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Normalization Condition and Normal and Abnormal Solutions of the Bethe-Salpeter Equation. II*

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The normalization integrals of the Bethe-Salpeter amplitudes are calculated in the Wick-Cutkosky model in order to check the conjecture that the sign of the norm is $(-1)^\kappa$ for $0<s<4$. It is explicitly verified for the following solutions with $n=l+1$: (1) κ arbitrary, s infinitesimal; (2) $\kappa=0$, $0<s\leq 2$; (3) $\kappa=1$, $0<s\leq 2+(n+2)^{-1}$; (4) $\kappa=0$, $4-s$ infinitesimal. Here κ, n, l are the conventional quantum numbers, and $s^{1/2}$ denotes the bound-state mass in units of the constituent-particle mass. Some speculations are presented concerning the existence of ghost states.

1. INTRODUCTION

IN a previous paper,[1] which will be quoted as I, we have discussed the normalization condition for the solutions of the Bethe-Salpeter equation in the Wick-Cutkosky model.[2,3] On the basis of the explicit results in the case of vanishing total momentum and in the case of infinitesimal bound-state mass with $n=l+1$ and $\kappa=0, 1, 2, 3$, where κ, n, l are the conventional quantum numbers,[3] we have conjectured that the sign of the norm of any solution is given by $(-1)^\kappa$ for $0<s<4$,[4] where $s^{1/2}$ is the bound-state mass in units of the mass of the constituent particles. The purpose of the present paper is to verify this conjecture in various cases. Throughout this paper, we shall employ the notations used in I.

The normalization condition for a solution $\phi_{\kappa nlm}(p,k)$ is

$$I_{\kappa n}(s)=\lambda_{\kappa n}(s)/\lambda_{\kappa n}'(s)\,, \tag{1.1}$$

where

$$I_{\kappa n}(s)\equiv -i\int d^4p[1-(p+k)^2][1-(p-k)^2] \times\bar{\phi}_{\kappa nlm}(p,k)\phi_{\kappa nlm}(p,k) \tag{1.2}$$

with $s=4k^2$. The eigenvalue $\lambda_{\kappa n}(s)$ is positive-definite because the Wick-rotated kernel[2] is of positive type. Its derivative $\lambda_{\kappa n}'(s)$ is expected to be negative-definite[5] because the binding energy should be a monotonically increasing function of the coupling constant. Thus, the right-hand side of (1.1) is always negative. Therefore, if $I_{\kappa n}(s)/|B_{\kappa n}(s)|^2<0$ the bound state has a positive norm, but if $I_{\kappa n}(s)/|B_{\kappa n}(s)|^2>0$ it must be a negative-norm state, where $B_{\kappa n}(s)$ is a normalization constant.

For simplicity, we consider the bound states with $n=l+1$, $(0<s<4)$:

$$\phi_{\kappa nlm}(p,k)=B_{\kappa n}(s)\mathcal{Y}_{lm}(\mathbf{p})\int_{-1}^{1}dz \times\frac{(-i)g_{\kappa n}(z,s)}{[1-\frac{1}{2}(1+z)(p+k)^2-\frac{1}{2}(1-z)(p-k)^2-i\epsilon]^{n+2}}. \tag{1.3}$$

Substituting (1.3) and its conjugate formula in (1.2), we obtain[1]

$$I_{\kappa n}(s)=-|B_{\kappa n}(s)|^2 \times(\pi(2n)!/2^{2n+1}[(n+1)!]^2)J_{\kappa n}(s)\,, \tag{1.4}$$

where

$$J_{\kappa n}(s)\equiv\int_{-1}^{1}dz\,g_{\kappa n}(z,s)\int_{-1}^{1}d\zeta\,g_{\kappa n}(\zeta,s)\int_0^1 dx\,x^{n+1}(1-x)^{n+1} \times\left\{\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta}[(\alpha+\beta)^2-\alpha\beta s]^{-n-1}\right\}_{\beta=1-\alpha}, \tag{1.5}$$

with

$$\alpha\equiv\tfrac{1}{2}(1+z)x+\tfrac{1}{2}(1+\zeta)(1-x)\,. \tag{1.6}$$

In the next section, we explicitly calculate $J_{\kappa n}(s)$ in the case of s infinitesimal but κ arbitrary. In Sec. 3, using qualitative properties of $g_{\kappa n}(z,s)$ only, we show that $J_{0n}(s)>0$ for $s\leq 2$ and $J_{1n}(s)<0$ for $0<s\leq 2+(n+2)^{-1}$. In Sec. 4, $J_{0n}(s)$ is explicitly calculated in the case of the infinitesimal binding energy $s\simeq 4$. Some discussion and speculations are presented in the final section.

