tion of the final group. Moreover, for the 2+1 group, the spinors with upper and lower indices are equivalent so that one can get only those representations of the unitary group which are obtained only from one kind of spinor, i.e., representations by symmetric tensors. For example, for SU_3 or NU_3^2, multispinors formed out of the three-dimensional fundamental representations give only symmetric "quark" compounds. These are precisely the representations with one of the λ equal to zero in Eq. (3.5). All representations of SU_3 (NU_3^2) can be obtained by using both of the fundamental representations (i.e., quark-antiquark compounds). Because we start with one kind of 2+1 spinors, we see therefore why only special representations by symmetric tensors of the unitary groups are realized for the oscillator. Thus, for $N=3$, the interesting octet representation, for example, does not occur in the case of oscillator. To obtain an octet of SU_n, the basic dynamical objects, out of which one forms the compounds, must have at least two fundamental representations.

VI. APPLICATIONS TO OTHER SYSTEMS

It should be remarked that the essential features of the procedure we have followed are quite generally valid. Suppose we have a quantum-mechanical system with some energy or mass spectrum. If we form "direct products" of such systems we will get, in the same way as was discussed in Sec. III, a unitary degeneracy group and a larger noncompact dynamical group to describe all the states of the composite system. For the two- and three-dimensional oscillator we interpret part of the degeneracy with the spin. The original basic system can already have a spin degree of freedom.[3] In fact, we wish to start with a basic group which is larger than the 2+1 group and which has at least two fundamental spinor representations (corresponding to quarks and antiquarks), in order to get also the octet representation of SU_3, for example. The spin degeneracy will combine with the SU_N degeneracy due to the direct product to form a noncompact SU_{2N} group containing both the spin quantum number and the quantum numbers of the composite system.

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Function in Quantum Mechanics Which Corresponds to a Given Function in Classical Mechanics

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A generalization of the procedures of Weyl and McCoy for making transition from a classical function to a quantum-mechanical operator function corresponding to it has been carried out, in order to include terms involving more than one pair of conjugate variables. An application of the method is made to the case of two spinless interacting charges—the Darwin interaction. This result is verified by solving directly for the quantum-mechanical Hamiltonian operator, starting from equations which were derived from the classical equations by replacing the classical Poisson brackets by commutators and by requiring the equality of the energy and the Hamiltonian operator. An alternative expression for a quantized Hamiltonian, derived in the Appendix, reduces to the form given originally by Born and Jordan for a single set of conjugate variables.

I. INTRODUCTION

THE straightforward prescription for obtaining a quantum-mechanical Hamiltonian from a classical Hamiltonian $h(q_i,p_i)$ is to replace the canonical variables q_i and p_i by the corresponding operators Q_i and P_i, respectively, where $P_i=-i\hbar\partial/\partial Q_i$, thus leading to the quantum-mechanical Hamiltonian $H(Q_i,P_i)$.

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This prescription follows from the hypothesis of quantum theory that the commutator $[Q_i,P_i]=Q_iP_i-P_iQ_i=ih$. Unfortunately, there are many cases where the above procedure fails. For example, the familiar classical interaction Hamiltonian for a nonrelativistic particle of charge e in an electromagnetic field described by the vector potential $\mathbf{a}$ can be written in three equivalent forms, namely, (a) $-(e/2mc)(\mathbf{p}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{p})$, (b) $-(e/mc)\mathbf{p}\cdot\mathbf{a}$, and (c) $-(e/mc)\mathbf{a}\cdot\mathbf{p}$. However, these are not equivalent in going to quantum mechanics, since the operators[1] P_i and A_i do not commute. In this case since the forms (b) and (c) do not lead to Hermitian quantum-mechanical operators, they are rejected on the basis of this criterion. Since both P_i and A_i are themselves Hermitian, form (a) leads to a Hermitian interaction Hamiltonian. In more complicated cases, the requirement that the Hamiltonian operator be Hermitian is not a sufficient criterion for determining a quantization procedure. For instance, the classical function $p_iq_i^2q_kp_k$ can be put in the three symmetrical forms, (i) $\frac{1}{2}(p_iq_i^2q_kp_k+p_kq_kq_i^2p_i)$, (ii) $\frac{1}{2}(q_i^2q_kp_ip_k+p_kp_iq_kq_i^2)$, and (iii) $\frac{1}{4}(p_iq_i^2q_kp_k+p_kq_kq_i^2p_i+q_i^2q_kp_ip_k+p_kp_iq_kq_i^2)$, each one of which leads to a different Hermitian operator.

