Relativistic Formulation of the $SU(6)$ Symmetry Scheme*

BUNJI SAKITA AND KAMESHWAR C. WALI
Argonne National Laboratory, Argonne, Illinois
(Received 22 March 1965)

A relativistic formulation of the $SU(6)$ symmetry scheme is presented, starting with the basic assumption that the fields corresponding to elementary particles are tensors of $M(12)$ [or $\tilde{U}(12)$ or $SU(12)_{\mathcal{L}}$]. In particular a mixed second-rank tensor and a totally symmetric third-rank tensor are associated with the meson and baryon fields, respectively. It is shown that if these fields are required to satisfy prescribed free-field equations of motion, then one is led to a particle supermultiplet structure which corresponds to the $35\oplus1$ and 56-dimensional representations of $SU(6)$ for the mesons and baryons. It is also shown that the spin-dependent and $SU(3)$-spin-dependent mass splittings can be included in the theory and that solutions in terms of physical particle fields can be obtained. Effective trilinear meson-meson and meson-baryon vertex functions, using these solutions and an interaction Lagrangian which is invariant under $M(12)$, are calculated in the lowest order perturbation. We would like to note especially the following results: (a) From the known pion-nucleon coupling constant, the width of the pion-nucleon (3,3) resonance is calculated to be 94 MeV. (b) The ratio of the magnetic form factors for the neutron and proton is $-\frac{2}{3}$ for all momentum transfers and $\mu_P=(1+2M_P/m_p)$ nuclear magnetons. (c) The charge form factor of the neutron is zero for all momentum transfers.

I. INTRODUCTION

THERE has been considerable interest recently in the $SU(6)$ symmetry scheme for elementary particles.[1] It is conceived as an extension of Wigner's nuclear-supermultiplet theory[2] to elementary-particle phenomena. Unlike other higher symmetry schemes,[3] the $SU(6)$ theory proposes to treat the ordinary spin on the same footing as the isotopic spin and hypercharge. Clearly such a formulation is possible only if the space-time variables and the spin variables are completely decoupled. This is possible only in a nonrelativistic theory as in the case of Wigner's supermultiplet theory. Since Lorentz transformations mix the intrinsic spin and the orbital angular momentum in an intricate manner, it is not obvious whether the $SU(6)$ theory can be extended to the relativistic domain. It is therefore not surprising that several attempts[4] have been made towards an understanding of this problem.

The problems connected with a relativistic formulation of the $SU(6)$ theory may be discussed from a purely mathematical point of view of finding an appropriate group of invariance. For this purpose, we recall that the irreducible representations of $SU(6)$ can be decomposed into irreducible representations of $SU(2)\otimes SU(3)$, where the $SU(2)$ can be identified as the ordinary spin group and the $SU(3)$ as the familiar internal symmetry group $SU(3)$. If the theory has to contain orbital angular momentum and spin mixed in a Lorentz-invariant manner, the spin groups $SU(2)$ has to be extended to $SL(2,C)$ which is the covering group of the restricted Lorentz group. A fully relativistic $SU(6)$ theory must include in addition to the homogeneous Lorentz transformations, space and time translations. The required group G therefore must contain $SU(6)$ and the Poincare group as subgroups in such a manner that the intersection of $SL(2,C)\otimes SU(3)$ and $SU(6)$ is $SU(2)\otimes SU(3)$. It has been shown[5] that G must then contain the group $SL(6,C)$. Now depending on how the translations are imbedded in the group, one obtains two types of structures for G: (i) G is given by $P'\otimes Q$, where P' is a group isomorphic to the physical Poincaré group and $Q\supset SL(6,C)$[6]; (ii) G is a semidirect product of T_{36} by $SL(6,C)$ where T_{36} is the group of translations in a 36-dimensional space.[7]

Once a group G is given, its unitary representations on Hilbert space provide a set of symmetry transformations on the physical states which are characterized by the bases of the representations. The basis of an irreducible representation gives a set of physical states which are commonly identified as the particles belonging to a

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* Work performed under the auspices of the U. S. Atomic Energy Commission.

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[6] L. Michel, Second Coral Gables Conference on Symmetry Principles at High Energy, January 1965 (to be published).

[7] L. Michel and B. Sakita (Ref. 5); W. Rühl (Ref. 4), T. Fulton and J. Wess (Ref. 4).

supermultiplet of the system. One can construct a unitary representation of G in case (i) by using Wigner's method for P'. Since G is a direct product of P' by Q, the basis of such a representation must be a tensor product of the basis of P' and the basis of the unitary representation of Q. But $Q \supset SL(6,C)$ which is noncompact. An irreducible unitary representation of Q is of infinite dimensions. This corresponds, therefore, physically to an infinite number of particles belonging to a supermultiplet.

In case (ii), G contains additional translations other than the usual four space-time ones. Clearly the physical interpretation of the extra translations is not easy. Further if one identifies the usual space-time translations with four of the translations in T_{36}, the physical mass is no longer invariant under $SL(6,C)$. It will change continuously under the transformations of $SL(6,C)$. Physically this corresponds to a continuous mass distribution for a given particle state. Since the physical world does not appear to admit either an infinite number of one-particle states for fixed four-momentum or a continuous mass distribution for a given particle state, we are forced to conclude that there is no physically interesting group of invariance which contains the Poincaré group and the $SU(6)$ group in a nontrivial way.[8]

A more physical description of the $SU(6)$ theory is provided by the quark model. A relativistic quark model can be constructed along the lines of the three-field Thirring model. The fundamental field in this model can be described by a 12-component spinor ψ_A. A pair of indices $i\alpha$ can be assigned to A, where i runs from 1 to 4 and can be identified as the Dirac spinor index. α takes the values 1 to 3 and corresponds to the $SU(3)$-spin index. If one decomposes a Dirac spinor field into two two-component spinors (Weyl decomposition), ψ decomposes into two six-component fields ϕ and χ. The fields ϕ and χ then provide vector and conjugate-vector representations of $SL(6,C)$ (Sec. II). As pointed out by several authors,[9] one can construct an interaction Lagrangian which is invariant under $SL(6,C)$. However, it is impossible to construct a free Lagrangian which is also invariant under $SL(6,C)$ without encountering the difficulties mentioned earlier in connection with the group of invariance. Without a free Lagrangian, the standard quantization procedures and the particle interpretation of the fields cannot be carried out.

In spite of this apparent difficulty, the quark model suggests an alternative approach. If we consider a model of noninteracting quarks and construct a free Lagrangian in terms of ψ which is invariant under $P\otimes SU(3)$ ($P\equiv$Poincaré group), we can obtain free field equations of motion (Dirac equation) and commutation relations. The solutions to these equations can be interpreted as particle states which form a basis of an irreducible representation of $SU(6)$ for a fixed momentum $\mathbf{q}$. This suggests the possibility that the basic fields are tensors of $SL(6,C)$ whereas the solutions to appropriate equations of motion for these fields give the desired particle-multiplet structure, even though the equations themselves are not covariant under $SL(6,C)$. The purpose of the present paper is to examine the possibility and consequences of such an approach.[10,11]

If one assumes that the elementary particles are the bound states of one or several quarks and antiquarks, the bound-state wave function (or field in a phenomenological Lagrangian theory) can be described as a product of the fundamental fields ψ's and $\bar{\psi}$'s. In the following discussion, however, it is not necessary to assume explicitly such a quark model. We shall only assume that the fields associated with the elementary particles transform like the products of ψ's and $\bar{\psi}$'s. In particular, the meson field is represented by a second rank mixed tensor $\Phi_A{}^B$ (144 components). A totally symmetric third rank tensor Ψ_{ABC} (364 components) is associated with the baryon field. These tensor representations of $SL(6,C)$ together with their properties under space reflections are discussed in Sec. II. It also contains the interaction Lagrangian which is assumed to be invariant under $SL(6,C)$ and space reflections. The interaction Lagrangian assumed in the present discussion is invariant under a larger group of transformations $M(12)$ [or $\tilde{U}(12)$ or $SU(12)_{\mathcal{L}}$].[12] Section III is devoted to the decomposition of Φ and Ψ into appropriate auxiliary fields and to the discussion of the symmetry properties with respect to the interchange of Dirac and $SU(3)$ spin indices. In Sec. IV, the free field equations of motion for the meson and baryon fields are given. It is shown that the meson field equations admit solutions which correspond to a nonet of 0^- and a nonet of 1^- mesons. The baryon field equations lead to solutions which correspond to a decuplet of $\frac{3}{2}^+$ and an octet of $\frac{1}{2}^+$ baryons. The desired mass splittings are introduced and the solutions for Φ and Ψ are obtained in terms of physical particle fields. These solutions are used to calculate effective vertex parts in the lowest order perturbation calculation in Sec. V. The relations between different coupling constants and some of their consequences are also discussed. Finally, the concluding section is devoted to a summary and the discussion of some of the difficulties of the theory.

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[12] K. Bardakci, J. M. Cornwall, P. G. O. Freund, and B. W. Lee, Phys. Rev. Letters **14**, 48 (1965); R. Delbourgo, A. Salam, and J. Strathdee (to be published).

II. ELEMENTARY FIELDS AND INTERACTIONS

Consider the group 6×6 complex matrices of determinant one, which we shall denote $SL(6,C)$. To every matrix A ($\det A=1$) belonging to this group, there corresponds a linear unimodular transformation $T(A)$ over a six-dimensional complex vector space. A vector ϕ in this space is a fundamental representation of $SL(6,C)$ and undergoes the transformation

$$\phi \to \phi' = A\phi. \tag{2.1}$$

Tensors in this vector space are called tensors of $SL(6,C)$. Let us also consider a conjugate vector space, in which a vector χ is transformed under $SL(6,C)$ as follows:

$$\chi \to \chi' = (A^{-1})^\dagger \chi. \tag{2.2}$$

Tensors in the latter vector space are called conjugate tensors of $SL(6,C)$. The scalar products $\phi^*\chi$ and $\chi^*\phi$ are invariant under the transformations of $SL(6,C)$.

