle C(\mathbf{k}_{m+1})\cdots C(\mathbf{k}_M)e^{-N/2}N^{(M-m)/2}/(M-m)!. \tag{3.9}$$

Again it is only the absorption vertices that involve the classical field, so that the quantum-mechanical transition amplitude is again not equivalent to a semiclassical amplitude.

The remainder of the procedure follows very much along the lines of Sec. II. We must evaluate the probability $P(M,N)$ that a total of M photons will be observed, of which one will be in the scattered region Δ_S, and $M-1$ will be in the forward region Δ_F,

$$P(M,N)=\frac{1}{(M-1)!}\int_{\Delta_S}d\mathbf{k}_1\int_{\Delta_F}d\mathbf{k}_2\cdots\int_{\Delta_F}d\mathbf{k}_M|\psi(K_M)|^2. \tag{3.10}$$

After the squaring operation is performed, and the identical terms grouped, we find

$$P(M,N)=e^{-N}\sum_{j=1}\frac{1}{(j-1)!}\sum_{a=0}\sum_{b=0}\frac{N^{M-j-a-b}}{(M-j-a-b)!}$$
$$\times\int_{\Delta_S}d\mathbf{k}_1\int_{\Delta_F}d\mathbf{k}_2\cdots\int_{\Delta_F}d\mathbf{k}_j[\sum_{n=0}^{\infty}U_{an}(k_1\cdots k_j)][\sum_{n'=0}^{\infty}U_{bn'}{}^*(k_1\cdots k_j)], \tag{3.11}$$

where

$$U_{an}(k_1\cdots k_j)=\int dx_1\cdots dx_{j+a}dy_1\cdots dy_n\langle \mathbf{p}',\mathbf{k}_1\cdots\mathbf{k}_j|f_j(x_1\cdots x_{j+a}|y_1\cdots y_n)$$
$$\times A^{(-)}(x_1)\cdots A^{(-)}(x_j)\tilde{A}^{(-)}(x_{j+1})\cdots\tilde{A}^{(-)}(x_{j+a})\tilde{A}^{(+)}(y_1)\cdots\tilde{A}^{(+)}(y_n)|\mathbf{p}\rangle, \tag{3.12}$$

and the function f_j is again defined by (2.23). The quantity $U_{an}(k_1\cdots k_j)$ is readily interpretable in terms of $S(\tilde{A})$. It is the sum of contributions to $\langle \mathbf{p}',\mathbf{k}_1\cdots\mathbf{k}_j|\times S(\tilde{A})|\mathbf{p}\rangle$ from all diagrams with n classical absorption (positive frequency) vertices, a classical emission (negative frequency) vertices, and j photon emission vertices. The existence of summations in (3.11) ranging over all possible numbers of absorption vertices is a direct consequence of the fact that $|\mathbf{p},C\rangle$ has amplitudes for all possible numbers of photons.

Finally, we define

$$P(N)=\sum_{M=1}^{\infty}P(M,N), \tag{3.13}$$

which is the probability of finding one photon in Δ_S and all the rest in Δ_F, with no restrictions on the number found in Δ_F. It follows immediately from (3.11) that

$$P(N)=\sum_{j=1}\frac{1}{(j-1)!}\int_{\Delta_S}d\mathbf{k}_1\int_{\Delta_F}d\mathbf{k}_2\cdots$$
$$\times\int_{\Delta_F}d\mathbf{k}_j|\sum_{a,n}U_{an}(k_1\cdots k_j)|^2. \tag{3.14}$$

Since the summation now ranges over *all* possible numbers of absorption *and* emission vertices, $P(N)$ is seen to be *exactly* equivalent to the semiclassical description,

$$P(N)=\sum_{j=1}\frac{1}{(j-1)!}\int_{\Delta_S}d\mathbf{k}_1\int_{\Delta_F}d\mathbf{k}_2\cdots$$
$$\times\int_{\Delta_F}d\mathbf{k}_j|\langle \mathbf{p}',\mathbf{k}_1\cdots\mathbf{k}_j|S(\tilde{A})|\mathbf{p}\rangle|^2. \tag{3.15}$$

Equation (3.15) is an exact equality, independent of the mean number N of photons in the beam, and independent of the shape $C(\mathbf{k})$ of the photon wave packets.