The problem of obtaining a method for quantizing a classical Hamiltonian was given some attention in the early literature. Weyl[2] proposed a method for resolving ambiguities which was based on group-theory considerations. Utilizing Weyl's results, McCoy[3] has derived an explicit expression for the quantum-mechanical operator corresponding to a polynominal in the conjugate coordinates of a classical system of one degree of freedom. His procedure is formally correct for infinite series. However, the classical Hamiltonian derived by Kerner,[4] and Daughaday and Nigam,[5] using the Fokker[6] and Wheeler-Feynman[6] scheme of classical electrodynamics of action-at-a-distance, contains terms involving more than one pair of conjugate variables, and Weyl's[2] and McCoy's[3] methods need to be extended to handle this case. This generalization has been carried out in this paper (Sec. III). As an example, in Sec. IV, the method is applied to obtain the quantum-mechanical Hamiltonian corresponding to the classical Hamiltonian for two interacting spinless charges (the Darwin interaction[7]). The result obtained is then verified, in Sec. V, by directly solving for the quantum-mechanical Hamiltonian from equations which are obtained from the classical equations by replacing the classical Poisson brackets by commutators and by requiring the equality of the energy and the Hamiltonian operator. The procedures adopted in obtaining the classical and the quantum-mechanical Hamiltonians in Secs. IV and V, make use of the assumption of the expansion of the Hamiltonian in a power series in the coupling parameter. In the Appendix, an alternative method of quantization has been carried out based on expressing a classical function by a nest of Poisson brackets and the replacement of classical Poisson brackets by commutators. This turns out to be a generalization of the method of quantization suggested by Born and Jordan.[8]

II. METHODS OF WEYL AND McCOY

Weyl[2] obtained the following general rule for carrying a function over from classical to quantum mechanics. Let $f(p,q)$ denote a function of the canonical variables p and q, of classical mechanics. In making the transition to quantum mechanics the variables p and q are replaced by the Hermitian operators P and Q which satisfy the commutation relation:

$$[P,Q]=PQ-QP=-ih. \tag{1}$$

The function $f(p,q)$ is expressed as a Fourier integral,

$$f(p,q)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d\sigma d\tau\, e^{i(\sigma p+\tau q)}\xi(\sigma,\tau). \tag{2}$$

The quantum-mechanical operator function corresponding to $f(p,q)$ is $F(P,Q)$ and is obtained by making the replacement $p\rightarrow P$, $q\rightarrow Q$ in Eq. (2) thus giving:

$$F(P,Q)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d\sigma d\tau\, e^{i(\sigma P+\tau Q)}\xi(\sigma,\tau). \tag{3}$$

McCoy[3] has obtained an explicit expression for the quantum-mechanical function $F(P,Q)$ corresponding to a polynomial function $f(p,q)$ of classical mechanics. His result is as follows: Any polynomial $F(P,Q)$ can be written in a form in which all the Q factors in each term occur on the left-hand side by making use of the commutation relation (1). This form of the function $F(P,Q)$ will be denoted by $F_Q(P,Q)$. The function of the classical commuting variables obtained by replacing P and Q in $F_Q(P,Q)$ by p and q, respectively, is denoted by $F_Q(p,q)$. In a similar manner $F_P(P,Q)$ and $F_P(p,q)$ may be defined. The classical functions $F_Q(p,q)$ and $F_P(p,q)$ are related to the classical function $f(p,q)$ by the following transformations:

$$\begin{aligned}F_Q(p,q)&=e^{-\frac{1}{2}ih(\partial^2/\partial p\partial q)}f(p,q)\\&=f(p,q)+(-\tfrac{1}{2}ih)\frac{\partial^2 f}{\partial p\partial q}\\&\quad+\frac{1}{2!}(-\tfrac{1}{2}ih)^2\frac{\partial^4 f}{\partial p^2\partial q^2}+\cdots\end{aligned} \tag{4}$$