Let these tensors be functions of a space-time point so that they can be regarded as tensor fields. One can then construct a twelve-component tensor field ψ from ϕ and χ:

$$\psi = \begin{pmatrix}\phi\\ \chi\end{pmatrix}. \tag{2.3}$$

ψ is then a 12-component tensor representation of $SL(6,C)$. If we want to identify ψ as the quark field mentioned in the previous section, we must identify the $SL(2,C)$ group of $SL(2,C)\otimes SU(3)$ as the covering group of the restricted Lorentz group[13] and the $SU(3)$ group as the internal symmetry $SU(3)$ group. The decomposition of $\psi_A (A=1, 2, \cdots, 12)$ into the representations of $SL(2,C)\otimes SU(3)$ can be done by assigning a pair of indices $i\alpha$ to A. The index i ($i=1, \cdots, 4$) is the Dirac spinor index and α ($\alpha=1, \cdots, 3$) is the $SU(3)$-spin index. Since to every i there corresponds a four-component spinor, the properties of ψ under spatial reflections can be easily included in the formalism in the usual fashion:

$$\psi \to \psi' = \eta\gamma_4\psi, \tag{2.4}$$

where η is a phase. We also note that if we use the explicit representations

$$\gamma_4 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \gamma_5 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{2.5}$$

then $\bar\psi\psi$ and $\bar\psi\gamma_5\psi$ are invariant under $SL(6,C)$, where

$$\bar\psi^A = \bar\psi^{i\alpha} = \psi_{i'\alpha}{}^*(\gamma_4)_{i'}{}^i. \tag{2.6}$$

As stated in the previous section, we assume that the fields corresponding to physical particles transform as products of ψ's and $\bar\psi$'s under the $SL(6,C)$ transformations. We consider, in particular, a second-rank mixed tensor-field $\Phi_A{}^B$ for the mesons and a totally symmetric third-rank tensor-field Ψ_{ABC} for the baryons. These fields are assumed to be local and their properties under space reflections are given by

$$\Phi \to \Phi' = \gamma_4 \otimes \tilde\gamma_4 \Phi, \tag{2.7}$$

$$\Psi \to \Psi' = \gamma_4 \otimes \gamma_4 \otimes \gamma_4 \Psi. \tag{2.8}$$

Written out explicitly, the above equations imply

$$\Phi_{i\alpha}{}'^{j\beta} = (\gamma_4)_i{}^{i'}\Phi_{i'\alpha}{}^{j'\beta}(\gamma_4)_{j'}{}^j \tag{2.9}$$

$$\Psi_{i\alpha,j\beta,k\gamma}{}' = (\gamma_4)_i{}^{i'}(\gamma_4)_j{}^{j'}(\gamma_4)_k{}^{k'}\Psi_{i'\alpha,j'\beta,k'\gamma}. \tag{2.10}$$

We construct the interaction Lagrangian $L_{\rm int}$ in terms of these fields by requiring that it be invariant under the $SL(6,C)$ transformations and also under the space reflections. If we restrict ourselves to trilinear meson-meson and meson-baryon interactions, $L_{\rm int}$ can be written in the form[14]

$$L_{\rm int} = \tfrac{1}{4}m_0 ig\, {\rm Tr}(\Phi\Phi\Phi) + iG\bar\Psi^{ADC}\Phi_A{}^B\Psi_{BDC}. \tag{2.11}$$

$L_{\rm int}$ in (2.11) is invariant under a larger group of transformations $M(12)$ [or $\tilde U(12)$],[12] which is the group of 12×12 complex matrices M satisfying the following condition:

$$M^\dagger[\gamma_4\otimes \mathbf{1}]M = \gamma_4 \otimes \mathbf{1}, \tag{2.12}$$

where $\mathbf{1}$ is a 3×3 unit matrix so that $[\gamma_4\otimes\mathbf{1}]$ is a 12×12 matrix and $M^\dagger$ is the Hermitian conjugate of M. $M(12)$ is a 144 parameter Lie group which contains $SL(6,C)$ in its subgroups. The quark spinor ψ is a fundamental representation of $M(12)$ and $\bar\psi$ is the contragradient representation so that $\bar\psi\psi$ is invariant. However $\bar\psi\gamma_5\psi$ which is invariant under $SL(6,C)$ is not invariant under $M(12)$.

III. DECOMPOSITION OF THE ELEMENTARY FIELDS

The meson field Φ consists of 144 components which are complex. In order to have antiparticles in the same multiplet as the particles, we must impose the reality condition:

$$\gamma_4 \otimes \tilde\gamma_4 \Phi = -\Phi^\dagger. \tag{3.1}$$

For fixed $SU(3)$ indices, Φ is a 4×4 matrix in the Dirac space. Consequently it can be expanded in terms of the sixteen independent Dirac matrices:

$$\Phi = 1\otimes\mathbf{S} + \gamma_\mu\otimes\mathbf{V}_\mu + \tfrac{1}{2}\sigma_{\mu\nu}\otimes\mathbf{T}_{\mu\nu} + \gamma_\mu\gamma_5\otimes\mathbf{A}_\mu + \gamma_5\otimes\mathbf{P}. \tag{3.2}$$

$\mathbf{S}$, $\mathbf{V}_\mu$, $\cdots$, $\mathbf{P}$ are 3×3 matrices in $SU(3)$-spin space.[15] Their properties under space-reflections can be easily

[13] R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and All That* (W. A. Benjamin, Inc., New York, 1964).

[14] Invariance under $SL(6,C)$ and space reflections permits also the interactions ${\rm Tr}(\Phi\gamma_5\Phi\gamma_5\Phi)$, $\bar\Psi(\gamma_5\Phi\gamma_5\otimes1\otimes1)\Psi$, $\bar\Psi(\gamma_5\Phi\otimes\gamma_5\otimes1)\Psi$ and $\bar\Psi(\Phi\otimes\gamma_5\otimes\gamma_5)\Psi$, which are not $M(12)$ invariant. However, for the present discussion we restrict our phenomenological $L_{\rm int}$ to the form given in (2.11).

[15] We use the bold-face letters for matrices acting in $SU(3)$-spin space.

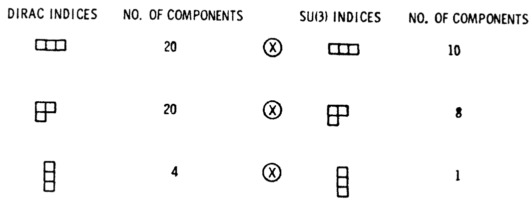

FIG. 1. Young diagrams corresponding to a totally symmetric third-rank tensor.

deduced from those of Φ defined in (2.7). They represent a nonet of scalar, vector, tensor, axial vector, and pseudoscalar fields, respectively.

The baryon field Ψ has 364 components, since it is a totally symmetric third rank tensor, i.e.,

$$\Psi_{i\alpha,j\beta,k\gamma}=\Psi_{j\beta,i\alpha,k\gamma}=\Psi_{i\alpha,k\gamma,j\beta}. \tag{3.3}$$

The condition (3.3) can be satisfied by suitably combining different symmetry properties of the Dirac and $SU(3)$ spin indices. It is convenient to describe the different possible combinations with the help of Young diagrams. The number of possible combinations is three and they are as shown in Fig. 1. The first line in Fig. 1 represents a set of decuplet fields which are completely symmetric with respect to the interchange of the $SU(3)$ indices as well as Dirac spinor indices. We denote these by $D[=D_{ijk,\alpha\beta\gamma}]$. The second line represents a set of octet fields each of which is a third-rank Dirac spinor with mixed symmetry properties. We denote these by $\chi[=\chi_{ijk,\alpha}{}^{\delta},\ \chi_{ijk,\alpha}{}^{\alpha}=0]$. The mixed symmetry implies

$$\chi_{ijk}=-\chi_{ikj}, \tag{3.4}$$

$$\chi_{ijk}+\chi_{jki}+\chi_{kij}=0. \tag{3.5}$$

Finally the last line represents a singlet field which is a completely antisymmetric third-rank Dirac spinor. We denote this by $G[=G_{ijk}]$ where

$$G_{ijk}=-G_{jik}=-G_{ikj}. \tag{3.6}$$

One can, therefore, decompose Ψ uniquely as follows:

$$\Psi_{i\alpha,j\beta,k\gamma}=D_{ijk,\alpha\beta\gamma}+B_{ijk,\alpha\beta\gamma}+\frac{1}{\sqrt{6}}\epsilon_{\alpha\beta\gamma}G_{ijk}, \tag{3.7}$$

where

$$B_{ijk,\alpha\beta\gamma}=\tfrac{1}{3}[\chi_{ijk,\alpha}{}^{\delta}\epsilon_{\delta\beta\gamma}+\chi_{jki,\beta}{}^{\delta}\epsilon_{\delta\gamma\alpha}+\chi_{kij,\gamma}{}^{\delta}\epsilon_{\delta\alpha\beta}]. \tag{3.8}$$

It is also easy to construct the projection operators $\mathbf{P}_D$, $\mathbf{P}_B$, and $\mathbf{P}_G$ by means of which we can obtain the fields D, B, and G from Ψ. To construct these operators, we note that

$$\tfrac{1}{2}\sum_{i=0}^{8}\boldsymbol{\lambda}_i\otimes\boldsymbol{\lambda}_i V=V^T, \tag{3.9}$$

where λ_i ($i=1\cdots 8$) are the 3×3 matrix representations of the $SU(3)$ generators[16] and $\boldsymbol{\lambda}_0=(\sqrt{\tfrac{2}{3}})\mathbf{1}$. V is a second rank tensor in $SU(3)$ space and V^T is its transpose. Then

$$\mathbf{P}_D=\tfrac{1}{6}\sum_{i=0}^{8}[\boldsymbol{\lambda}_i\otimes\boldsymbol{\lambda}_i\otimes\mathbf{1}+\mathbf{1}\otimes\boldsymbol{\lambda}_i\otimes\boldsymbol{\lambda}_i+\boldsymbol{\lambda}_i\otimes\mathbf{1}\otimes\boldsymbol{\lambda}_i]+\tfrac{1}{6}\boldsymbol{\varepsilon},$$
$$\mathbf{P}_G=\tfrac{1}{6}\boldsymbol{\varepsilon}, \tag{3.10}$$

and

$$\mathbf{P}_B=\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}-P_D-P_G, \tag{3.11}$$

where $\boldsymbol{\varepsilon}$ is given by

$$(\boldsymbol{\varepsilon})_{\alpha\beta\gamma}{}^{\alpha'\beta'\gamma'}=\epsilon_{\alpha\beta\gamma}\epsilon^{\alpha'\beta'\gamma'}. \tag{3.12}$$

To make our computations more explicit,

$$(\boldsymbol{\varepsilon})_{\alpha\beta\gamma}{}^{\alpha'\beta'\gamma'}D_{ijk,\alpha'\beta'\gamma'}=\epsilon_{\alpha\beta\gamma}\epsilon^{\alpha'\beta'\gamma'}D_{ijk,\alpha'\beta'\gamma'}=0,$$

since D is symmetric in α, β, γ. Also

$$(\boldsymbol{\varepsilon})_{\alpha\beta\gamma}{}^{\alpha'\beta'\gamma'}B_{ijk,\alpha'\beta'\gamma'}=\epsilon_{\alpha\beta\gamma}[\chi_{ijk,\alpha}{}^{\alpha}]=0$$

and

$$\tfrac{1}{6}(\boldsymbol{\varepsilon})_{\alpha\beta\gamma}{}^{\alpha'\beta'\gamma'}\epsilon_{\alpha'\beta'\gamma'}G_{ijk}=\epsilon_{\alpha\beta\gamma}G_{ijk}.$$