2. INFINITESIMAL MASS

In this section, we shall calculate $J_{\kappa n}(s)$ in the case of s infinitesimal. Let

$$g_{\kappa n}(z,s)=\sum_{r=0}^{\infty}s^r g_{\kappa n}^{(r)}(z)\,, \tag{2.1}$$

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[1] N. Nakanishi, Phys. Rev. 138, B1182 (1965).

[2] G. C. Wick, Phys. Rev. 96, 1124 (1954).

[3] R. E. Cutkosky, Phys. Rev. 96, 1135 (1954).

[4] As was noted in I, it is more general to say that the sign of the norm coincides with the "p_0-parity" so as to include the case $s<0$. The considerations in Secs. 2 and 3 are easily extended to the case $s<0$.

[5] This is physically very reasonable, but within the present author's knowledge there is no general proof of $\lambda_{\kappa n}'(s)<0$ even in the Wick-Cutkosky model. It has been verified explicitly, however, in the following cases: (1) s is infinitesimal (Ref. 1); (2) $4-s$ is infinitesimal (Ref. 3); (3) $\kappa+n\leq 3$ and $0\leq s\leq 4$ (numerically) (Ref. 3).

Note added in proof. As pointed out by E. Predazzi (unpublished report. His counterexample to our conjecture is erroneous), $\lambda_{\kappa n}'(s)<0$ follows immediately from a classical theorem [see, for example, P. I. Richard, *Manual of Mathematical Physics* (Pergamon Press, Inc., New York, 1959), p. 408] because the coefficient function $(1-z^2)^{-n}[1-\frac{1}{4}s(1-z^2)]^{-1}$ of λ in the Sturm-Liouville eigenvalue equation of $g_{\kappa n}(z,s)$ is an increasing function of s.

and

$$G_{\kappa n}{}^{(r,j)} \equiv \int_{-1}^{1} dz\, z^j g_{\kappa n}{}^{(r)}(z). \tag{2.2}$$

Then it has been shown in I that

$$G_{\kappa n}{}^{(r,j)} = 0 \quad \text{for } 2r+j<\kappa, \tag{2.3}$$

and

$$2r(2\kappa+2n-2r+1)G_{\kappa n}{}^{(r,\kappa-2r)} = -(\kappa+n)(\kappa+n+1)\sum_{q=0}^{r-1}(-\tfrac{1}{4})^{r-q}G_{\kappa n}{}^{(q,\kappa-2q)}. \tag{2.4}$$

It has followed from (2.3) that the leading term of $J_{\kappa n}(s)$ is of order s^κ, and we can neglect higher order terms in s or in z and ζ in the integrand of (1.5). Since

$$\alpha(1-\alpha) = \tfrac{1}{4} - \tfrac{1}{4}[zx+\zeta(1-x)]^2, \tag{2.5}$$

we see

$$J_{\kappa n}(s) = J_{\kappa n}{}^{(0)}s^\kappa + O(s^{\kappa+1}), \tag{2.6}$$

with[1]

$$J_{\kappa n}{}^{(0)} \equiv \sum_{r+q+j=\kappa} \int_{-1}^{1} dz\, g_{\kappa n}{}^{(r)}(z) \int_{-1}^{1} d\zeta\, g_{\kappa n}{}^{(q)}(\zeta) \times \int_0^1 dx\, x^{n+1}(1-x)^{n+1}[(n+j)!/n!j!](2n+2j+2) \times (2n+2j+3)(-\tfrac{1}{4})^j[zx+\zeta(1-x)]^{2j}. \tag{2.7}$$

The integral over x is easily carried out by expanding the last factor of (2.7).

$$J_{\kappa n}{}^{(0)} = \sum_{r+q+j=\kappa} \sum_{m=-j}^{j} G_{\kappa n}{}^{(r,j+m)} G_{\kappa n}{}^{(q,j-m)} \times \frac{(-1)^j(n+j)!(2j)!(n+j+m+1)!(n+j-m+1)!}{2^{2j}n!j!(j+m)!(j-m)!(2n+2j+1)!}. \tag{2.8}$$

Because of (2.3), only the terms satisfying

$$j+m=\kappa-2r, \quad j-m=\kappa-2q \tag{2.9}$$

survive in (2.8).

