the relative energy p_0.[12] They will, therefore, vanish *on* the mass shell apart from the infrared divergence difficulty in the Wick-Cutkosky model. Thus, in general, we may expect that our ghosts do not appear in the S-matrix.[13] This is quite a nice feature for positive use of ghosts, because one can get rid of Lehmann's theorem[14] without violating any one of relativity, causality, and unitarity. Our ghosts might, therefore, be useful for possible removal of the ultraviolet-divergence difficulty of the present field theory.

[12] Here, of course, we confine ourselves to considering the case in which the two particles have an equal mass. The unequal-mass case is subject to future investigation. In that case, we have to note that the relativistic relative momentum p_μ is not defined unambiguously.

[13] *Note added in proof.* With unequal masses m_1 and m_2, we can show that the solutions with odd κ vanish when

$$m_2(m_1^2-v)=m_1(m_2^2-w),$$

at least for $n=l+1$. Thus the unitarity will not be violated also in the unequal-mass case.

[14] H. Lehmann, Nuovo Cimento **11**, 342 (1954).

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"Simple" Alternatives to $SU(6)$*

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A search is made for simple classical Lie groups which could serve as alternatives to $SU(6)$. Among the groups with less than 140 generators, the most likely possibilities are found to be $R(11)$, with 55 generators, and $Sp(16)$, with 136 generators; but even these two have features which seem to discriminate against their use as particle symmetry groups.

I. INTRODUCTION

THE group $SU(6)$ shows every indication of becoming a successful symmetry group for particle physics.[1] But before we take $SU(6)$ entirely for granted, it seems appropriate to inquire into the existence of other groups which might serve equally well in the role of uniting spin and "unitary spin."

The present paper is a consideration of the most obvious class of alternatives to $SU(6)$: simple Lie groups which contain $SU(3)\otimes R(3)$ [or $SU(3)/Z_3 \otimes R(3)$] as a subgroup and offer a reasonable fit to the spectrum of known particles. The present search will be limited to the unitary, orthogonal, and symplectic groups[2]; thus our project is essentially the study of representations of $SU(3)\otimes R(3)$ by unitary, orthogonal, and symplectic matrices. The general process of determining representations of semisimple groups is well known[3]; and the problem of determining which are unitary, orthogonal, and symplectic has been attacked in generality by Malcev.[4] Thus we need only specialize these results and compare with the known spectrum of particles.

II. GROUPS CONTAINING $SU(3)\otimes R(3)$ AS A SUBGROUP

In order for $SU(3)\otimes R(3)$ to be a subgroup of the orthogonal group $R(N)$, it is necessary and sufficient that there exist a representation of $SU(3)\otimes R(3)$ by $N\times N$ orthogonal matrices; corresponding statements apply for the symplectic and unitary groups $Sp(N)$ and $SU(N)$. An arbitrary representation of $SU(3)\otimes R(3)$ can be written in the form

$$\sum_i \mathbf{X}_i\otimes \mathbf{Y}_i, \tag{1}$$

where $\mathbf{X}_i$ is an irreducible representation of $SU(3)$, $\mathbf{Y}_i$ is an irreducible representation of $R(3)$, and the sum is a direct sum. The irreducible representations of $SU(3)$ and of $R(3)$ are well known. In order to determine whether they are orthogonal, symplectic, or unitary, we shall make use of the following results quoted by Malcev[4]:

I. (Theorem 4 of Ref. 4.) The sum of two mutually contragredient representations is both orthogonal and symplectic.

II. (Lemma 1 of Ref. 4.) The Kronecker product of two contragredient representations is orthogonal. The Kronecker product of two orthogonal or two symplectic representations is orthogonal; the Kronecker product of an orthogonal and a symplectic representation is symplectic.

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[1] Note, however, the difficulties with relativistic extension of $SU(6)$; see, e.g., S. Coleman, Phys. Rev. **138**, B1262 (1965); M. A. B. Bég and A. Pais, Phys. Rev. Letters **14**, 509 (1965).

[2] That is, the "exceptional" groups G_2, F_4, E_6, E_7, and E_8 will not be considered.