Consequently,

$$(\mathbf{P}_D)_{\alpha\beta\gamma}{}^{\alpha'\beta'\gamma'}D_{ijk,\alpha'\beta'\gamma'}=\tfrac{1}{3}[D_{ijk,\beta\alpha\gamma}+D_{ijk,\alpha\gamma\beta}+D_{ijk,\gamma\beta\alpha}]=D_{ijk,\alpha\beta\gamma},$$

$$(\mathbf{P}_D)_{\alpha\beta\gamma}{}^{\alpha'\beta'\gamma'}B_{ijk,\alpha'\beta'\gamma'}=\tfrac{1}{3}[B_{ijk,\beta\alpha\gamma}+B_{ijk,\alpha\gamma\beta}+B_{ijk,\gamma\beta\alpha}]=\chi_{ijk,\alpha\beta\gamma}+\chi_{jki,\alpha\beta\gamma}+\chi_{kij,\alpha\beta\gamma}=0,$$

$$(\mathbf{P}_D)_{\alpha\beta\gamma}{}^{\alpha'\beta'\gamma'}\epsilon_{\alpha'\beta'\gamma'}G_{ijk}=-\epsilon_{\alpha\beta\gamma}G_{ijk}+\epsilon_{\alpha\beta\gamma}G_{ijk}=0.$$

We also note that the projection operator that gives an octet component $\chi_{ijk,\alpha}{}^{\delta}$ from B is given by

$$(\mathbf{P}_0)_{\alpha}{}^{\alpha'\beta'\gamma'\delta}=\delta_{\alpha}{}^{\alpha'}\epsilon^{\delta\beta'\gamma'}-\tfrac{1}{3}\delta_{\alpha}{}^{\delta}\epsilon^{\alpha'\beta'\gamma'}. \tag{3.13}$$

The symmetry properties of D, χ, and G with respect to the Dirac indices i, j, and k can be exhibited by a further decomposition using the charge conjugation matrix C which has the following properties:

$$C^T=-C,\quad C^{-1}\gamma_\mu C=-\gamma_\mu{}^T. \tag{3.14}$$

From (3.14) it follows that $\gamma_\mu C$ and $\sigma_{\mu\nu}C$ are symmetric matrices whereas $\gamma_\mu\gamma_5 C$ and $\gamma_5 C$ are antisymmetric matrices. One can therefore write D_{ijk} in the form

$$D_{ijk}=\tfrac{1}{2}\psi_{\mu,i}(\gamma_\mu C)_{jk}+\tfrac{1}{4}\psi_{\mu\nu,i}(\sigma_{\mu\nu}C)_{jk}, \tag{3.15}$$

which insures the symmetry of D_{ijk} under the interchange of j and k. Since D_{ijk} is totally symmetric,

$$(C^{-1})^{ij}D_{ijk}=(C^{-1}\gamma_\mu\gamma_5)^{ij}D_{ijk}=(C^{-1}\gamma_5)^{ij}D_{ijk}=0. \tag{3.16}$$

From (3.15) and (3.16) we obtain

$$\gamma_\mu\psi_\mu=0;\quad \psi_\mu=\gamma_\rho\psi_{\rho\mu}. \tag{3.17}$$

In (3.15) ψ_μ has 16 components and antisymmetric tensor $\psi_{\mu\nu}$ has 24 components. There are 20 relations among them on account of (3.17). Hence (3.15) and (3.17) correctly represent a totally symmetric third-rank

[16] M. Gell-Mann, Phys. Rev. 125, 1067 (1962).

Dirac spinor D_{ijk} with 20 independent components. Considerations along similar lines permit the expansion of χ and G as follows:

$$\chi_{ijk}=\tfrac{1}{2}[\xi_i C_{jk}+\eta_{\mu,i}(\gamma_\mu\gamma_5 C)_{jk}+\psi_i(\gamma_5 C)_{jk}],$$

where

$$\xi=\gamma_\mu\gamma_5\eta_\mu+\gamma_5\psi, \tag{3.18}$$

and

$$G_{ijk}=\tfrac{1}{2}[\xi_i' C_{jk}+\eta_{\mu,i}'(\gamma_\mu\gamma_5 C)_{jk}+\psi_i'(\gamma_5 C)_{jk}],$$

where

$$\eta_\mu'=\gamma_\mu\gamma_5\xi', \quad \psi'=-\gamma_5\xi'. \tag{3.19}$$

IV. FREE-FIELD EQUATIONS OF MOTION FOR MESONS AND BARYONS

The spinor fields described so far can lead to no physical consequences unless they are made to represent physical particle states. As stated in the Introduction, we may accomplish this by requiring that Φ and Ψ satisfy prescribed free field equations of motion. The guiding principle in the choice of such equations is provided by one of the most remarkable successes of the $SU(6)$ symmetry, namely the assignment of representations for the low lying baryonic and mesonic states. The octet of $\frac{1}{2}^+$ and the decuplet of $\frac{3}{2}^+$ baryons can be identified as belonging to the 56-dimensional representation. The octet 0^- and the nonet 1^- mesons can be fitted into the 35-dimensional representation whereas the X^0 meson[17] can belong to the singlet representation. We shall choose the wave equations which lead to solutions that correspond to the $SU(6)$ supermultiplet states.

The meson field Φ is required to satisfy

$$\tfrac{1}{2}[\gamma\cdot\partial\otimes 1'\otimes 1\otimes 1'-1\otimes\tilde{\gamma}\cdot\partial\otimes 1\otimes 1']\Phi+m\Phi=0, \tag{4.1}$$

which is the Duffin-Kemmer equation[18] rewritten in a form more convenient for our purpose. To introduce the desired mass splittings, we regard m as a matrix in both Dirac-spin and $SU(3)$-spin space. It can be chosen in a number of different ways. One of the forms which leads to the well-known empirical relations for the masses of the pseudoscalar and vector mesons is as follows:

$$\begin{aligned} m=m_0&(1\otimes 1'\otimes 1\otimes 1') \\ &+\tfrac{1}{2}m_1(\gamma_\lambda\otimes\tilde{\gamma}_\lambda+\gamma_\lambda\gamma_5\otimes\tilde{\gamma}_\lambda\tilde{\gamma}_5)\otimes(1\otimes 1') \\ &+\tfrac{1}{2}m'(1\otimes 1'+\gamma_5\otimes\tilde{\gamma}_5)\otimes(\boldsymbol{\delta}\otimes 1'+1\otimes\boldsymbol{\delta}') \\ &+m_S\sum_{A=1}^{16}\gamma_A\otimes\tilde{\gamma}_5\tilde{\gamma}_A\tilde{\gamma}_5\otimes\sum_{i=0}^{8}\lambda_i\otimes\lambda_i, \end{aligned} \tag{4.2}$$

where $\boldsymbol{\delta}$ is given by

$$\boldsymbol{\delta}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}.$$

We shall continue to use the direct-product notation with the convention regarding matrix multiplication as defined in (2.7) and (2.9). (Also see footnote 15.) In Eq. (4.2), the first term gives a common mass m_0 to all mesons. The second term is a spin-dependent term and is introduced to split the vector mesons from the pseudoscalar ones. The third term is responsible for Gell-Mann–Okubo (GMO)[19] splittings as well as singlet and octet mixing. The last term splits the singlet pseudoscalar meson from others, since it is proportional to the projection operator of the singlet, pseudoscalar term. If we assume a sufficiently large value for m_S, there would be little mixing between the singlet $0^-(X^0)$ and the octet $0^-(\eta^0)$ states. Since the GMO mass sum rule is satisfied very well for the octet of 0^- mesons, we assume this to be the case.

If we insert the expansion (3.2) for Φ in (4.1), multiply by the Dirac matrices 1, γ_μ, $\sigma_{\mu\nu}$, $\gamma_\mu\gamma_5$, and γ_5, respectively, and take trace, we obtain the following set of equations in $SU(3)$ space:

$$[(m_0+4m_1)1\otimes 1'+m'(\boldsymbol{\delta}\otimes 1'+1\otimes\boldsymbol{\delta}')]\mathbf{S}=0, \tag{4.3}$$

$$[m_0 1\otimes 1']\mathbf{V}_\mu+1\otimes 1'\partial_\lambda\mathbf{T}_{\lambda\mu}=0, \tag{4.4}$$

$$[m_0 1\otimes 1'+m'(\boldsymbol{\delta}\otimes 1'+1\otimes\boldsymbol{\delta}')]\mathbf{T}_{\mu\lambda} +1\otimes 1'(\partial_\mu\mathbf{V}_\lambda-\partial_\lambda\mathbf{V}_\mu)=0, \tag{4.5}$$

$$[m_0 1\otimes 1']\mathbf{A}_\mu+1\otimes 1'\partial_\mu\mathbf{P}=0, \tag{4.6}$$

$$[(m_0-4m_1)1\otimes 1'+m'(\boldsymbol{\delta}\otimes 1'+1\otimes\boldsymbol{\delta}')]\mathbf{P} +1\otimes 1'\partial_\mu\mathbf{A}_\mu=0. \tag{4.7}$$

From these equations it follows that

$$\mathbf{S}=0, \tag{4.8}$$

$$\partial_\mu\mathbf{V}_\mu=0, \tag{4.9}$$

$$[(1\otimes 1')\Box-m_0\{m_0 1\otimes 1' +m'(\boldsymbol{\delta}\otimes 1'+1\otimes\boldsymbol{\delta}')\}]\mathbf{V}_\mu=0, \tag{4.10}$$

$$[(1\otimes 1')\Box-m_0\{(m_0-4m_1)1\otimes 1' +m'(\boldsymbol{\delta}\otimes 1'+1\otimes\boldsymbol{\delta}')\}]\mathbf{P}=0. \tag{4.11}$$

Equations (4.9), (4.10), and (4.11) represent the free wave equations of motion for a nonet of 1^- mesons and an octet of 0^- mesons. The physical states can be obtained by diagonalizing the mass matrix in (4.10) and

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[17] G. R. Kalbfleisch *et al.*, Phys. Rev. Letters **12**, 527 (1964); M. Gundzik *et al.*, *ibid.* **12**, 546 (1964).

[18] R. J. Duffin, Phys. Rev. **54**, 1114 (1938); N. Kemmer, Proc. Roy. Soc. (London) **A173**, 91 (1939). Please note that (for typographical reasons) primes are used with identity and $\boldsymbol{\delta}$ matrices instead of tildes to indicate the multiplication from the right [Eqs. (2.7)–(2.10)].