It is necessary to evaluate $G_{\kappa n}{}^{(r,\kappa-2r)}$. The summation in the right-hand side of (2.4) is easily carried out by means of the formula (2.4) itself but with $r-1$ in place of r. Putting

$$N = \kappa+n, \tag{2.10}$$

we have

$$2r(2N-2r+1)G_{\kappa n}{}^{(r,\kappa-2r)} = \tfrac{1}{4}(N-2r+3) \times (N-2r+2)G_{\kappa n}{}^{(r-1,\kappa-2r+2)}, \tag{2.11}$$

and therefore

$$G_{\kappa n}{}^{(r,\kappa-2r)} = \frac{(N+1)!(2N-2r-1)!!}{2^{3r}r!(N-2r+1)!(2N-1)!!}G_{\kappa n}, \tag{2.12}$$

where $G_{\kappa n} \equiv G_{\kappa n}{}^{(0,\kappa)}$ was given in I, and

$$(2k+1)!! \equiv \prod_{j=0}^{k}(2j+1). \tag{2.13}$$

Substitution of (2.12) in (2.8) with (2.9) yields

$$J_{\kappa n}{}^{(0)} = \frac{(-1)^\kappa[(N+1)!]^2}{2^{N+\kappa}(N-\kappa)![(2N-1)!!]^2}(G_{\kappa n})^2 \sum_{r=0}^{[\kappa/2]}\sum_{q=0}^{[\kappa/2]} \frac{(-1)^{r+q}}{2^{r+q}} \times \frac{(2N-2r-1)!!(2N-2q-1)!!(2\kappa-2r-2q-1)!!}{r!q!(\kappa-2r)!(\kappa-2q)!(2N-2r-2q+1)!!}. \tag{2.14}$$

Now, we have to evaluate the sum

$$S_{N\kappa r} \equiv \sum_{q=0}^{[\kappa/2]} \frac{(-1)^q(2N-2q-1)!!(2\kappa-2r-2q-1)!!}{2^q q!(\kappa-2q)!(2N-2r-2q+1)!!}, \tag{2.15}$$

which may be rewritten as

$$S_{N\kappa r} = \frac{(2N-1)!!(2\kappa-2r-1)!!}{\kappa!(2N-2r+1)!!} \times \sum_{q=0}^{\infty} \frac{(-\frac{1}{2}\kappa)_q(-\frac{1}{2}\kappa+\frac{1}{2})_q(-N+r-\frac{1}{2})_q}{q!(-N+\frac{1}{2})_q(-\kappa+r+\frac{1}{2})_q}, \tag{2.16}$$

where

$$(a)_n \equiv \prod_{j=0}^{n-1}(a+j). \tag{2.17}$$

Hence

$$S_{N\kappa r} = \frac{(2N-1)!!(2\kappa-2r-1)!!}{\kappa!(2N-2r+1)!!}\, {}_3F_2(-\tfrac{1}{2}\kappa, -\tfrac{1}{2}\kappa+\tfrac{1}{2}, -N+r-\tfrac{1}{2}; -N+\tfrac{1}{2}, -\kappa+r+\tfrac{1}{2}; 1), \tag{2.18}$$

where ${}_3F_2$ denotes a generalized hypergeometric function. Since the sum of its two denominator parameters exceeds that of its three numerator parameters just by one, we can apply Saalschütz's theorem[6]:

$$ {}_3F_2(-n, a, b; c, 1-c+a+b-n; 1) = [(c-a)_n(c-b)_n]/[(c)_n(c-a-b)_n], \tag{2.19}$$

where n is a non-negative integer, and a, b, c may depend on n. When $\kappa=2m$ where m is an integer, (2.18) becomes

$$S_{N\kappa r} = \frac{(2N-1)!!(4m-2r-1)!!(-N+m)_m(1-r)_m}{(2m)!(2N-2r+1)!!(-N+\frac{1}{2})_m(m-r+\frac{1}{2})_m}. \tag{2.20}$$

Since $r=0, 1, 2, \cdots, m$ as is seen from (2.14), we see

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[6] E. D. Rainville, *Special Functions* (The Macmillan Company, New York, 1960), p. 87 with a remark in p. 88.