[3] See, for example, E. B. Dynkin, Supplement to "Maximal Subgroups of the Classical Groups," *American Mathematical Society Translations Series 2* (American Mathematical Society, Providence, Rhode Island, 1957), Vol. 6, pp. 319–362 [Trudy Moskov. Mat. Obšč. **1**, 39 (1952)].

[4] A. I. Malcev, "On Semi-simple Subgroups of Lie Groups," *American Mathematical Society Translation Number 33* (American Mathematical Society, New York, 1950) [Izv. Akad. Nauk. SSR **8**, 143 (1944)].

Note also that an irreducible representation which is neither orthogonal nor symplectic is still equivalent to a unitary representation, when the group is compact.

We shall assume the following restrictions on physical grounds.

III. Particles with odd-half-integer spin shall not occur in the same representation as particles with even spin (to allow otherwise would presumably lead to problems with statistics).

IV. Particles with nonzero triality shall not occur in the same representation as particles with zero triality (even if such particles exist, they presumably have a mass considerably higher than the most familiar baryons and mesons).

From these we can deduce certain restrictions on the $\mathbf{X}_i$ and $\mathbf{Y}_i$ in the expressions (1) which map $SU(3)\otimes R(3)$ into certain classes of groups. All representations of a given unitary or symplectic group can be obtained from Kronecker products of the fundamental representation(s) (that is, the representations by $N\times N$ unitary or symplectic matrices, respectively). Thus, if the form (1) represents a mapping into a unitary or symplectic group, at least one of the $\mathbf{Y}_i$ must correspond to odd-half-integral spin, in order to have any representations with this feature. By III above, then:

(a) For a mapping into a unitary or symplectic group all the $\mathbf{Y}_i$ must have even dimension.

(b) In general, by III above, the $\mathbf{Y}_i$ in any expression (1) must be either all of even dimension or all of odd dimension.

(c) By IV above, all the $\mathbf{X}_i$ in any expression (1) must have the same triality.[5]

(d) Now we note that the representations **1** and **8** of $SU(3)$ are orthogonal. The representation $(\mathbf{3}+\bar{\mathbf{3}})$ is both orthogonal and symplectic (by I), but it is not allowed by (c); the same is true of $(\mathbf{6}+\bar{\mathbf{6}})$ or of $(\mathbf{3}+\bar{\mathbf{3}})+(\mathbf{6}+\bar{\mathbf{6}})$. There are no other symplectic or orthogonal representations of $SU(3)$ with dimension less than 20.

The representation **2** of $R(3)$ is symplectic, being in fact the fundamental representation of $Sp(2)$. From II, then:

(e) All even-dimensional representations of $R(3)$ are symplectic, and all odd-dimensional representations are orthogonal.

From (a) to (e), and II, we can now find the possible terms $\mathbf{X}_i\otimes\mathbf{Y}_i$ which can occur in (1); taking these in all possible combinations, we find the sums listed in

Table I. Possible expressions $\Sigma_i\, \mathbf{X}_i\otimes\mathbf{Y}_i$.[a]

Orthogonal ($N\leq 20$) N Odd (B_n)	Orthogonal ($N\leq 20$) N Even (D_n)	Symplectic (C_n) ($N\leq 18$)	Unitary (A_n)[b] ($N\leq 14$)
$8\otimes1+1\otimes3$	$(8\otimes1+1\otimes3+1\otimes3)$	$8\otimes2$	$3\otimes2$
$8\otimes1+1\otimes5$	$(8\otimes1+1\otimes3+1\otimes5)$	$8\otimes2+1\otimes2$	$3\otimes4$
$(8\otimes1+1\otimes7)$	$8\otimes1+1\otimes3+1\otimes7$		$6\otimes2$
$(8\otimes1+1\otimes9)$	$8\otimes1+1\otimes3+1\otimes9$		$3\otimes2+3\otimes2$
$8\otimes1+1\otimes11$	$(8\otimes1+1\otimes5+1\otimes5)$		
$8\otimes1+1\otimes3$ $+1\otimes3+1\otimes3$	$8\otimes1+1\otimes5+1\otimes7$		
$8\otimes1+1\otimes3$ $+1\otimes3+1\otimes5$	$(8\otimes1+1\otimes3+1\otimes3$ $+1\otimes3+1\otimes3)$		

[a] We have not listed expressions containing terms of the form $1\otimes1$.
[b] A second, trivially related, set is obtained by replacing 3 and 6 by $\bar{3}$ and $\bar{6}$, respectively.