[19] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); S. Okubo, Progr. Theoret. Phys. (Kyoto) **27**, 949 (1962).

(4.11), and are described by

$$\mathbf{V}=\begin{pmatrix} \frac{\rho^0+\omega^0}{\sqrt{2}} & \rho^+ & K^{*+} \\ \rho^- & \frac{\omega^0-\rho^0}{\sqrt{2}} & K^{*0} \\ K^{*-} & \bar{K}^{*0} & \phi^0 \end{pmatrix},$$

$$\mathbf{P}=\begin{pmatrix} \frac{\pi^0}{\sqrt{2}}+\frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}}+\frac{\eta}{\sqrt{6}} & K^- \\ K^- & \bar{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix}.$$

The masses corresponding to these particles are as follows:

$$\begin{aligned} m_\rho^2=m_\omega^2=m_0^2;\quad m_{K^*}{}^2=m_0(m_0+m_0');\\ m_\phi^2=m_0(m_0+2m_0'),\\ m_\pi^2=m_0(m_0-4m_1);\quad m_K^2=m_0(m_0-4m_1+m_0');\\ m_\eta^2=m_0(m_0-4m_1+\tfrac{4}{3}m_0'). \end{aligned} \tag{4.12}$$

It is clear that the squares of the 0^- meson masses satisfy the GMO sum rule for an octet and that

$$m_\phi^2-m_{K^*}{}^2=m_{K^*}{}^2-m_\rho^2=m_K^2-m_\pi^2. \tag{4.13}$$

From Eqs. (4.5), (4.6), and (4.12),

$$T_{\mu\lambda,\alpha}{}^\beta=[1/(m_T)_\alpha{}^\beta](\partial_\lambda V_{\mu,\alpha}{}^\beta-\partial_\mu V_{\lambda,\alpha}{}^\beta),$$
$$A_{\mu,\alpha}{}^\beta=(1/m_\rho)\partial_\mu P_\alpha{}^\beta,$$

where $(m_T)_\alpha{}^\beta$ are given by the elements of the matrix

$$m_T=\begin{pmatrix} m_\rho & m_\rho & m_{K^*}{}^2/m_\rho \\ m_\rho & m_\rho & m_{K^*}{}^2/m_\rho \\ m_{K^*}{}^2/m_\rho & m_{K^*}{}^2/m_\rho & m_\phi{}^2/m_\rho \end{pmatrix}. \tag{4.14}$$

The solution of the wave equation (4.1) can therefore be explicitly written as

$$\Phi_{i\alpha}{}^{i\beta}=\Bigg[(\gamma_\mu)_i{}^jV_{\mu,\alpha}{}^\beta-\frac{1}{2(m_T)_\alpha{}^\beta}(\sigma_{\mu\nu})_i{}^jF_{\mu\nu,\alpha}{}^\beta +(\gamma_5)_i{}^j\left(P_\alpha{}^\beta+\frac{X^0\delta_\alpha{}^\beta}{\sqrt{3}}\right) -(\gamma_\mu\gamma_5)_i{}^j\left(\frac{P_\alpha{}^\beta}{m_\rho}+\frac{X^0}{m_S}\frac{\delta_{\alpha\beta}}{\sqrt{3}}\right)\Bigg], \tag{4.15}$$

where m_S is the mass of X^0 and $F_{\mu\nu}=\partial_\mu V_\nu-\partial_\nu V_\mu$.

For the baryons, we assume that Ψ satisfies the generalized Bargmann-Wigner equations[20]:

$$[\gamma\cdot\partial+M]\Psi=0, \tag{4.16}$$

where M is again a matrix. The desired mass splittings can be introduced by choosing M to be a matrix in $SU(3)$ space[21] alone. The required form is

$$\mathbf{M}=[M_0\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}+(M'-M_1-3M_2)\mathbf{P}_D +(\mathbf{P}_D+\mathbf{P}_B)(M_1\mathbf{G}_1+M_2\mathbf{G}_2)], \tag{4.17}$$

where $\mathbf{P}_D$ and $\mathbf{P}_B$ are the projection operators defined in Eqs. (3.10) and (3.11). The first term gives a common mass to all the baryons. The second term splits the decuplet masses from the octet masses. $\mathbf{G}_1$ and $\mathbf{G}_2$ in the third term are introduced so that they not only give the GMO splittings, but also give the observed relations between the decuplet spacings and the octet spacings. They are given by

$$\mathbf{G}_1=\boldsymbol{\delta}\otimes\mathbf{1}\otimes\mathbf{1}+\mathbf{1}\otimes\boldsymbol{\delta}\otimes\mathbf{1}+\mathbf{1}\otimes\mathbf{1}\otimes\boldsymbol{\delta},$$

$$\mathbf{G}_2=\sum_{i=0}^{8}[\boldsymbol{\lambda}_i\otimes(\boldsymbol{\delta\lambda}_i\otimes\mathbf{1}+\mathbf{1}\otimes\boldsymbol{\delta\lambda}_i) +\text{symmetrizing terms}]. \tag{4.18}$$

To be more explicit,

$$(G_1\Psi)_{\alpha\beta\gamma}=[\Psi_{3\beta\gamma}\delta_\alpha{}^3+\Psi_{\alpha3\gamma}\delta_\beta{}^3+\Psi_{\alpha\beta3}\delta_\gamma{}^3], \tag{4.19}$$

$$(G_2\Psi)_{\alpha\beta\gamma}=[(\Psi_{\beta3\gamma}+\Psi_{\gamma\beta3})\delta_\alpha{}^3+(\Psi_{\alpha\gamma3}+\Psi_{3\alpha\gamma})\delta_\beta{}^3 +(\Psi_{3\beta\alpha}+\Psi_{\alpha3\beta})\delta_\gamma{}^3], \tag{4.20}$$

where for convenience we have suppressed the Dirac indices.

As in the meson case, if we insert the expansion (3.7) in (4.16), it follows that

$$\epsilon_{\alpha\beta\gamma}G_{ijk}=0, \tag{4.21}$$

$$[\gamma\cdot\partial+(M_0+M'-M_1-3M_2) +(M_1+2M_2)(\delta_\alpha{}^3+\delta_\beta{}^3+\delta_\gamma{}^3)]_i{}^{i'}D_{i'jk,\alpha\beta\gamma}=0, \tag{4.22}$$

$$[\gamma\cdot\partial+(M_0-M_2)]_i{}^{i'}\chi_{i'jk,\alpha}{}^\beta +(M_1+\tfrac{1}{2}M_2)(\delta_\alpha{}^3-\delta_3{}^\beta)\chi_{ijk,\alpha}{}^\beta +\tfrac{3}{2}M_2[(\delta_\alpha{}^3+\delta_3{}^\beta)\chi_{ijk,\alpha}{}^\beta-\tfrac{2}{3}\delta_\alpha{}^\beta\chi_{ijk,3}{}^3]=0. \tag{4.23}$$

With the identification,

$$\begin{aligned} &D_{111}=N^{*++}, && D_{112}=N^{*+}/\sqrt{3}, && D_{122}=N^{*0}/\sqrt{3},\\ &D_{222}=N^{*-}, && D_{113}=Y^{*+}/\sqrt{3}, && D_{123}=Y^{*0}/\sqrt{6},\\ &D_{223}=Y^{*-}/\sqrt{3}, && D_{133}=\Xi^{*0}/\sqrt{3}, && D_{233}=\Xi^{*-}/\sqrt{3},\\ & && D_{333}=\Omega^-, && \end{aligned}$$

[20] V. Bargmann and E. P. Wigner, Proc. Natl. Acad. Sci. U. S. 34, 211 (1948).

[21] We split the decuplet mass from the octet by using the $SU(3)$ spin-dependent term. We may, of course, use the ordinary spin-dependent splitting. Because of the over-all symmetry of Ψ, however, it can be proved that both methods are equivalent when the mass matrix is operated on Ψ.

$$\chi=\begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}}+\frac{\Lambda}{\sqrt{6}} & \Sigma^+ & P \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}}+\frac{\Lambda}{\sqrt{6}} & N \\ \Xi^- & \Xi^0 & -\frac{2\Lambda}{\sqrt{6}} \end{pmatrix},$$

the masses of the decuplet and the octet states are given by the mass formulas

$$M_D=(M_0+M_2)+M'-(M_1+\tfrac{1}{2}M_2)Y -M_2[I(I+1)-\tfrac{1}{4}Y^2], \quad (4.24)$$

$$M_O=(M_0+M_2)-(M_1+\tfrac{1}{2}M_2)Y -M_2[I(I+1)-\tfrac{1}{4}Y^2]. \quad (4.25)$$

If we rewrite (4.22) as [no summation is intended for repeated $SU(3)$ indices].

$$[\gamma\cdot\partial+M_{\alpha\beta\gamma}]_i^{i'}D_{i'jk,\alpha\beta\gamma}=0, \quad (4.26)$$

where

$$M_{\alpha\beta\gamma}=(M_0+M'-M_1-3M_2) +(M_1+2M_2)(\delta_\alpha{}^3+\delta_\beta{}^3+\delta_\gamma{}^3),$$

and substitute (3.15) in (4.26), we obtain

$$[\gamma\cdot\partial+M_{\alpha\beta\gamma}]\psi_{\mu,\alpha\beta\gamma}=0; \quad (4.27)$$
$$[\gamma\cdot\partial+M_{\alpha\beta\gamma}]\psi_{\mu\nu,\alpha\beta\gamma}=0.$$

Since $D_{ijk,\alpha\beta\gamma}$ it totally symmetric, Eq. (4.26) can also be written as

$$[\gamma\cdot\partial+M_{\alpha\beta\gamma}]_i^{i'}D_{ji'k,\alpha\beta\gamma}=0.$$

If we now substitute the expansion

$$D_{ji'k,\alpha\beta\gamma}=[\tfrac{1}{2}\psi_{\mu,j}(\gamma_\mu C)_{i'k}+\tfrac{1}{4}\psi_{\mu\nu,j}(\sigma_{\mu\nu}C)_{i'k}]_{\alpha\beta\gamma},$$

$$[(\gamma\cdot\partial+M_{\alpha\beta\gamma})\gamma_\mu C]_{ik}\psi_{\mu,\alpha\beta\gamma} +[(\gamma\cdot\partial+M_{\alpha\beta\gamma})\sigma_{\mu\nu}C]_{ik}\psi_{\mu\nu,\alpha\beta\gamma}=0. \quad (4.28)$$

From (4.28), if we multiply by $C^{-1}\sigma_{\mu\nu}$ and take the trace we obtain

$$\psi_{\mu\nu,\alpha\beta\gamma}=\frac{1}{M_{\alpha\beta\gamma}}(\partial_\nu\psi_\mu-\partial_\mu\psi_\nu). \quad (4.29)$$

The equations

$$[\gamma\cdot\partial+M_{\alpha\beta\gamma}]\psi_{\mu,\alpha\beta\gamma}=0;\quad \gamma_\mu\psi_{\mu,\alpha\beta\gamma}=0 \quad (4.30)$$

are the Rarita-Schwinger equations for a $J=\frac{3}{2}$ particle. Equation (4.29) enables us to express the antisymmetric tensor $\psi_{\mu\nu}$ in terms of the Rarita-Schwinger fields ψ_μ.[22]

[22] The equivalence between the Bargmann-Wigner wave equation and the Rarita-Schwinger equation for the spin-$\frac{3}{2}$ particle has been shown by C. G. Oliveira and A. Vidal, Notas de Fisica IX, No. B, 226 (1962).