IV. DISCUSSION

Brown and Kibble, and Goldman calculate the amplitude for emission of a photon to lowest order in the quantized electromagnetic field. For the initial and final states of the electron they use solutions of the electron wave equation in the presence of the classical electromagnetic field $\tilde{A}$. Kibble, in his subsequent paper, has shown that the amplitude obtained in this way is

identical to that obtained by summing the diagrams of $\langle \mathbf{p}',\mathbf{k}|S(\tilde{A})|\mathbf{p}\rangle$, provided only that radiative corrections are ignored. This completes the connection between the treatments of BK, G and the fully quantum-mechanical picture, except for a comment on the interpretation of the electron-momentum variables. It follows from Kibble's work that the quantities taken by BK, G to be the initial and final momenta of the free electron $\mathbf{p}$ and $\mathbf{p}'$, respectively, are the same quantities that appear in the matrix element $\langle \mathbf{p}',\mathbf{k}|S(\tilde{A})|\mathbf{p}\rangle$. It is now abundantly clear from the original quantum-mechanical expressions (2.2) and (3.6), in which the incident beam is initially isolated from the electron by the use of photon wave packets, that these are indeed the free-electron momenta. To emphasize the initial separation, one may also describe the incident electron with a wave packet $C'(\mathbf{p})$, as in (2.4). The effect will be to replace the above matrix element by $\int d\mathbf{p}\langle \mathbf{p}',\mathbf{k}|S(\tilde{A})|\mathbf{p}\rangle C'(\mathbf{p})$ and, consequently, to replace the BK, G initial-state wave function, corresponding to a unique electron momentum, by a similarly weighted superposition. However, if $C'(\mathbf{p})$ is sharply spiked, the calculation remains unchanged.

On the basis of the fully quantum-mechanical picture one may easily resolve the paradox concerning energy-momentum conservation, which appears to exist in the monochromatic limit. The problem is that since the Compton shift is a direct consequence of energy-momentum conservation in the scattering of a photon from an electron, then any additional shift would appear to be in conflict with the conservation laws, regardless of the mere *presence* of other photons. The resolution is simply based on the fact that each incident photon has an energy-momentum spread Δk_B, corresponding to the volume in momentum (or wave vector) space Δ_B spanned by the wave packet $C(\mathbf{k})$.

A photon which "forward scatters" does so, then, only in the sense that its initial and final momenta are both within Δ_B. These momenta are not, in general, identical. Nevertheless, one cannot tell by observation, even *in principle*, whether or not such a photon has "physically" scattered, since its measured final momentum is within the range of uncertainty of its initial momentum. Although true forward-scattering diagrams (those in which the initial and final momenta of a scattered photon are identical) do exist, their contributions to the transition probability have measure zero, since they correspond simply to points in the volume integrals over initial and final momenta. Therefore, the photons that interact but do not leave the beam may, nevertheless, partake in the overall energy-momentum conservation of the *system*. If the incident beam contains a definite number N of photons, then there are transition amplitudes for which the latitude on energy-momentum conservation (by the electron and the photon scattered out of the beam) is as much as

$$(\delta k)_U = N\Delta k_B = N(2\pi/L), \tag{4.1}$$

where L is the linear dimension of the wave packet. Since the *sum* of the energy-momentum uncertainties of the photons in the beam[18] $(\delta k)_U$ depends on the number of photons per unit length rather than on the linear dimension L alone, it evidently remains nonzero even as the monochromatic limit is approached. We may verify that $(\delta k)_U$ is actually large enough by comparing it with the magnitude of the intensity-dependent shift $(\delta k)_S$ predicted for a plane-wave beam,[1]

$$(\delta k)_S = \frac{(\alpha\rho\lambda_C^2/\pi)\sin^2(\frac{1}{2}\theta)}{[1+(2k_B/m)\sin^2(\frac{1}{2}\theta)][1+(2k_B/m+\alpha\rho\lambda_C^2/\pi k_B)\sin^2(\frac{1}{2}\theta)]}, \tag{4.2}$$

where α is the fine-structure constant, ρ is the photon density, λ_C is the electron Compton wavelength, m is the electron mass, and θ is the scattering angle. First we may rewrite $(\delta k)_U$ as

$$(\delta k)_U = 2\pi A\rho, \tag{4.3}$$

where A is the area of the beam in the plane transverse to the propagation direction, and then note that for the plane-wave approximation to be applicable the area A must be much greater than the effective cross sectional area seen by the electron. In this case the latter may be well represented by the Thomson cross section $\sigma_T = (8\pi/3)r_0^2 = (8\pi/3)\alpha^2\lambda_C^2$, so that the inequality

$$(\delta k)_U \gg (16\pi^2/3)\alpha^2\lambda_C^2\rho, \tag{4.4}$$

must hold, and it follows immediately that $(\delta k)_U/(\delta k)_S \gg 1$. Thus, there is ample energy-momentum uncertainty among the photons in the incident beam to allow for the predicted frequency shift, even though the monochromatic limit is approached.