that $(1-r)_m=0$ except for $r=0$. Thus,

$$J_{\kappa n}{}^{(0)}=\frac{(-1)^\kappa[(N+1)!]^2(G_{\kappa n})^2}{2^{N+\kappa}\kappa!(N-\kappa)!(2N-1)!!}S_{N\kappa 0}\,, \tag{2.21}$$

with

$$S_{N\kappa 0}=\frac{(2N-\kappa)!}{2^{N-\kappa}(N-\kappa)!(2N+1)!!}\,. \tag{2.22}$$

In the case of $\kappa=2m+1$, we again find that $S_{N\kappa r}=0$ for $r\neq 0$ and that $S_{N\kappa 0}$ is given by (2.22). Thus, for arbitrary κ, (1.4) with (2.6) leads to

$$\frac{I_{\kappa n}(s)}{|B_{\kappa n}(s)|^2}=-\frac{(-1)^\kappa\pi(2n)![(\kappa+n+1)!]^2(\kappa+2n)!(G_{\kappa n})^2}{2^{2\kappa+4n+1}\kappa!(n!)^2[(n+1)!]^2(2\kappa+2n-1)!!(2\kappa+2n+1)!!}s^\kappa+O(s^{\kappa+1})\,. \tag{2.23}$$

When $\kappa=0$, 1, 2, 3, (2.23) naturally coincides with our previous result given in I. The sign of the norm is $(-1)^\kappa$ as was expected. Since $\lambda_{\kappa n}'(0)/\lambda_{\kappa n}(0)$ was given in I, we have the explicit expression for $|B_{\kappa n}(s)|^2$ in the leading order.

3. FINITE MASS

We shall discuss the sign of $J_{\kappa n}(s)$ for $\kappa=0$, 1 without evaluating it explicitly. Carrying out the differentiations in (1.5) in closed form, we find

$$J_{\kappa n}(s)=(n+1)\int_{-1}^1 dz\, g_{\kappa n}(z,s)\int_{-1}^1 d\zeta\, g_{\kappa n}(\zeta,s)\times\int_0^1 dx\, x^{n+1}(1-x)^{n+1}\Phi_n(y,s)\,, \tag{3.1}$$

where

$$\Phi_n(y,s)\equiv\frac{(2n+3)(2-s)+ys[(n+1)s+2]}{(1-ys)^{n+3}}\,, \tag{3.2}$$

with

$$y\equiv\alpha(1-\alpha)\,. \tag{3.3}$$

Since $0\leq y\leq\frac{1}{4}$, it is evident that

$$\Phi_n(y,s)\geq 0\quad\text{for } s\leq 2\,. \tag{3.4}$$

Since $g_{0n}(z,s)$ has a definite sign in $-1<z<1$,[3] it is immediately seen from (3.1) that

$$J_{0n}(s)>0\quad\text{for } s\leq 2\,. \tag{3.5}$$

Thus, the normal solutions have a positive norm at least for $0<s\leq 2$. For larger values of s, we have to know the detailed form of $g_{0n}(z,s)$.

Next, we consider the case $\kappa=1$. It is known that $g_{1n}(z,s)$ is an odd function of z and has a definite sign in $0<z<1$.[3] Therefore

$$J_{1n}(s)=2(n+1)\int_0^1 dz\, g_{1n}(z,s)\int_0^1 d\zeta\, g_{1n}(\zeta,s)\times\int_0^1 dx\, x^{n+1}(1-x)^{n+1}[\Phi_n(y,s)-\Phi_n(\tilde{y},s)]\,, \tag{3.6}$$

with

$$y\equiv\tfrac{1}{4}-\tfrac{1}{4}[zx+\zeta(1-x)]^2\,,\qquad \tilde{y}\equiv\tfrac{1}{4}-\tfrac{1}{4}[zx-\zeta(1-x)]^2\,. \tag{3.7}$$

Since $\tilde{y}\geq y$, if

$$(\partial/\partial y)\Phi_n(y,s)>0\,, \tag{3.8}$$

then we have $J_{1n}(s)<0$. It is straightforward to obtain

$$(\partial/\partial y)\Phi_n(y,s)=(n+2)s[1-ys]^{-n-4}\times\{2[2n+5-(n+2)s]+ys[(n+1)s+2]\}\,. \tag{3.9}$$

The right-hand side of (3.9) is positive for $0<y\leq\frac{1}{4}$ if

$$0<s\leq(2n+5)/(n+2)\,. \tag{3.10}$$

Thus

$$J_{1n}(s)<0\quad\text{for } 0<s\leq 2+(n+2)^{-1}\,, \tag{3.11}$$

namely, the abnormal solutions with $\kappa=1$ have a negative norm at least for $0<s\leq 2+(n+2)^{-1}$.