Thus $D_{ijk,\alpha\beta\gamma}$ is given by

$$D_{ijk,\alpha\beta\gamma}=\left[\tfrac{1}{2}(\gamma_\mu C)_{jk}\psi_{\mu i,\alpha\beta\gamma}-\frac{1}{4M_{\alpha\beta\gamma}}(\sigma_{\mu\nu}C)_{jk}\Lambda_{\mu\nu i,\alpha\beta\gamma}\right],$$

where

$$\Lambda_{\mu\nu}=(\partial_\mu\psi_\nu-\partial_\nu\psi_\mu). \quad (4.31)$$

Similar considerations lead to

$$\chi_{ijk,\alpha}{}^\beta=\left[\tfrac{1}{2}(\gamma_5 C)_{jk}\psi_{i,\alpha}{}^\beta-\frac{1}{2M_\alpha{}^\beta}(\gamma_\mu\gamma_5 C)_{jk}\partial_\mu\psi_{i,\alpha}{}^\beta\right], \quad (4.32)$$

where ψ satisfies the Dirac equation

$$(\gamma\cdot\partial+M_\alpha{}^\beta)\psi_{i,\alpha}{}^\beta+\tfrac{1}{2}(M_\Sigma-M_\Lambda)\delta_\alpha{}^\beta\psi_{i,3}{}^3=0 \quad (4.33)$$

and $M_\alpha{}^\beta$ are given by the elements of the matrix

$$\begin{pmatrix} M_\Sigma & M_\Sigma & M_N \\ M_\Sigma & M_\Sigma & M_N \\ M_\Xi & M_\Xi & \tfrac{1}{2}(3M_\Lambda-M_\Sigma)\end{pmatrix}.$$

V. EFFECTIVE VERTEX FUNCTIONS

Since we have the explicit solutions of the equations of motion for free fields, we can obtain effective trilinear, quadrilinear, $\cdots$interactions in the lowest order perturbation. For the present discussion we confine ourselves to trilinear meson-meson and meson-baryon interactions. The results without the mass dissymmetries were given in Ref. 10. The solutions for the meson and baryon fields in the previous section provide a natural way of considering the modifications of the effective interactions due to mass differences. The main purpose of this section is to consider these modifications and some of their consequences. The relations between the various coupling constants are summarized in a more explicit manner in the Appendix.

Meson-Meson Interactions

If the solution (4.15) is inserted in L_{int}

$$L_{\text{int}}(\text{mesons})=m_\rho ig\tfrac{1}{4}\,\text{Tr}(\Phi\Phi\Phi),$$

the following effective VPP, VVP, and VVV interactions are obtained:

$$L_{\text{int}}(VPP)=\tfrac{9}{2}ig\,\text{Tr}(\mathbf{V}_\mu\mathbf{P}\overleftrightarrow{\partial}_\mu\mathbf{P}), \quad (5.1)$$

$$L_{\text{int}}(VVP)=3ig\epsilon_{\mu\nu\rho\sigma}\left\{\text{Tr}\left[\partial_\mu\mathbf{V}_\nu\left(\frac{\partial_\rho\mathbf{V}_\sigma}{m}\right)\mathbf{P}\right]+\text{Tr}\left[\left(\frac{\partial_\mu\mathbf{V}_\nu}{m}\right)\partial_\rho\mathbf{V}_\sigma\mathbf{P}\right]+m_\rho\,\text{Tr}\left[\left(\frac{\partial_\mu\mathbf{V}_\nu}{m}\right)\left(\frac{\partial_\rho\mathbf{V}_\sigma}{m}\right)\mathbf{P}\right]\right\}, \quad (5.2)$$

TABLE I. Decuplet-decuplet and decuplet-baryon currents. $H=[(M_i+M_f)^2+q^2]/2M_iM_f$; $I_{\mu\nu}=(H\delta_{\mu\nu}+q_\mu q_\nu/M_iM_f)$.

	$\bar{D}D$	$\bar{D}B$
S	$I_{\mu\nu}\bar{\psi}_\mu\psi_\nu$	0
V	$I_{\mu\nu}\bar{\psi}_\mu\gamma_\rho\psi_\nu$	$\frac{2}{3}H\bar{\psi}_\rho\gamma_5\psi+\frac{2}{3M_i}p_\mu\left(\frac{p_\rho'}{M_f}\bar{\psi}_\mu\gamma_5\psi-i\bar{\psi}_\mu\gamma_\rho\gamma_5\psi\right)$
T	$I_{\mu\nu}\bar{\psi}_\mu\sigma_{\rho\sigma}\psi_\nu$	$\frac{1}{3}H(\bar{\psi}_\sigma\gamma_\rho\gamma_5\psi-\bar{\psi}_\rho\gamma_5\psi)+\frac{1}{3M_iM_f}p_\mu\bar{\psi}_\mu(p_\sigma'\gamma_\rho-p_\rho'\gamma_\sigma)\gamma_5\psi$
A	$I_{\mu\nu}\bar{\psi}_\mu\gamma_5\gamma_\rho\psi_\nu$	$\frac{2}{3}H\bar{\psi}_\rho\psi+\frac{2}{3}\frac{p_\rho'p_\mu}{M_iM_f}\bar{\psi}_\mu\psi$
P	$I_{\mu\nu}\bar{\psi}_\mu\gamma_5\psi_\nu$	$-\frac{2}{3M_i}i\bar{\psi}_\mu p_\mu\psi$

$$L_{\rm int}(VVV)=3g\left[m_\rho\,{\rm Tr}\left(\left[\frac{\mathbf{F}_{\mu\nu}}{m}\right]\mathbf{V}_\mu\mathbf{V}_\nu\right)+\tfrac{1}{2}m_\rho^2\left\{{\rm Tr}\left[\left(\frac{\mathbf{F}_{\mu\nu}}{m}\right)\left(\frac{\mathbf{V}_\mu}{m}\right)\mathbf{V}_\nu\right]+{\rm Tr}\left[\left(\frac{\mathbf{F}_{\mu\nu}}{m}\right)\mathbf{V}_\mu\left(\frac{\mathbf{V}_\nu}{m}\right)\right]-{\rm Tr}\left[\mathbf{F}_{\mu\nu}\left(\frac{\mathbf{V}_\mu}{m}\right)\left(\frac{\mathbf{V}_\nu}{m}\right)\right]\right\}-\tfrac{2}{3}m_\rho\,{\rm Tr}\left[\left(\frac{\partial_\mu\mathbf{V}_\nu}{m}\right)\left(\frac{\partial_\nu\mathbf{V}_\rho}{m}\right)\left(\frac{\partial_\rho\mathbf{V}_\mu}{m}\right)\right]\right], \quad (5.3)$$

where $\mathbf{V}/m$ is a 3×3 matrix whose elements are given by

$$\left(\frac{\mathbf{V}}{m}\right)_\alpha^\beta=\frac{\mathbf{V}_\alpha^\beta}{(m_T)_\alpha^\beta}$$

with $(m_T)_\alpha^\beta$ as defined in (4.14).

If the $SU(3)$ mass splittings are neglected, the vector meson field is coupled to a conserved $SU(3)$ meson current. Among the many relations between the different coupling constants listed in the Appendix, we consider

$$g_{\omega\rho\pi}^2=\frac{4}{m_\rho^2}g_{\rho\pi\pi}^2. \quad (5.4)$$

From the Gell-Mann, Sharp, and Wagner[23] model for the $\omega\to3\pi$ decay, the known $\rho\to2\pi$ decay rate and (5.4), the width Γ of $\omega\to3\pi$ is $\Gamma=5.4$ MeV. This is to be compared with the experimental value of 9.4 MeV. It would be interesting to see whether the difference could be accounted for by a direct $\omega\to3\pi$ contribution that can arise from ${\rm Tr}(\Phi\Phi\Phi\Phi)$ interaction.

To obtain the effective meson-baryon couplings, it is convenient to define a current

$$J_A{}^B=\bar{\Psi}^{BDC}\Psi_{ADC}. \quad (5.5)$$

If we substitute the free field solutions (3.7), (4.31), and (4.32) for Ψ in (5.5), then

$$J_A{}^B=J_A{}^B(\bar{D}D)+J_A{}^B(\bar{D}B)+J_A{}^B(\bar{B}D)+J_A{}^B(\bar{B}B). \quad (5.6)$$

Each term in Eq. (5.6) can be separated into the $SU(3)$ singlet and octet parts. Further the space-time properties of each of the currents allows the decomposition into the usual scalar, vector, tensor, axial vector, and pseudoscalar parts:

$$J_{i\alpha}{}^{j\beta}(\ )=\tfrac{1}{4}[(1)_i{}^jJ^S(\ )+(\gamma_\mu)_i{}^jJ_\mu{}^V(\ )+(\sigma_{\mu\nu})_i{}^jJ_{\nu\mu}{}^T(\ )+(\gamma_\mu\gamma_5)_i{}^jJ_\mu{}^A(\ )+(\gamma_5)_i{}^jJ^P(\ )]_\alpha{}^\beta. \quad (5.7)$$

The results[24] are given in Tables I and II. Table I contains the contributions to the current $J_A{}^B(\bar{D}D)$ and $J_A{}^B(\bar{D}B)$. $J_A{}^B(\bar{B}D)$ can be obtained from Table I since $J_A{}^B(\bar{B}D)=[J_A{}^B(\bar{D}B)]^\dagger$. Table II contains the baryonic current separated into $SU(3)$ parts that transform like the symmetric (D) octet, antisymmetric (F) octet and the singlet (S). The effective $\bar{D}DV$, $\bar{D}DP$, $\bar{D}BV$, $\bar{D}BP$, $\bar{B}BV$, and $\bar{B}BP$ couplings are obtained by combining these currents with the free-field solutions for mesons (4.15). Thus,

$$L_{\rm int}(\text{baryons-mesons})=L_{\rm int}(\bar{D}DM)+L_{\rm int}(\bar{D}BM)+L_{\rm int}(\bar{B}DM)+L_{\rm int}(\bar{B}BM),$$

where

$$L_{\rm int}(\ )=iG\left[J^P(\ )P+\frac{1}{m_\rho}J^A(\ )\partial_\mu P+J_\mu{}^V(\ )V_\mu-J_{\mu\nu}{}^T(\ )\left(\frac{F_{\mu\nu}}{m}\right)\right]. \quad (5.8)$$

From Tables I and II and (5.8),

$$\frac{1}{iG}L_{\rm int}(\bar{D}DM)=\left(H\delta_{\mu\nu}+\frac{q_\mu q_\nu}{M_iM_f}\right)\left\{\left(1+\frac{M_i+M_f}{m_\rho}\right)[\bar{\psi}_\mu\gamma_5\psi_\nu P]+[\bar{\psi}_\mu\gamma_\rho\psi_\nu V_\rho]+2i\left[\bar{\psi}_\mu\sigma_{\rho\sigma}q_\sigma\psi_\nu\left(\frac{V_\rho}{m}\right)\right]\right\}, \quad (5.9)$$

[23] M. Gell-Mann, D. Sharp, and W. G. Wagner, Phys. Rev. Letters 8, 261 (1962).