The principal conclusion of this paper, viz., that under certain conditions the fully quantum-mechanical description is equivalent to a semiclassical description, may be thought of as a particular verification of the correspondence principle. Actually it is somewhat broader in the sense that the equivalence has been shown to hold for coherent beams regardless of the mean number of photons involved. On the other hand, it is somewhat more restricted in the following sense. For the scattering of a coherent beam, the Brown and Kibble

[18] $(\delta k)_U$ is not the same as what is normally called the energy-momentum uncertainty of the ensemble of photons comprising the beam. The latter is equal to $N^{1/2}\Delta k_B$ by virtue of the quite general relation that the standard deviation of a sum of N independent but identically distributed stochastic variables is equal to $N^{1/2}$ times the standard deviation of a single variable.

closed form results may be used to provide an analytic continuation from the domain in which the series expansion in powers of the photon density converges, to the domain of arbitrarily high photon densities. However, one should recall that for the scattering of a beam with a definite number of photons the closed form solution is only an alternative representation of the convergent, infinite series, which *approximates* the true finite series (2.21). It is by no means obvious that at ultra-high intensities, where the infinite series fails to converge, the closed form solution continues to constitute a satisfactory approximation to the true finite series. It would perhaps not be surprising if the correspondence principle were inapplicable in this latter case, since it is well known that an ensemble of photons of definite number is not classical, because of the uncertainty relation that exists between number and phase for bosons.[19]

V. CONCLUSIONS

It has been shown that under certain conditions the fully quantum-mechanical description of the Compton scattering of a photon beam is equivalent to a semiclassical description, i.e., one in which the incident beam is treated classically, but the electron, and all virtual and scattered photons are treated quantum mechanically. When taken to first order in the quantized electromagnetic field, the semiclassical description becomes that used by Brown and Kibble, and Goldman,[2] so that their prediction of an intensity-dependent increment to the Compton frequency shift is vindicated. The conditions under which the equivalence holds depend upon the nature of the incident beam. If it is in a coherent state, the equivalence is exact under all conditions. If it contains a definite number of photons, the equivalence is approximate, and requires that the total number of photons be much greater than one, and that the expansion in powers of the photon density converge. The demonstration of the equivalence is based upon the explicit use of wave packets to describe the photons and to guarantee the boundary conditions, viz., the initial and final isolation of the beam from the electron. The equivalence has been shown to hold in the sense that the transition probabilities, but not the transition amplitudes, are the same in the two treatments.

APPENDIX

The coherent states[7–9] can be shown to be the *only* kind for which the fully quantum-mechanical picture of Compton scattering is *exactly* equivalent to the semiclassical picture. To do so we take an arbitrary superposition of n-photon states to describe the incident beam, calculate the transition probability, and require that the latter be equal to the corresponding semiclassical expression. The conditions imposed on the weight functions of the n-photon states are such that the initial state must be coherent.

The incident beam is described by the state vector

$$\Phi=\sum_n g(n)\int d\mathbf{l}_1\cdots d\mathbf{l}_n C(\mathbf{l}_1)\cdots \times C(\mathbf{l}_n)\,|\,\mathbf{p},\mathbf{l}_1\cdots\mathbf{l}_n\rangle(n!)^{-1/2}, \quad \text{(A1)}$$

with the normalization conditions

$$\int d\mathbf{l}\,|C(\mathbf{l})|^2=\sum_n |g(n)|^2=1. \quad \text{(A2)}$$

The probability P of finding one photon in Δ_S and an arbitrary number in Δ_F can be evaluated as in the text, and found to be

$$P=\sum_{j=1}^{\infty}\frac{1}{(j-1)!}\sum_{abmn=0}^{\infty}\sum_{s=0}^{\infty}\frac{h(s-j-a+n)h^*(s-j-a+m)}{(s-j-a-b)!}N^{-(a+m+b+n)/2} \times\int_{\Delta_S}d\mathbf{k}_1\int_{\Delta_F}d\mathbf{k}_2\cdots\int_{\Delta_F}d\mathbf{k}_j U_{an}(k_1\cdots k_j)U_{bm}^*(k_1\cdots k_j), \quad \text{(A3)}$$

where $U_{an}(k_1\cdots k_j)$ is defined by Eq. (3.12), and

$$h(n)=(n!)^{1/2}g(n). \quad \text{(A4)}$$

The semiclassical probability expression is given by Eq. (3.14). In order for P to be equal to it to within a constant multiplicative factor we must have

$$N^{-(a+m+b+n)/2}\sum_{s=0}^{\infty}\frac{h(s-j-a+n)h^*(s-j-a+m)}{(s-j-a-b)!}=\text{const.}$$