4. INFINITESIMAL BINDING ENERGY

We shall consider the case of infinitesimal binding energy $s\simeq 4$. Since our evaluation is completely relativistic, it is different from the nonrelativistic approximation.[7]

As seen from (3.1) with (3.2), the normalization integral becomes singular as $s\to 4$, and the numerator of (3.2) contains both positive and negative terms. Its computation should, therefore, be made very carefully. In the following, we shall calculate $J_{0n}(4)$ alone. The analytic calculation for $\kappa\geq 1$ seems to be extremely difficult.

Let

$$X\equiv zx+\zeta(1-x)\,, \tag{4.1}$$

so that $y=\frac{1}{4}(1-X^2)$, and

$$\sigma\equiv\tfrac{1}{4}s\,. \tag{4.2}$$

[7] Sato has shown that the norm of the normal solutions with $l=0$, 1 is positive in the nonrelativistic approximation in the case in which the exchanged-meson mass is much larger than the binding energy. [I. Sato, J. Math. Phys. 4, 24 (1963)].

Then (3.2) is rewritten as

$$\tfrac{1}{2}\Phi_n(\tfrac{1}{4}(1-X^2),\,s)=\frac{2(n+2)\beta}{\sigma^{n+2}(\beta+X^2)^{n+3}}-\frac{2(n+1)\sigma+1}{\sigma^{n+2}(\beta+X^2)^{n+2}},\quad(4.3)$$

where

$$\beta\equiv(1-\sigma)/\sigma.\quad(4.4)$$

In order to avoid the singularity, β will be regarded as a finite (positive) quantity until the final stage of the calculation.

It is convenient to linearize the denominators in (4.3) in the following way:

$$\frac{1}{(\beta+X^2)^{n+j+1}}=\frac{(-1)^{n+j}}{(n+j)!}\left(\frac{\partial}{\partial\beta}\right)^{n+j}(\beta+X^2)^{-1},\quad(4.5)$$

with $j=1$ or 2, and

$$\begin{aligned}(\beta+X^2)^{-1}&=\beta^{-1/2}\,\mathrm{Im}(X-i\beta^{1/2})^{-1}\\&=(2n+3)\beta^{-1/2}\\&\quad\times\mathrm{Im}\int_0^\infty d\alpha\frac{\alpha^{2n+2}}{(\alpha-i\beta^{1/2}+X)^{2n+4}}.\end{aligned}\quad(4.6)$$

With (4.1), Euler's formula (or Feynman identity) leads to

$$\int_0^1 dx\frac{x^{n+1}(1-x)^{n+1}}{(\alpha-i\beta^{1/2}+X)^{2n+4}}=\frac{[(n+1)!]^2}{(2n+3)!}\times\frac{1}{(\alpha-i\beta^{1/2}+z)^{n+2}(\alpha-i\beta^{1/2}+\zeta)^{n+2}}.\quad(4.7)$$

Hence, (3.1) is now rewritten as[8]

$$J_{\kappa n}(s)=\frac{2(n+1)}{\sigma^{n+2}}\sum_{j=1}^{2}A_j\frac{(-1)^{n+j}[(n+1)!]^2}{(n+j)!(2n+2)!}\left(\frac{\partial}{\partial\beta}\right)^{n+j}\times K_{\kappa n}(\beta,s),\quad(4.8)$$

with

$$\begin{aligned}A_1&\equiv-[2(n+1)\sigma+1],\\A_2&\equiv2(n+2)\beta,\end{aligned}\quad(4.9)$$

and

$$K_{\kappa n}(\beta,s)\equiv\beta^{-1/2}\,\mathrm{Im}\int_0^\infty d\alpha\,\alpha^{2n+2}\times\left[\int_{-1}^{1}dz\frac{g_{\kappa n}(z,s)}{(\alpha-i\beta^{1/2}+z)^{n+2}}\right]^2.\quad(4.10)$$

Hereafter, we shall confine ourselves to considering the case $\kappa=0$ only. It is known that[3]

$$g_{0n}(z,4)=(1-|z|)^n,\quad(4.11)$$

where the arbitrary coefficient has been fixed to unity. Then the z integral in (4.10) is easily carried out.

$$K_{0n}(\beta,4)=\left(\frac{2}{n+1}\right)^2\beta^{-1/2}\,\mathrm{Im}\int_0^\infty d\alpha\times\frac{\alpha^{2n+2}}{[(\alpha-i\beta^{1/2})^2-1]^2(\alpha-i\beta^{1/2})^{2n}}.\quad(4.12)$$