[24] The expressions for different currents on Tables I and II reduce to the corresponding ones in Ref. 10 when the mass splittings are neglected. Please note the following errors in Table I of Ref. 10. In the axial vector (A) current of $\bar{D}D$, it should be $\gamma_5\gamma_\rho$ instead of $\gamma_\rho\gamma_5$. In the $\bar{D}B$ contribution there should be a negative sign before the pseudoscalar (P) current.

TABLE II. Baryon-baryon current.

	F	D	S
S	$\frac{1}{3}H\bar{\psi}\psi$	0	$\frac{1}{3}H\bar{\psi}\psi$
V	$\frac{1}{3}\left(H-\frac{1}{6}\frac{q^2}{M_iM_f}\right)\bar{\psi}\gamma_\rho\psi-\frac{(M_i+M_f)}{18M_iM_f}i\bar{\psi}\sigma_{\rho\sigma}q_\sigma\psi+\frac{(M_f-M_i)}{18M_iM_f}iq_\rho\bar{\psi}\psi$	$\frac{q^2}{6M_iM_f}\bar{\psi}\gamma_\rho\psi+\frac{(M_i+M_f)}{6M_iM_f}i\bar{\psi}\sigma_{\rho\sigma}q_\sigma\psi-\frac{(M_f-M_i)}{6M_iM_f}iq_\rho\bar{\psi}\psi$	$\frac{1}{3}\left(H-\frac{1}{3}\frac{q^2}{M_iM_f}\right)\bar{\psi}\gamma_\rho\psi-i\frac{(M_i+M_f)}{9M_iM_f}\bar{\psi}\sigma_{\rho\sigma}q_\sigma\psi+\frac{M_f-M_i}{9M_iM_f}iq_\rho\bar{\psi}\psi$
T	$\frac{1}{3}H\bar{\psi}\sigma_{\rho\sigma}\psi-\frac{1}{18M_iM_f}(p_\rho'p_\sigma-p_\sigma'p_\rho)\bar{\psi}\psi$	$\frac{1}{6}H\bar{\psi}\sigma_{\rho\sigma}\psi+\frac{1}{6M_iM_f}(p_\rho'p_\sigma-p_\sigma'p_\rho)\bar{\psi}\psi$	$\frac{1}{18}H\bar{\psi}\sigma_{\rho\sigma}\psi-\frac{1}{9M_iM_f}(p_\rho'p_\sigma-p_\sigma'p_\rho)\bar{\psi}\psi$
A	$(2/9)H\bar{\psi}\gamma_5\gamma_\rho\psi$	$\frac{1}{3}H\bar{\psi}\gamma_5\gamma_\rho\psi$	$\frac{1}{9}H\bar{\psi}\gamma_5\gamma_\rho\psi$
P	$(2/9)H\bar{\psi}\gamma_5\psi$	$\frac{1}{3}H\bar{\psi}\gamma_5\psi$	$\frac{1}{9}H\bar{\psi}\gamma_5\psi$

$$\frac{1}{iG}\{L_{\rm int}(\bar{D}BM)+L_{\rm int}(\bar{B}DM)\}$$
$$=-\frac{2}{3}\left(1+\frac{M_i+M_f}{m_p}\right)i\left\{\frac{p_\mu}{M_i}[\bar{\psi}_\mu\psi P]+\frac{p_\mu'}{M_f}[\bar{\psi}\psi_\mu P]\right\}+\frac{2}{3}\left(1+\frac{M_i+M_f}{m}\right)$$
$$\times\left\{H([\bar{\psi}_\mu\gamma_5\psi V_\mu]+[\bar{\psi}\gamma_5\psi_\mu V_\mu])-\frac{q_\mu(p_\rho'+p_\rho)}{2M_iM_f}([\bar{\psi}_\mu\gamma_5\psi V_\rho]-[\bar{\psi}\gamma_5\psi_\mu V_\rho])\right.$$
$$\left.+iq_\mu\left(\frac{1}{M_i}[\bar{\psi}_\mu\gamma_\rho\gamma_5\psi V_\rho]+\frac{1}{M_f}[\bar{\psi}\gamma_\rho\gamma_5\psi_\mu V_\rho]\right)\right\},\quad(5.10)$$

$$\frac{1}{iG}L_{\rm int}(\bar{B}BM)=\frac{1}{3}H\left(1+\frac{M_i+M_f}{m_p}\right)\{[\bar{\psi}\gamma_5\psi P]_D+\tfrac{2}{3}[\bar{\psi}\gamma_5\psi P]_F+\tfrac{1}{3}[\bar{\psi}\gamma_5\psi P]_S\}$$
$$+\frac{\sqrt{3}}{9}H\left(1+\frac{M_i+M_f}{m_S}\right)\mathrm{Tr}[\bar{\psi}\gamma_5\psi]X^0+\sum_{W=F,D,S}\{f_1^W[\bar{\psi}\gamma_\rho\psi V_\rho]_W+f_2^W i[\bar{\psi}\sigma_{\rho\sigma}q_\sigma\psi V_\rho]_W\}.\quad(5.11)$$

In order to write the above interactions in as simple a form as possible we have used the following definitions and abbreviations:

$$H=((M_i+M_f)^2+q^2)/2M_iM_f,\qquad(5.12)$$

where M_i and M_f refer to the initial and final baryon masses. $q_\mu=p_\mu'-p_\mu$ where p_μ and p_μ' are the four momenta of the initial and final baryons.

$$[\bar{\psi}_\mu O\psi_\nu M]=\bar{\psi}^{\alpha\beta\gamma}O\psi_{\nu,\alpha\beta'\gamma}M_\alpha{}^{\alpha'},$$
$$[\bar{\psi}_\mu O\psi M]=\bar{\psi}_\mu{}^{\alpha\beta\gamma}O\psi_\beta{}^\delta\epsilon_{\delta\gamma\alpha'}M_\alpha{}^{\alpha'},$$
$$[\bar{\psi}O\psi_\mu M]=\epsilon^{\delta\gamma\alpha}\bar{\psi}_\delta{}^\beta O\psi_{\alpha'\beta\gamma}M_\alpha{}^{\alpha'},$$
$$[\bar{\psi}O\psi M]_F=+\mathrm{Tr}(\bar{\psi}OM\psi)-\mathrm{Tr}(\bar{\psi}O\psi M)\qquad(5.13)$$
$$[\bar{\psi}O\psi M]_D=\mathrm{Tr}(\bar{\psi}O\psi M)+\mathrm{Tr}(\bar{\psi}OM\psi)-\tfrac{2}{3}\mathrm{Tr}(\bar{\psi}O\psi)\,\mathrm{Tr}M,$$
$$[\bar{\psi}O\psi M]_S=\mathrm{Tr}[\bar{\psi}O\psi]\,\mathrm{Tr}M,$$

where O is any Dirac matrix and M represents either the pseudoscalar mesons P or the vector mesons V. It should be noted that in the present model, for the pseudoscalar mesons,

$$M_\alpha{}^{\alpha'}=P_\alpha{}^{\alpha'}+(1/\sqrt{3})X_0\delta_\alpha{}^{\alpha'}$$

and for the vector mesons,

$$M_\alpha{}^{\alpha'}=V_\alpha{}^{\alpha'},$$

where P and V are defined in the previous section. In the case of vector-meson interactions,

$$(1/m)[\bar{\psi}O\psi V]=[\bar{\psi}O\psi(V/m)],$$

and $f_{1,2}{}^W$ are given in Table III.

If the $SU(3)$ mass splittings are neglected, the Yukawa-type meson-baryon coupling constants have a D/F ratio of $\frac{3}{2}$ which agrees with the corresponding results in the nonrelativistic $SU(6)$ theory.[25] From the

[25] F. Gürsey, A. Pais, L. A. Radicati, Phys. Rev. Letters **13**, 299 (1964).

TABLE III. Expressions for $f_{1,2}{}^W$ defined in Eq. (5.11).

$i \backslash W$	$f_i{}^W$ F	D	S
1	$\frac{H}{3}-\frac{q^2}{18M_iM_f}\left(1+\frac{M_i+M_f}{m}\right)$	$\frac{q^2}{6M_iM_f}\left(1+\frac{M_i+M_f}{m}\right)$	$\frac{H}{3}-\frac{q^2}{9M_iM_f}\left(1+\frac{M_i+M_f}{m}\right)$
2	$-\frac{M_i+M_f}{18M_iM_f}+\frac{2}{9}\frac{1}{m}\left(H+\frac{q^2}{4M_iM_f}\right)$	$\frac{M_i+M_f}{6M_iM_f}\left(1+\frac{M_i+M_f}{m}\right)$	$-\frac{1}{9}\frac{M_i+M_f}{M_iM_f}+\frac{1}{9}\frac{1}{m}\left(H+\frac{q^2}{M_iM_f}\right)$

results in the Appendix,

$$g_{N^{*++},P\pi^+}{}^2=\frac{4}{9}\frac{[1+(M_{N^*}+M_N)/m_\rho]^2}{M_N{}^2}G^2, \quad (5.14)$$

$$g_{PP,\pi^0}{}^2=\frac{50}{81}\left(1-\frac{m_\pi{}^2}{4M^2}\right)^2\left(1+\frac{2M_N}{m_\rho}\right)^2G^2, \quad (5.15)$$

$$g_{PP,\rho^0}{}^2=\frac{2}{9}\left[1-\frac{1}{3}\frac{m_\rho}{M_N}-\frac{5}{12}\left(\frac{m_\rho}{M_N}\right)^2\right]^2G^2. \quad (5.16)$$

The requirement that $g_{PP,\pi^0}{}^2=15$ leads to a value of $G^2=2.05$. The width of N^* and $g_{PP,\rho^0}{}^2$ calculated using this value of G^2 are 94 MeV and 0.21. In spite of significant changes in the expressions for these coupling constants due to relativistic effects, the values obtained are in good agreement with the corresponding ones in the nonrelativistic $SU(6)$.[25] Since in the present considerations, the vertex functions are characterized by momentum-dependent form factors, a direct comparison with the experimental values needs further investigation.