The constant is immediately required to be unity by the normalization condition (A2). The equation may be simplified in appearance by defining

$$t=s-j-a-b,\quad \mu=n+b,\quad \nu=m+a, \quad \text{(A5)}$$

and

$$h(n)=e^{-N/2}N^{n/2}f(n), \quad \text{(A6)}$$

to obtain

$$\sum_{t=0}^{\infty}\frac{N^t}{t!}f(t+\mu)f^*(t+\nu)=e^N, \quad \text{(A7)}$$

where μ and ν are arbitrary non-negative integers.

[19] W. Heitler, *The Quantum Theory of Radiation* (Clarendon Press, Oxford, 1954), 3rd ed., Chap. 2, Sec. 7.

The solutions to Eq. (A7) determine the class of quantum states for which the equivalence of quantum and semiclassical descriptions obtains. An obvious solution is

$$f(t)=e^{i\phi}, \tag{A8}$$

or

$$g(n)=(n!)^{-1/2}N^{n/2}e^{-N/2}e^{i\phi}, \tag{A9}$$

which defines the coherent states, as can be seen by comparison with (3.4). We shall now show that this solution is unique. This can *not* be done by expanding the right-hand side of (A7) in a power series and equating the coefficients of N^t on both sides. The reason is that $f(t)$ may itself be a function of N. We proceed instead by rewriting (A7) as

$$\sum_{t=0}^{\infty}\frac{N^t}{t!}f(t+r+\mu)f^*(t+r+\nu)=e^N, \tag{A10}$$

where r, μ, and ν are arbitrary non-negative integers. Then we multiply both sides of the equation by $(-N)^r/r!$ and sum over r,

$$\sum_{r=0}^{\infty}\frac{(-N)^r}{r!}\sum_{t=0}^{\infty}\frac{N^t}{t!}f(t+r+\mu)f^*(t+r+\nu)=1. \tag{A11}$$

Next we make the change of variables $t=s-r$, and invert the order of summation,

$$\sum_{r=0}^{\infty}\sum_{s=r}^{\infty}=\sum_{s=0}^{\infty}\sum_{r=0}^{s}, \tag{A12}$$

to obtain

$$\sum_{s=0}^{\infty}N^s f(s+\mu)f^*(s+\nu)\sum_{r=0}^{s}\frac{(-1)^r}{r!(s-r)!}=1. \tag{A13}$$

The sum over r is unity for $s=0$, and it is the binomial expansion of $(1-1)^s/s!=0$ for $s>0$, so that

$$f(\mu)f^*(\nu)=1. \tag{A14}$$

Since μ and ν are arbitrary, the solution is $f(t)=e^{i\phi}$ with ϕ constant.

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Regge-Pole Models for High-Energy πN, KN, and $\bar{K}N$ Scattering*

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It is shown that πN, KN, and $\bar{K}N$ elastic-scattering and charge-exchange data at high energy and small momentum transfer can be well fitted by assuming that the amplitudes are dominated by a few Regge poles in the crossed channel. The constraints imposed by the factorization principle are included. Unitary symmetry (SU_3) is approximately satisfied. Sample predictions of πp polarization and $K^+ + n \to K^0 + p$ charge exchange are made.

1. INTRODUCTION

THIS paper shows that the present pion-nucleon and kaon-nucleon data, at high energy and small momentum transfer, are consistent with the dominance of a few Regge poles in the crossed channel. Explicit models are constructed which give good fits to the data in the range of incident momentum 6 to 20 GeV/c and squared momentum transfer $|t|<1$ (GeV/c)^2. Possible branch points in the complex angular-momentum plane are neglected. Mandelstam[1] has shown that such branch points are probably not negligible at asymptotic energies; however, there seems to be a good chance that over a considerable energy range—perhaps up to 100 GeV or more—their effects are not important.[2]

There have already been several Regge-pole models[3,4] (some including a cut[5,6]) for the pion-nucleon and kaon-nucleon systems. However, the authors have not included the helicity-flip terms, have largely ignored the question of isospin dependence, and have not at-

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