Table I. The limitation that the sum (1) shall have dimension $N\leq 20$, 18, or 14 has been imposed on the assumption that we are not interested in groups with more than 200 generators.[6] Note that orthogonal terms of the form $\mathbf{1}\otimes\mathbf{Y}$ with $Y\geq 13$ need not be considered since they must be accompanied by the term $\mathbf{8}\otimes\mathbf{1}$, in order that $SU(3)$ be nontrivially represented; similarly for symplectic terms of the form $\mathbf{1}\otimes\mathbf{Y}$ with $Y\geq 4$.

However, it turns out that some of the expressions listed in the first two columns of Table I define a mapping into a group in such a way that none of the representations yield spin one-half states; such expressions have been enclosed in parentheses. Consider, for example, the mapping of $SU(3)\otimes R(3)$ into $R(14)$ by means of the representation $(\mathbf{8}\otimes\mathbf{1})+(\mathbf{1}\otimes\mathbf{3}+\mathbf{1}\otimes\mathbf{3})$. These matrices are a subgroup of the fundamental (or "vector") representation of $R(14)$, of course; but they are also a subgroup of the fundamental representation of $R(8)\otimes R(6)$. Thus we can do our mapping by way of $R(8)\otimes R(6)$:

$$SU(3)\otimes R(3)\rightarrow R(8)\otimes R(6)\rightarrow R(14)\,, \qquad (2)$$

where the second mapping is the obvious one. Now it is easy to show, by use of generalized Dirac matrices,[7] that under the mapping

$$R(M)\otimes R(N-M)\rightarrow R(N) \qquad (3)$$

any spinor representation of $R(N)$ decomposes into a direct product of the spinor representations of $R(M)$ and $R(N-M)$ (possibly two terms). So in the example being considered, the spinor representations of $R(14)$ will contain no spin one-half particles unless the spinor representation will contain spin one-half particles if the carries the $\mathbf{3}+\mathbf{3}$ reducible representation of $R(3)$ into the fundamental representation of $R(6)$]; furthermore, since all representations of $R(14)$ can be obtained from direct products of the spinor representations, no representation will contain spin one-half particles if the

[5] This may be an overly strong conclusion to draw from present experimental evidence, since there is the possibility that the fundamental representation (1) contains a mixture of triality states while the representations corresponding to observed particles do not. But since the observed zero-triality representations would then result from the decomposition of products of mixed-triality representations, this possibility does not seem likely, and will be ignored.

[6] The groups $R(2n+1)$, $R(2n)$, $Sp(2n)$, and $SU(n+1)$ (corresponding to the algebras B_n, D_n, C_n, and A_n, respectively) have $n(2n+1)$, $n(2n-1)$, $n(2n+1)$, and $n(n+2)$ generators, respectively.

[7] See, for example, D. Joseph, Phys. Rev. **126**, 319 (1962).

TABLE II. Allowable simple groups containing $SU(3)\otimes R(3)$ or $[SU(3)/Z_3]\otimes R(3)$.[a]

Compact group	Number of generators	Algebra	$\Sigma_i\, X_i \otimes Y_i$
$SU(6)$	35	A_5	$3\otimes 2$
$R(11)$	55	B_5	$8\otimes 1+1\otimes 3$
$R(13)$	78	B_6	$8\otimes 1+1\otimes 5$
$R(17)$	136	B_8	$8\otimes 1+1\otimes 3+1\otimes 3+1\otimes 3$
$Sp(16)$	136	C_8	$8\otimes 2$
$SU(12)$	143	A_{11}	$3\otimes 4$ or $6\otimes 2$ or $3\otimes 2+3\otimes 2$

[a] Trivial extensions such as $SU(7)\supset SU(6)\supset SU(3)\otimes R(3)$ are not included; these correspond to adding terms of the form $1\otimes 1$ to the expressions for $\Sigma_i\, X_i\otimes Y_i$ listed in Tables I and II.

spinor representation does not. Finally, direct examination shows that each of the spinor representations of $R(6)$ decomposes into the $\mathbf{3}+\mathbf{1}$ (reducible) representation of $R(3)$. Thus this mapping into $R(14)$ would lead to no spin one-half particles, and is therefore rejected.