It is interesting to note that in the zero-momentum-transfer limit, γ_μ-type coupling of the vector meson to the baryons is pure F-type. Thus the vector-meson field is coupled to a conserved $SU(3)$ current. Further, both the F- and D-type strangeness non-changing currents are conserved even in the presence of $SU(3)$ mass splittings.

Although our treatment has been without reference to quarks, it is helpful to consider the quark model to specify the electromagnetic interaction. In the quark model, the electromagnetic current is described by a bilinear form of the quark fields. As is evident from Table III, if we take only the minimal electromagnetic interaction for the quarks, we shall not obtain the correct value for the magnetic moment of the proton. We assume, therefore, that there is in addition an anamalous magnetic moment interaction. Now the bilinear form in quark fields is a 144-dimensional tensor representation of $M(12)$ and the minimal and anamalous parts transform as a vector and a tensor, respectively. If we assume these transformation properties under $M(12)$, then the electromagnetic interaction has the general form

$$F^V(q^2)J_\rho{}^{(V)}A_\rho-\frac{1}{m}F^T(q^2)J_{\rho\sigma}{}^T\mathfrak{F}_{\rho\sigma}. \quad (5.17)$$

The contributions of these terms to the charge $[F_{\rm ch}(q^2)]$ and magnetic $[F_{\rm mag}(q^2)]$ form factors[26] are given in Table IV. From these results it follows that[27]

$$\frac{F_{\rm mag}{}^N(q^2)}{F_{\rm mag}{}^P(q^2)}=\frac{\mu_N}{\mu_P}=-\tfrac{2}{3} \quad (5.18)$$

and

$$F_{\rm ch}{}^N(q^2)=0, \quad (5.19)$$

for all values of q^2. These consequences are certainly consistent with the presently available experimental information. Another very interesting property of these form factors is that they satisfy the required threshold condition.[28]

In the low momentum transfer region, the electromagnetic structure of the proton and neutron may be dominated by the vector mesons. If we therefore make the further assumption that the electromagnetic field couples to the baryon in exactly the same manner as the $V_1{}^1$ component of the vector meson octet in the low momentum transfer region, we can calculate the absolute values of the total magnetic moments of all the baryons from Table III. We only note here that

$$\mu_P=1+\frac{2M_N}{m_\rho} \quad \text{and} \quad \mu_N=-\tfrac{2}{3}\mu_P \quad (5.20)$$

in units of nuclear magnetons. Equation (5.20) gives $\mu_\rho\simeq 3.4$ as compared with the experimental value of 2.79.

VI. SUMMARY AND DISCUSSION

The starting point of our formulation was the assumption that fields corresponding to elementary particles are tensors of $M(12)$. The particle multiplet structure itself, however, was derived by the requirement that these fields satisfy prescribed free-field equations of motion which are not covariant with respect to $M(12)$.

[26] F. J. Ernst, R. G. Sachs and K. C. Wali, Phys. Rev. **119**, 1105 (1960).

[27] K. J. Barnes, P. Carruthers, and Frank von Hippel, Phys. Rev. Letters **14**, 82 (1965).

[28] i.e., $F_{\rm ch}(q^2)=2MF_{\rm mag}(q^2)$ at $q^2=-4M^2$. V. Barger and R. Carhart, Phys. Rev. **136**, B281 (1964). This condition is trivially satisfied since $F_{\rm ch}(-4M^2)=F_{\rm mag}(-4M^2)=0$.

TABLE IV. q^2 is the square of the momentum transfer. The form vectors $F^V(q^2)$ and $F^T(q^2)$ that multiply the vector and tensor contributions respectively are omitted from the table.

	Contribution of J^V		Contribution of J^T	
	F	D	F	D
$F_{\text{ch}}(q^2)$	$\frac{(M_i+M_f)^2}{6M_iM_f}\left(1+\frac{q^2}{(M_i+M_f)^2}\right)$	0	$-\frac{(M_i+M_f)}{6M_iM_f}\frac{q^2}{m}\left(1+\frac{q^2}{(M_i+M_f)^2}\right)$	0
$F_{\text{mag}}(q^2)$	$\frac{M_i+M_f}{9M_iM_f}\left(1+\frac{q^2}{(M_i+M_f)^2}\right)$	$\frac{M_i+M_f}{6M_iM_f}\left(1+\frac{q^2}{(M_i+M_f)^2}\right)$	$\frac{(M_i+M_f)^2}{9M_iM_f}\frac{1}{m}\left(1+\frac{q^2}{(M_i+M_f)^2}\right)$	$\frac{(M_i+M_f)^2}{6M_iM_f}\frac{1}{m}\left(1+\frac{q^2}{(M_i+M_f)^2}\right)$

It is remarkable that there exist tensors[29] and equations of motion (Bargmann-Wigner equations) which lead to a supermultiplet structure which corresponds exactly to that of $SU(6)$ symmetry. It was also shown that the observed mass splittings can be easily incorporated in the equations of motion leading to solutions in terms of physical masses.

The interaction Lagrangian was assumed to be invariant under $M(12)$ and effective vertex functions were obtained in the lowest order perturbation calculation using the solutions of the field equations. The relations between various coupling constants were discussed in the text and are also summarized in the Appendix in some cases of interest. In the few cases where an attempt has been made to compare the results with experiments, the results are certainly consistent with available information.

So far we have restricted our attention mainly to vertex parts. The problem of obtaining effective matrix elements for scattering processes remains to be discussed. Let us consider meson-baryon scattering as an example. There is a set of one-particle exchange diagrams such as vector-meson exchange diagrams, which have singularities near the physical region. If we take the $M(12)$ invariant vertex, this set of diagrams does not have an over-all $M(12)$ invariance except in the special case when the four-momentum in the propagator is zero (i.e., forward scattering). Thus, in general, even for the Born terms we do not obtain $M(12)$ invariant results. There are more complicated diagrams, which are in general responsible for the short-range forces. It is reasonable to assume that there are stronger symmetries in the short-range forces. From this basis we would like to propose the following model for the scattering problem. The Born terms for two-particle scattering amplitudes are to be obtained from one-particle exchange graphs which represent long-range forces and from the $M(12)$ invariant direct graphs which represent short-range forces. The latter contributions can be derived from the following effective Lagrangian:

$$L=G_1\bar{\Psi}^{ABC}\bar{\Psi}_{ABC}\Phi_D{}^E\Phi_E{}^D+G_2\bar{\Psi}^{ABC}\Phi_A{}^{A''}\Phi_{A''}{}^{A'}\bar{\Psi}_{A'BC}+G_3\bar{\Psi}^{ABC}\Phi_A{}^{A'}\Phi_B{}^{B'}\bar{\Psi}_{A'B'C}.$$

The complete scattering amplitude can then be obtained from these Born terms by using unitarity relations in a suitable way, e.g., dispersion relations with subtractions.

Perhaps it is also worth pointing out some of the difficulties in the formulation. $M(12)$ contains $SL(6,C)$ which is noncompact. Hence, it is not possible to consider this group as the group of transformations acting on physical states because of the difficulties discussed in the Introduction. Since the free-field equations are not covariant with respect to $M(12)$, the physical states do not form a unitary representations of $M(12)$. On the other hand, our procedure of calculating the effective vertex functions is equivalent to assuming formal invariance under $M(12)$ for these functions. Clearly such an invariance is not maintained when higher order corrections are taken into account, since the internal lines violate $M(12)$ invariance. Hence, it is hard to understand the various relationships between different coupling constants from the point of view of conventional renormalization procedures. Our assumption therefore has to be regarded only as a working hypothesis and its justification has to be sought on some dynamical basis.

ACKNOWLEDGMENT

The authors would like to thank Dr. W. D. McGlinn for many helpful discussions and a critical reading of the manuscript.

APPENDIX

The purpose of this Appendix is to collect together the relationships between diffeerent coupling constants for some cases of interest. In each case an interaction Lagrangian which conserves ordinary isotopic spin and hypercharge is written. The coupling constants are identified with the appropriate vertex functions of the present model with all the particles on the mass shell.

[29] Similarly, other tensors with appropriate symmetry properties have the content of $SU(6)$ multiplets; R. Delbourgo and M. A. Rashid, International Centre for Theoretical Physics, Trieste (to be published).

VPP Interactions

$$L_{\rm int}=i\left[\frac{1}{i}g_{\rho\pi\pi}\boldsymbol{\varrho}_\mu\cdot\boldsymbol{\pi}\times\overleftrightarrow{\partial}_\mu\boldsymbol{\pi}+g_{\rho KK}\boldsymbol{\varrho}_\mu\cdot K^\dagger\boldsymbol{\tau}\overleftrightarrow{\partial}_\mu K+g_{K^*K\pi}(K_\mu{}^{*\dagger}\boldsymbol{\tau}K\cdot\overleftrightarrow{\partial}_\mu\boldsymbol{\pi}-\text{H.c.})\right.$$
$$\left.+G_{K^*K\eta}(K_\mu{}^{*\dagger}K\overleftrightarrow{\partial}_\mu\eta-\text{H.c.})+g_{\omega KK}\omega_\mu K^\dagger\overleftrightarrow{\partial}_\mu K+g_{\phi KK}\phi_\mu K^\dagger\overleftrightarrow{\partial}_\mu K\right]. \tag{A1}$$

Comparison with (5.1) gives

$$g_{\rho\pi\pi}=(9/\sqrt{2})g \tag{A2}$$

and

$$g_{\rho\pi\pi}=2g_{\rho KK}=2g_{K^*K\pi}=-(2/\sqrt{3})g_{K^*K\eta}=-2g_{\omega KK}=\sqrt{2}g_{\phi KK}. \tag{A3}$$

Relations (A3) are consequences of just $SU(3)$ symmetry and ω-ϕ mixing.