Since the integrand has no singularity in the lower half-plane, we can rotate the α contour into the negative imaginary axis. Then

$$\begin{aligned}K_{0n}(\beta,4)&=\frac{4}{(n+1)^2}\beta^{-1/2}\int_0^\infty d\alpha'\times\frac{\alpha'^{2n+2}}{[(\alpha'+\beta^{1/2})^2+1]^2(\alpha'+\beta^{1/2})^{2n}}\quad(4.13)\\&=\frac{4}{(n+1)^2}\beta\int_0^1 du\frac{(1-u)^{2n+2}}{(\beta+u^2)^2},\end{aligned}$$

with $u=\beta^{1/2}/(\alpha'+\beta^{1/2})$. Hence

$$J_{0n}(4)\simeq\frac{2^3n!(n+1)!}{(2n+2)!}\sum_{j=1}^{2}A_j\int_0^1 du(1-u)^{2n+2}\times L_{n+j}(u,\beta)\quad(4.14)$$

for $\beta\to0^+$, where

$$\begin{aligned}L_{n+j}(u,\beta)&\equiv\frac{(-1)^{n+j}}{(n+j)!}\left(\frac{\partial}{\partial\beta}\right)^{n+j}\frac{\beta}{(\beta+u^2)^2}\\&=-\frac{n+j}{(\beta+u^2)^{n+j+1}}+\frac{(n+j+1)\beta}{(\beta+u^2)^{n+j+2}}.\end{aligned}\quad(4.15)$$

Substituting (4.15) and (4.9) with $\sigma=1$ in (4.14), and using ϵ in place of β (i.e., $\epsilon=1-\tfrac{1}{4}s$), we have

$$J_{0n}(4)\simeq\frac{2^3n!(n+1)!}{(2n+2)!}\int_0^1 du(1-u)^{2n+2}M_n(u,\epsilon),\quad(4.16)$$

with

$$M_n(u,\epsilon)\equiv\frac{\epsilon^2-(5n+8)\epsilon u^2+(n+1)(2n+3)u^4}{(\epsilon+u^2)^{n+4}}.\quad(4.17)$$

One can, of course, carry out the integration of (4.16) explicitly, but the result will be extremely complicated. Since we need only the limit of $\epsilon\to0^+$, it is much preferable to use the following technique.

It is well known that

$$\delta(u)\simeq(1/\pi)[\epsilon^{1/2}/(\epsilon+u^2)]\quad(4.18)$$

[8] By definition, the differentiation with respect to β in (4.8) should be performed as if β were independent of s.

with $\epsilon\to 0^+$, and hence

$$\delta''(u)\simeq\frac{2\epsilon^{1/2}}{\pi}\cdot\frac{-\epsilon+3u^2}{(\epsilon+u^2)^3}. \quad (4.19)$$

Consequently,

$$\begin{aligned}[-\epsilon+(2N+3)u^2]/(\epsilon+u^2)^{N+3} &=\frac{(-1)^N}{(N+1)!\epsilon^N}\left[\left(\frac{\partial}{\partial\alpha}\right)^N\frac{-\alpha\epsilon+3u^2}{(\alpha\epsilon+u^2)^3}\right]_{\alpha=1} \\ &\simeq\frac{(-1)^N}{(N+1)!\epsilon^N}\left\{\left(\frac{\partial}{\partial\alpha}\right)^N\left[\alpha^{-2}\frac{\pi}{2\epsilon^{1/2}}\delta''(u/\alpha^{1/2})\right]\right\}_{\alpha=1} \\ &=\frac{\pi(-1)^N}{2\epsilon^{N+\frac{1}{2}}(N+1)!}\left[\left(\frac{\partial}{\partial\alpha}\right)^N\alpha^{-1/2}\right]_{\alpha=1}\delta''(u) \\ &=[\pi(2N)!/2^{2N+1}N!(N+1)!]\delta''(u)\epsilon^{-N-\frac{1}{2}}. \end{aligned} \quad (4.20)$$

Since (4.17) is rewritten as

$$\begin{aligned}M_n(u,\epsilon)=(n+1)[-\epsilon+(2n+3)u^2]/(\epsilon+u^2)^{n+3} \\ -(n+2)\epsilon[-\epsilon+(2n+5)u^2]/(\epsilon+u^2)^{n+4},\end{aligned} \quad (4.21)$$