If we arrange the possibilities of Table I according to the number of generators, up to 150, we obtain Table II.

III. FURTHER CONSIDERATION OF $R(11)$ AND $Sp(16)$

Since our purpose is to compare the predictions of $R(11)$ and $Sp(16)$ with those of $SU(6)$, we first display a list of the representations of $SU(6)$, in Table III. At first glance it might appear that $SU(6)$ is undesirable because of the large number of representations with nonzero triality, representations which have not so far been experimentally observed. However, these can be removed by specifying the symmetry group to be $SU(6)/Z$ [rather than $SU(6)$ itself], where Z is the appropriate discrete normal subgroup; we then allow only single-valued (true) representations of this group. Such a procedure is entirely analogous to that used to eliminate quarks from the $SU(3)$ scheme by modifying that group to be $SU(3)/Z_3$.

Table IV shows two apparent defects in the representations yielded by $R(11)$; first, we get not one, but two, octets of spin one-half baryons; and second, there are more spin-zero mesons than we would wish (at least at present). This doubling of the baryon octet carries over into all odd-half-integer spin representations, since these can all be obtained from direct products of the "spinor" representation with other representations. One thought might be to relate this doubling to the existence of antibaryons; however, there is no generator of $R(11)$ which distinguishes between such pairs—and hence no apparent source of an additive conserved quantity which could be identified with baryon number.

On the other hand, since $R(11)$ is based on the octet,

TABLE III. Representations of $SU(6)\,(A_5)$ with dimension ≤ 500.[a,b]

Dimension	Maximum weight	$SU(3)\otimes R(3)$ decomposition	Identification
6	[10000] (fundamental)	(3,2)	(Nonzero triality)
20	[00100]	(1,4)+(8,2)	Baryon–two-meson states
35	[10001] (adjoint)	(8,3)+(1,3)+(8,1)	Mesons
56	[30000]	(10,4)+(8,2)	Decuplet and octet baryons
70	[11000]	(8,4)+(10,2)+(8,2)+(1,2)	$Y_0^*(1405)$, "new γ octet",?
175	[00200]	(1,7)+(8,5)+(27,3)+(8,3)+(1,3)+(10,1)+($\overline{10}$,1)	...
189	[01010]	(8,5)+(1,5)+(10,3)+($\overline{10}$,3) +2(8,3)+(27,1)+(8,1)+(1,1)	Spin-two mesons plus?
280	[20010]	(10,5)+(8,5)+(27,3)+(10,3)+2(8,3) +(1,3)+(10,1)+($\overline{10}$,1)+(8,1)	...
405	[20002]	(27,5)+(8,5)+(1,5)+(27,3)+(10,3) +($\overline{10}$,3)+2(8,3)+(27,1)+(8,1)+(1,1)	...
462	[60000]	(28,7)+···	...
490	[03000]	(10,7)+(27,5)+(8,5)+(35,3)+(10,3) +($\overline{10}$,3)+(8,3)+(28,1)+(27,1)+(1,1)	Two-baryon states, including deuteron

Some products:

$56\otimes 35=1134'+700+70+56$

$56\otimes 56=1134_s+462_s+1050_a+490_a$

$35\otimes 35=405_s+189_s+35_s+1_s+\overline{280}_a+280_a+35_a$

$20\otimes 20=175_s+35_s+189_a+1_a$

[a] Mostly from F. Gürsey and L. A. Radicati, Phys. Rev. Letters 13, 173 (1964); M. A. B. Bég and V. Singh, Phys. Rev. Letters 13, 509 (1964); and F. J. Dyson and N.-H. Xuong, Phys. Rev. Letters 13, 815 (1964).

[b] Not listed above are the following representations corresponding to nonzero triality: 15, 21, 84, 105, 105′, 120, 126, 210, 210′, 252, 315, 336, 384, and 420.

TABLE IV. Representations of $R(11)\,(B_5)$ with dimension ≤ 1500.