[1] Capital letters are used to denote operators.
[2] H. Weyl, *The Theory of Groups and Quantum Mechanics* (E. P. Dutton and Company, Inc., New York, 1931), p. 274.
[3] N. H. McCoy, Proc. Natl. Acad. Sci. U. S. **18**, 674 (1932).
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[8] M. Born and P. Jordan, Z. Physik **34**, 874 (1925).

and

$$F_P(p,q)=e^{\frac{1}{2}i\hbar(\partial^2/\partial p\partial q)}f(p,q)$$
$$=f(p,q)+(\tfrac{1}{2}i\hbar)\frac{\partial^2 f}{\partial p\partial q}+\frac{1}{2!}(\tfrac{1}{2}i\hbar)^2\frac{\partial^4 f}{\partial p^2\partial q^2}+\cdots. \quad (5)$$

In order to obtain $F_Q(P,Q)$ from $F_Q(p,q)$ we need only to write $F_Q(p,q)$ with the q factors on the left in each term and then replace p, q by P, Q, respectively.

III. GENERALIZATION OF THE METHOD OF WEYL AND McCOY

We now generalize McCoy's[3] method. We take the case of two sets of variables and the general result for any number of pairs of conjugate coordinates is then found from this case by induction.

Let $f(p_1,q_1,p_2,q_2)$ be a function of the two sets of canonical variables p_1, q_1 and p_2, q_2. Its Fourier integral representation is given by

$$f(p_1,q_1,p_2,q_2)=\int\int\int\int d\sigma_1 d\tau_1 d\sigma_2 d\tau_2 \times e^{i(\sigma_1 p_1+\tau_1 q_1+\sigma_2 p_2+\tau_2 q_2)}\xi(\sigma_1,\tau_1,\sigma_2,\tau_2). \quad (6)$$

In analogy to the case of a single pair of conjugate variables, the quantum-mechanical Hermitian function $F(P_1,Q_1,P_2,Q_2)$ which corresponds to $f(p_1,q_1,p_2,q_2)$ is assumed to be

$$F(P_1,Q_1,P_2,Q_2)=\int\int\int\int d\sigma_1 d\tau_1 d\sigma_2 d\tau_2 \times e^{i(\sigma_1 P_1+\tau_1 Q_1+\sigma_2 P_2+\tau_2 Q_2)}\xi(\sigma_1,\tau_1,\sigma_2,\tau_2), \quad (7)$$
$$=\int\int d\sigma_1 d\tau_1 e^{i(\sigma_1 P_1+\tau_1 Q_1)}\times\int\int d\sigma_2 d\tau_2 e^{i(\sigma_2 P_2+\tau_2 Q_2)}\xi(\sigma_1,\tau_1,\sigma_2,\tau_2), \quad (8)$$

where in writing Eq. (8) we have made use of the fact that the conjugate variables P_1, Q_1 commute with P_2, Q_2. Next define $F_Q(P_1,Q_1,P_2,Q_2)$ as the form of the operator obtained when the commutation relations have been used to bring the Q_1 factors to the left of the P_1's and the Q_2 factors to the left of the P_2's. This rearrangement of Eq. (8) can be accomplished with the operator identity[9]

$$e^{i(\sigma P+\tau Q)}=e^{\frac{1}{2}i\sigma\tau\hbar}e^{i\tau Q}e^{i\sigma P}. \quad (9)$$

[9] Kermark and McCrea, Proc. Edinburgh Math. Soc. 2, 224 (1931); N. H. McCoy, *ibid.* 3, 121 (1932).

We then obtain

$$F_Q(P_1,Q_1,P_2,Q_2)=\int\int d\sigma_1 d\tau_1 e^{\frac{1}{2}i\sigma_1\tau_1\hbar}e^{i\tau_1 Q_1}e^{i\sigma_1 P_1}\times\left\{\int\int d\sigma_2 d\tau_2 e^{\frac{1}{2}i\sigma_2\tau_2\hbar}e^{i\tau_2 Q_2}e^{i\sigma_2 P_2}\xi(\sigma_1,\tau_1,\sigma_2,\tau_2)\right\}. \quad (10)$$