(4.16) becomes

$$J_{0n}(4)\simeq\frac{2^3n!(n+1)!}{(2n+2)!}\times\left[\frac{\pi(2n)!}{2^{2n+1}(n!)^2}-\frac{\pi(2n+2)!}{2^{2n+3}[(n+1)!]^2}\right]R_n\epsilon^{-n-\frac{1}{2}}, \quad (4.22)$$

with

$$R_n\equiv\int_0^1 du(1-u)^{2n+2}\delta''(u) =\tfrac{1}{2}(2n+2)(2n+1). \quad (4.23)$$

Thus

$$J_{0n}(4)\simeq(\pi/2^{2n})\epsilon^{-n-\frac{1}{2}}>0, \quad (4.24)$$

namely, the normal solutions have a positive norm at the limit of zero binding energy as was expected.

Finally, since[3]

$$\lambda_{0n}(4)\simeq(2n/\pi)\epsilon^{1/2}, \quad (4.25)$$

and hence

$$\lambda_{0n}(4)/\lambda_{0n}'(4)\simeq-8\epsilon, \quad (4.26)$$

from (1.1), (1.4), and (4.24) we obtain

$$|B_{0n}(4)|^2\simeq(2^{4n+4}[(n+1)!]^2/(2n)!\pi^2)\epsilon^{n+\frac{3}{2}} \quad (4.27)$$

with $\epsilon=1-\frac{1}{4}s$.

5. DISCUSSION

In this paper, we have explicitly demonstrated the validity of our conjecture that the sign of the norm is $(-1)^\kappa$ for $0<s<4$. We may thus claim that our conjecture is undoubtedly true. It is still very desirable, however, to check the sign of $J_{1n}(s)$ for $2+(2+n)^{-1}<s<4$. This computation will be carried out by means of a computer at least for small n.

In I, we have verified in the case $k_\mu=0$ that all the solutions appear in the scattering Green's function as the residues of poles. This is a very important result because it shows that the negative-norm solutions cannot be discarded. By the continuity of the residues in s, it is extremely likely that this completeness property remains valid also for $0<s<4$, although its explicit verification will be prohibitively difficult.

It is also very interesting to check our conjecture in some solvable models other than the Wick-Cutkosky model. Probably our conjecture will remain true in any consistent model as long as the normalization condition is well defined, because any speciality of the Wick-Cutkosky model seems to be inessential for our qualitative result.

Our ghost states in the Bethe-Salpeter equation are much different from the old-fashioned ghost of Källén and Pauli,[9] which was found in the one-particle Green's function in the Lee model. Since this model is of a very pathological nature, many authors have conjectured that the appearance of the ghost is due to the defect of the model, for example, the absence of the crossing-symmetry property. On the other hand, our ghosts have been found in a causal, relativistic field theory,[10] although it is not fully realistic. They originate from the relativistic covariance but are not related to the dynamics at the small distances where the present field theory is very doubtful. They may thus provide a more serious problem than the Källén-Pauli ghost did.

The existence of negative-norm states contradicts one of the axioms of the present field theory. From the conservative standpoint, it will be natural to impose an upper bound on the coupling constant so as to avoid the appearance of abnormal solutions. If one relies upon the results of the Wick-Cutkosky model semiquantitatively, then one finds that the upper bound $\lambda=\frac{1}{4}$ is much smaller than the value, $\lambda=6$, which yields the constant total cross section at high energies in the crossed channel.[11] It is unlikely that the situation essentially changes when higher order kernels are taken into account, because at $k_\mu=0$ the $l=1$ state is necessarily degenerate with an abnormal solution and $\lambda_{\kappa n}(s)$ is expected to be a decreasing function of s. This may imply that the Regge trajectory which can explain the observed high-energy total cross section would necessarily be accompanied by a "ghost trajectory."

As was pointed out in I,[4] the negative-norm states are precisely the solutions which are odd functions of

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[9] G. Källén and W. Pauli, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **30**, No. 7 (1955).

[10] Crossing symmetry between s and u channels may formally be taken into account in the sense of Bertocchi *et al.* [L. Bertocchi, S. Fubini, and M. Tonin, Nuovo Cimento **25**, 626 (1962)].

[11] N. Nakanishi, Nuovo Cimento **34**, 795 (1964).

the relative energy p_0.[12] They will, therefore, vanish *on* the mass shell apart from the infrared divergence difficulty in the Wick-Cutkosky model. Thus, in general, we may expect that our ghosts do not appear in the S-matrix.[13] This is quite a nice feature for positive use of ghosts, because one can get rid of Lehmann's theorem[14] without violating any one of relativity, causality, and unitarity. Our ghosts might, therefore, be useful for possible removal of the ultraviolet-divergence difficulty of the present field theory.