Dimension	Maximum weight	$SU(3)\otimes R(3)$ decomposition	Possible identification
11	[10000] (fundamental)	(1,3)+(8,1)	Heavy mesons?
32	[00001] (spinor)	2(8,2)	Baryon octet plus?
55	[01000] (adjoint)	(8,3)+(1,3)+(10,1)+($\overline{10}$,1)+(8,1)	Mesons
65	[20000]	(1,5)+(8,3)+(27,1)+(8,1)+(1,1)	...
165	[00100]	(10,3)+($\overline{10}$,3)+2(8,3)+(27,1)+(10,1) +($\overline{10}$,1)+(8,1)+2(1,1)	*S*-wave two-baryon states
275	[30000]	(1,7)+(8,5)+(27,3)+(8,3)+(1,3)+(64,1) +(27,1)+(10,1)+($\overline{10}$,1)+(8,1)+(1,1)	...
320	[10001]	2[(8,4)+(27,2)+(10,2)+($\overline{10}$,2)+2(8,2)+(1,2)]	Excited baryons?
330	[00010]	(27,3)+2(10,3)+2($\overline{10}$,3)+2(8,3) +(1,3)+2(27,1)+3(8,1)	*S*-wave two-baryon states
429	[11000]	(8,5)+(1,5)+(27,3)+(10,3)+($\overline{10}$,3) +3(8,3)+(1,3)+(35,1)+($\overline{35}$,1) +2(27,1)+(10,1)+($\overline{10}$,1)+3(8,1)	...
462	[00002]	3(27,3)+(10,3)+($\overline{10}$,3)+3(8,3)+(1,3) +(27,1)+2(10,1)+2($\overline{10}$,1)+2(8,1)+(1,1)	...
935	[40000]	...	...
1144	[02000]	...	...
1408	[01001]	2[(27,4)+(10,4)+($\overline{10}$,4)+2(8,4)+(1,4)] and many spin one-half states from (35,2) on down	Baryon decuplet and other excited baryons

Some products:

$32\otimes 11=320+32$

$32\otimes 55=1408+320+32$

$32\otimes 65=1760+320$ where $1760=[20001]$

$32\otimes 32=462_s+55_s+11_s+330_a+165_a+1_a$

rather than triplet, representation of $SU(3)$, it has the pleasant feature of eliminating quarks in a very direct fashion, without resort to the requirement of single-valuedness of the representation which must be used in $SU(6)/Z$ symmetry.

The first three representations of $R(13)$ and $R(17)$ are displayed in Table V to demonstrate that these groups show the same troublesome features as $R(11)$, but in a more elaborate fashion. It is clear that inclusion of terms of the form $\mathbf{1}\otimes\mathbf{1}$ in the expressions (1), which has not been considered so far, would only lead to coalescence of representations into larger collections of multiplets; and this would not alleviate the short-comings of the rotation groups.

The next notable group after $R(11)$ is $Sp(16)$, the representations of which are shown in Table VI. This group has the appealing feature of being based directly on what might be considered the most basic of the representations physically observed, namely, the baryon octet. And the list of "small" representations is pleasingly brief, compared with those for $SU(6)/Z$ and $R(11)$. Unfortunately, such a strong symmetry as this is almost bound to conflict with experiment in short order; and in fact, the D/F ratio turns out to be zero

TABLE V. Smallest representations of $R(13)$ and $R(17)$.

Group	Dimension	$SU(3)\otimes R(3)$ decomposition
$R(13)$	13 (fundamental)	(1,5)+(8,1)
(B_6)	64 (spinor)	2(8,4)
	78 (adjoint)	(1,7)+(8,5)+(1,3)+($\overline{10}$,1)+(10,1)+(8,1)
$R(17)$	17 (fundamental)	3(1,3)+(8,1)
(B_8)	136 (adjoint)	3(1,5)+3(8,3)+6(1,3)+(10,1)+($\overline{10}$,1)+(8,1)+3(1,1)
	256 (spinor)	4(8,4)+8(8,2)

TABLE VI. Representations of $Sp(16)(C_8)$ with dimension ≤ 2000.