VVP Interactions

$$L_{\rm int}=i\epsilon_{\mu\nu\lambda\delta}[g_{\rho\omega\pi}\partial_\mu\boldsymbol{\varrho}_\nu\partial_\lambda\omega_\delta\cdot\boldsymbol{\pi}+g_{K^*K^*\pi}\partial_\mu K_\nu{}^{*\dagger}\boldsymbol{\tau}\partial_\lambda K_\delta{}^*\cdot\boldsymbol{\pi}+g_{\rho K^*K}(\partial_\mu\boldsymbol{\varrho}_\nu\cdot\partial_\lambda K_\delta{}^{*\dagger}\boldsymbol{\tau}K-\text{H.c.})+g_{\omega K^*K}(\partial_\mu\omega_\nu\partial_\lambda K^{*\dagger}K-\text{H.c.})$$
$$+g_{\phi K^*K}(\partial_\mu\phi_\nu\partial_\lambda K_\delta{}^{*\dagger}K-\text{H.c.})+g_{\rho\rho\eta}\partial_\mu\boldsymbol{\varrho}_\nu\cdot\partial_\lambda\boldsymbol{\varrho}_\delta\eta+g_{\omega\omega\eta}\partial_\mu\omega_\nu\partial_\lambda\omega_\delta\eta+g_{\phi\phi\eta}\partial_\mu\phi_\nu\partial_\lambda\phi_\delta\eta+g_{K^*K\eta}\partial_\mu K^{*\dagger}\partial_\lambda K_\delta{}^*\eta] \tag{A4}$$
$$+g_{\rho\rho X_0}\partial_\mu\boldsymbol{\varrho}_\nu\cdot\partial_\lambda\boldsymbol{\varrho}_\delta X_0+g_{\omega\omega X_0}\partial_\mu\omega_\nu\partial_\lambda\omega_\delta X_0+g_{\phi\phi X_0}\partial_\mu\phi_\nu\partial_\lambda\phi_\delta X_0+g_{K^*K^*X_0}\partial_\mu K^*\partial_\lambda K^*{}_\delta X_0.$$

Comparison with (5.2) gives

$$g_{\rho\omega\pi}=(9\sqrt{2}/m_\rho)g \tag{A5}$$

and

$$g_{K^*K^*\pi}=\frac{1}{6}\left(\frac{m_\rho}{m_{K^*}}\right)^2\left[2+\left(\frac{m_\rho}{m_{K^*}}\right)^2\right]g_{\rho\omega\pi},\qquad g_{\phi\phi\eta}=-\frac{1}{3\sqrt{3}}\left(\frac{m_\rho}{m_\phi}\right)^2\left[2+\left(\frac{m_\rho}{m_\phi}\right)^2\right]g_{\rho\omega\pi},$$

$$g_{\rho K^*K}=\frac{1}{6}\left[1+2\left(\frac{m_\rho}{m_{K^*}}\right)^2\right]g_{\rho\omega\pi},\qquad g_{K^*K^*\eta}=-\frac{1}{6\sqrt{3}}\left(\frac{m_\rho}{m_{K^*}}\right)^2\left[2+\left(\frac{m_\rho}{m_{K^*}}\right)^2\right]g_{\rho\omega\pi},$$

$$g_{\omega K^*K^*}=+\frac{1}{6}\left[1+2\left(\frac{m_\rho}{m_{K^*}}\right)^2\right]g_{\rho\omega\pi},\qquad g_{\rho\rho X_0}=g_{\omega\omega X_0}=\frac{1}{3\sqrt{6}}\left[1+2\left(\frac{m_\rho}{m_S}\right)\right]g_{\rho\omega\pi}, \tag{A6}$$

$$g_{\phi K^*K}=\frac{1}{3\sqrt{2}}\left[\left(\frac{m_\rho}{m_{K^*}}\right)^2+\left(\frac{m_\rho}{m_\phi}\right)^2+\left(\frac{m_\rho}{m_{K^*}}\right)^2\left(\frac{m_\rho}{m_\phi}\right)^2\right]g_{\rho\omega\pi},\qquad g_{\phi\phi X_0}=\frac{1}{3\sqrt{6}}\left[\left(\frac{m_\rho}{m_\phi}\right)^2+2\left(\frac{m_\rho}{m_S}\right)\left(\frac{m_\rho}{m_\phi}\right)^2\right]g_{\rho\omega\pi},$$

$$g_{\rho\rho\eta}=g_{\omega\omega\eta}=\frac{1}{2\sqrt{3}}g_{\rho\omega\pi},\qquad g_{K^*K^*X_0}=\frac{2}{3\sqrt{6}}\left[\left(\frac{m_\rho}{m_{K^*}}\right)^2+2\left(\frac{m_\rho}{m_S}\right)\left(\frac{m_\rho}{m_{K^*}}\right)^2\right]g_{\rho\omega\pi},$$

VPP and **VVP** coupling constants are related because of (A2) and (A5):

$$g_{\rho\omega\pi}=(2/m_\rho)g_{\rho\pi\pi}. \tag{A7}$$

$\bar{D}BP$ Interactions

The space-time structure of the interaction is given by

$$\bar{\psi}_\mu(\partial\psi/\partial x_\mu)\phi, \tag{A8}$$

where ϕ represents the pseudoscalar meson field. If the coupling of the baryons and pseudoscalar mesons is decomposed into the conventional isotopic-spin representation,[30] then from (5.10)

$$g_{B^*,BP}=g_{B^*,BP}{}^0\frac{2}{3}\frac{1}{M_B}\left(1+\frac{M_{B^*}+M_B}{m_\rho}\right), \tag{A9}$$

[30] For the isotopic-spin decomposition of $\bar{D}BP$ and $B\bar{B}P$ interactions see, for example, A. W. Martin and K. C. Wali, Nuovo Cimento **31**, 1324 (1964); Phys. Rev. **130**, 2455 (1963).

where M_{B^*} and M_B refer to the masses of the members of the decuplet and the baryon octet respectively. $g_{B^*,BP}{}^0$ are determined by $SU(3)$ symmetry and they are given by

$$G=g_{N^*,N\pi}{}^0=-g_{N^*,\Sigma K}{}^0=-(\sqrt{6})g_{Y^*,\Sigma\pi}{}^0=-\sqrt{2}g_{Y^*,\Lambda\pi}{}^0=(\sqrt{6})g_{Y^*,N\bar{K}}{}^0=(\sqrt{6})g_{Y^*,\Xi K}{}^0$$
$$=\sqrt{2}g_{Y^*,\Sigma\eta}{}^0=(\sqrt{6})g_{\Xi^*,\Sigma\bar{K}}{}^0=-(\sqrt{6})g_{\Xi^*,\Xi\pi}{}^0=\sqrt{2}g_{\Xi^*,\Lambda\bar{K}}{}^0=-\sqrt{2}g_{\Xi^*,\Xi\eta}{}^0=g_{\Omega,\Xi\bar{K}}{}^0. \quad \text{(A10)}$$

$\bar{B}BP$ Interactions

With the usual decomposition of the interactions into isotopic spin representation and (5.11), we have

$$g_{BB'P}=g_{BB'P}{}^0\frac{1}{18}\left(\frac{(M_B+M_{B'})^2}{M_BM_{B'}}-\frac{m_P{}^2}{M_BM_{B'}}\right)\left(1+\frac{M_B+M_{B'}}{m_\rho}\right), \quad \text{(A11)}$$

where $g_{BB'P}{}^0$ are determined by exact $SU(3)$ symmetry and a D/F ratio of $\frac{3}{2}$. Thus

$$g_{NN\pi}{}^0=5(G/\sqrt{2}), \quad g_{\Lambda NK}{}^0=-3\sqrt{3}(G/\sqrt{2}), \quad g_{NN\eta}=\sqrt{3}(G/\sqrt{2}),$$
$$g_{\Lambda\Sigma\pi}{}^0=2\sqrt{3}(G/\sqrt{2}), \quad g_{\Sigma NK}{}^0=G/\sqrt{2}, \quad g_{\Lambda\Lambda\eta}=-2\sqrt{3}(G/\sqrt{2}),$$
$$g_{\Sigma\Sigma\pi}{}^0=4(G/\sqrt{2}), \quad h_{\Lambda\Xi K}=\sqrt{3}(G/\sqrt{2}), \quad g_{\Sigma\Sigma\eta}=2\sqrt{3}(G/\sqrt{2}),$$
$$g_{\Xi\Xi\pi}{}^0=-G/\sqrt{2}, \quad h_{\Sigma\Xi K}=-5(G/\sqrt{2}), \quad g_{\Xi\Xi\eta}=-3\sqrt{3}(G/\sqrt{2}).$$

The γ_μ- and $\sigma_{\mu\nu}$-type $\bar{\mathbf{B}}\mathbf{BV}$ interactions can be easily read from Table III.

PHYSICAL REVIEW VOLUME 139, NUMBER 5B 6 SEPTEMBER 1965

Summing Certain ϕ^4 Graphs Using Integral Equations*

S. Nussinov
University of Washington, Seattle, Washington
(Received 23 April 1965)

It is shown that the problem of summing certain graph chains occurring in ϕ^4 theory in which we have three (or four) particles in the intermediate state is reducible to an integral equation. For forward scattering and zero-mass field, this equation can be solved exactly using a method for solving the Bethe-Salpeter equation which has been recently suggested. As an example, the case of the truss-bridge diagrams is worked out in detail.

INTRODUCTION

An exact solution for the forward-scattering Bethe-Salpeter equation in ϕ^4 theory for zero internal masses and a kernel which is any arbitrary finite sum of irreducible primitively divergent graphs was recently obtained.[1] This was achieved by Wick-rotating and performing a four-dimensional partial-wave projection,[2] and finally utilizing the dilatational invariance by transforming to Mellin space[3] and obtaining by simple algebraic calculation an exact solution. It was shown that the inverse Mellin transform yields a partial-wave amplitude with fixed cuts.[4]

In the present paper we show that integral equations can be used to sum a certain class of diagrams—those shown schematically in Figs. 1 and 2. These diagrams do not contain two-particle intermediate states so that their formal sum does not lead to the ordinary Bethe-Salpeter equation. However, these diagrams may be divided into links by cutting across a line and a vertex or 2 vertices. This will allow us to write down simple

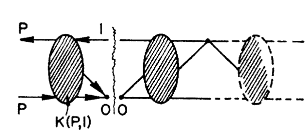

FIG. 1. Generalized diagrams which are separable by cutting across a line and a vertex.

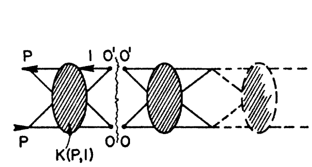

FIG. 2. Generalized diagram separable by cutting across two vertices.

* Supported in part by the U. S. Atomic Energy Commission under contract A.T. (45-1)-1388, program B.

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[2] J. D. Bjorken, J. Math. Phys. **5**, 192 (1964).

[3] P. Morse and H. Feshbach, *Methods of Theoretical Physics*, (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 1, p. 976 ff.

[4] The fixedness of the cut is shown in detail in a forthcoming paper.