[12] Here, of course, we confine ourselves to considering the case in which the two particles have an equal mass. The unequal-mass case is subject to future investigation. In that case, we have to note that the relativistic relative momentum p_μ is not defined unambiguously.

[13] *Note added in proof.* With unequal masses m_1 and m_2, we can show that the solutions with odd κ vanish when

$$m_2(m_1^2-v)=m_1(m_2^2-w),$$

at least for $n=l+1$. Thus the unitarity will not be violated also in the unequal-mass case.

[14] H. Lehmann, Nuovo Cimento **11**, 342 (1954).

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"Simple" Alternatives to $SU(6)$*

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A search is made for simple classical Lie groups which could serve as alternatives to $SU(6)$. Among the groups with less than 140 generators, the most likely possibilities are found to be $R(11)$, with 55 generators, and $Sp(16)$, with 136 generators; but even these two have features which seem to discriminate against their use as particle symmetry groups.

I. INTRODUCTION

The group $SU(6)$ shows every indication of becoming a successful symmetry group for particle physics.[1] But before we take $SU(6)$ entirely for granted, it seems appropriate to inquire into the existence of other groups which might serve equally well in the role of uniting spin and "unitary spin."

The present paper is a consideration of the most obvious class of alternatives to $SU(6)$: simple Lie groups which contain $SU(3)\otimes R(3)$ [or $SU(3)/Z_3 \otimes R(3)$] as a subgroup and offer a reasonable fit to the spectrum of known particles. The present search will be limited to the unitary, orthogonal, and symplectic groups[2]; thus our project is essentially the study of representations of $SU(3)\otimes R(3)$ by unitary, orthogonal, and symplectic matrices. The general process of determining representations of semisimple groups is well known[3]; and the problem of determining which are unitary, orthogonal, and symplectic has been attacked in generality by Malcev.[4] Thus we need only specialize these results and compare with the known spectrum of particles.

II. GROUPS CONTAINING $SU(3)\otimes R(3)$ AS A SUBGROUP

In order for $SU(3)\otimes R(3)$ to be a subgroup of the orthogonal group $R(N)$, it is necessary and sufficient that there exist a representation of $SU(3)\otimes R(3)$ by $N\times N$ orthogonal matrices; corresponding statements apply for the symplectic and unitary groups $Sp(N)$ and $SU(N)$. An arbitrary representation of $SU(3)\otimes R(3)$ can be written in the form

$$\sum_i \mathbf{X}_i\otimes \mathbf{Y}_i, \tag{1}$$

where $\mathbf{X}_i$ is an irreducible representation of $SU(3)$, $\mathbf{Y}_i$ is an irreducible representation of $R(3)$, and the sum is a direct sum. The irreducible representations of $SU(3)$ and of $R(3)$ are well known. In order to determine whether they are orthogonal, symplectic, or unitary, we shall make use of the following results quoted by Malcev[4]:

I. (Theorem 4 of Ref. 4.) The sum of two mutually contragredient representations is both orthogonal and symplectic.

II. (Lemma 1 of Ref. 4.) The Kronecker product of two contragredient representations is orthogonal. The Kronecker product of two orthogonal or two symplectic representations is orthogonal; the Kronecker product of an orthogonal and a symplectic representation is symplectic.

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[1] Note, however, the difficulties with relativistic extension of $SU(6)$; see, e.g., S. Coleman, Phys. Rev. **138**, B1262 (1965); M. A. B. Bég and A. Pais, Phys. Rev. Letters **14**, 509 (1965).

[2] That is, the "exceptional" groups G_2, F_4, E_6, E_7, and E_8 will not be considered.

[3] See, for example, E. B. Dynkin, Supplement to "Maximal Subgroups of the Classical Groups," *American Mathematical Society Translations Series 2* (American Mathematical Society, Providence, Rhode Island, 1957), Vol. 6, pp. 319–362 [Trudy Moskov. Mat. Obšč. **1**, 39 (1952)].

[4] A. I. Malcev, "On Semi-simple Subgroups of Lie Groups," *American Mathematical Society Translation Number 33* (American Mathematical Society, New York, 1950) [Izv. Akad. Nauk. SSR **8**, 143 (1944)].