Dimension	Maximum weight	$SU(3)\otimes R(3)$ decomposition	Possible identification
16	[10000000] (fundamental)	(8,2)	Baryon octet
119	[01000000]	$(\mathbf{10},3)+(\overline{\mathbf{10}},3)+(8,3)+(27,1)+(8,1)$	S-wave two-baryon states
136	[20000000] (adjoint)	$(27,3)+(8,3)+(1,3)+(\mathbf{10},1)+(\overline{\mathbf{10}},1)+(8,1)$	Mesons
544	[00100000]	···	S-wave three-baryon states
816	[30000000]	$(64,4)+(27,4)+(\mathbf{10},4)+(\overline{\mathbf{10}},4)+(8,4)$ $+(1,4)+336$ states with spin one-half	Decuplet and other excited baryons
1344	[11000000]	···	Excited baryons
1700	[00010000]	···	S-wave four-baryon states

Some products:

$\mathbf{16}\otimes\mathbf{136}=\mathbf{816}+\mathbf{1344}+\mathbf{16}$

$\mathbf{16}\otimes\mathbf{16}=\mathbf{136}_s+\mathbf{119}_a+\mathbf{1}_a$

$\{\mathbf{16}\}_a{}^3=\mathbf{544}+\mathbf{16}\quad\{\mathbf{16}\}_s{}^3=\mathbf{816}$

$\{\mathbf{16}\}_a{}^4=\mathbf{1700}+\mathbf{119}+\mathbf{1}$

TABLE VII. Representations of $R(9)(B_4)$ with dimension ≤ 150.

Dimension	Maximum weight	$SU(3)\otimes R(3)$ decomposition
9	[1000] (fundamental)	$(3,1)+(\bar{3},1)+(1,3)$
16	[0001] (spinor)	$(3,2)+(\bar{3},2)+2(1,2)$
36	[0100] (adjoint)	$(3,3)+(\bar{3},3)+(1,3)+(8,1)+(3,1)+(\bar{3},1)+(1,1)$
44	[2000]	$(1,5)+(3,3)+(\bar{3},3)+(8,1)+(6,1)+(\bar{6},1)+(1,1)$
84	[0010]	$(8,3)+2(3,3)+2(\bar{3},3)+(1,3)+(6,1)+(\bar{6},1)+(3,1)+(\bar{3},1)+3(1,1)$
126	[0002]	$(8,3)+(6,3)+(\bar{6},3)+2(3,3)+2(\bar{3},3)+3(1,3)+(8,1)+2(3,1)+2(\bar{3},1)+(1,1)$
128	[1001]	$(3,4)+(\bar{3},4)+2(1,4)+2(8,2)+(6,2)+(\bar{6},2)+3(3,2)+3(\bar{3},2)+2(1,2)$

for the pseudoscalar mesons since **136** occurs only in the symmetric product of **16** with itself. Even if we suppose that $Sp(16)$ is a badly broken symmetry group, this result is very disappointing after the rather remarkable agreement obtained from $SU(6)/Z$. Of course, the group $Sp(16)$ does prohibit quarks in the same direct fashion as $R(11)$.

Finally, the representations of $R(9)$ are displayed in Table VII, as an example of the result of dropping restriction (c) of Sec. 2. [Next to $SU(6)$, this is the smallest simple group containing $SU(3)\otimes R(3)$.]

IV. CONCLUSION

Thus, out of 21 simple classical groups having between 24 and 140 generators [$SU(3)\otimes R(3)$ itself has 24], only $R(11)$, $R(13)$, $R(17)$, and $Sp(16)$ fulfill certain minimal requirements for possible alternatives to $SU(6)$. Furthermore, the two most promising of these, $R(11)$ and $Sp(16)$, have qualitative features which make them currently less appealing than $SU(6)$, even aside from the larger numbers of generators involved: 55 and 136, respectively, compared with 35 for $SU(6)$.

There remain yet to be considered the three "exceptional" groups, F_4, E_6, and E_7, that have between 24 and 140 generators. These three groups do indeed contain $SU(3)\otimes R(3)$ (as do most of the classical groups having more than 24 generators); but more detailed considerations are left to the more mathematically inclined reader.[8]

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[8] The general problem of finding subgroups of the exceptional groups has been treated by E. B. Dynkin, "Semisimple Subalgebras of Semisimple Lie Algebras," *American Mathematical Society Translations Series 2*, (American Mathematical Society, Providence, Rhode Island, 1957), Vol. 6, pp. 111–244 [Mat. Sbornik N.S. **30**(72), 349 (1952)].