The corresponding classical expression can be written as

$$F_Q(p_1,q_1,p_2,q_2)=\int\int d\sigma_1 d\tau_1\left[1+(\tfrac{1}{2}i\sigma_1\tau_1\hbar)+\frac{1}{2!}(\tfrac{1}{2}i\sigma_1\tau_1\hbar)^2+\cdots\right]e^{i(\sigma_1 p_1+\tau_1 q_1)}\times\left\{\int\int d\sigma_2 d\tau_2\left[1+(\tfrac{1}{2}i\sigma_2\tau_2\hbar)+\frac{1}{2!}(\tfrac{1}{2}i\sigma_2\tau_2\hbar)^2+\cdots\right]\times e^{i(\sigma_2 p_2+\tau_2 q_2)}\xi(\sigma_1,\tau_1,\sigma_2,\tau_2)\right. \quad (11)$$

From Eq. (6) we obtain:

$$\frac{\partial^{2n}}{\partial p_1{}^n\partial q_1{}^n}\frac{\partial^{2l}}{\partial p_2{}^l\partial q_2{}^l}f=\int\int\int\int d\sigma_1 d\tau_1 d\sigma_2 d\tau_2(-\sigma_1\tau_1)^n(-\sigma_2\tau_2)^l\times e^{i(\sigma_1 p_1+\tau_1 q_1+\sigma_2 p_2+\tau_2 q_2)}\xi. \quad (12)$$

Substituting Eq. (12) in Eq. (11) we obtain

$$F_Q(p_1,q_1,p_2,q_2)=\sum_{n,l}\frac{1}{n!l!}(-\tfrac{1}{2}i\hbar)^{n+l}\frac{\partial^{2n}}{\partial p_1{}^n\partial q_1{}^n}\frac{\partial^{2l}}{\partial p_2{}^l\partial q_2{}^l}f(p_1,q_1,p_2,q_2)$$
$$=e^{-\frac{1}{2}i\hbar(\partial^2/\partial p_1\partial q_1)}e^{-\frac{1}{2}i\hbar(\partial^2/\partial p_2\partial q_2)}f(p_1,q_1,p_2,q_2). \quad (13)$$

The generalization of this expression to functions with n sets of canonical variables is as follows:

$$F_Q(p_1,q_1,\cdots p_n,q_n)=\left[\prod_{l=1}^{n}\exp(-\tfrac{1}{2}i\hbar(\partial^2/\partial p_l\partial q_l))\right]f(p_1,q_1,\cdots p_n,q_n). \quad (14)$$

The quantum-mechanical operator $F_Q(P_1,Q_1,\cdots P_n,Q_n)$ corresponding to $F_Q(p_1,q_1,\cdots p_n,q_n)$ is obtained by setting each q factor to the left of its conjugate p factors and by replacing p_1, q_1, $\cdots p_n$, q_n by P_1, Q_1, $\cdots P_n$, Q_n.

As an example, we consider the application of Eq. (14) to the term

$$f(p_1,q_1,\cdots p_n,q_n)=\varphi(q_1,q_2,\cdots q_n)p_1{}^\alpha p_2{}^\beta\cdots p_n{}^\gamma. \quad (15)$$

For this case, $F_Q(P_1,Q_1,\cdots P_n,Q_n)$ becomes

$$F_Q(P_1,Q_1,\cdots P_n,Q_n)$$
$$=\sum_{k_n=0}^{\gamma}(-\tfrac{1}{2}i\hbar)^{k_n}\Big\{\cdots\Big\{\sum_{k_2=0}^{\beta}(-\tfrac{1}{2}i\hbar)^{k_2}\Big\{\sum_{k_1=0}^{\alpha}(-\tfrac{1}{2}i\hbar)^{k_1}$$
$$\times\varphi_{k_1,k_2,\cdots k_n}(Q_1,\cdots Q_n)\binom{\alpha}{k_1}P_1{}^{\alpha-k_1}\Big\}$$
$$\times\binom{\beta}{k_2}P_2{}^{\beta-k_2}\Big\}\cdots\Big\}\binom{\gamma}{k_n}P_n{}^{\gamma-k_n}, \quad (16)$$

where

$$\binom{\alpha}{k_1}=\frac{\alpha!}{k_1!(\alpha-k_1)!}, \quad \varphi_{k_1}=\frac{\partial^{k_1}}{\partial q_1{}^{k_1}}\varphi,$$
$$\varphi_{k_1,k_2}=\frac{\partial^{k_1}}{\partial q_1{}^{k_1}}\frac{\partial^{k_2}}{\partial q_2{}^{k_2}}\varphi, \text{ etc.} \quad (17)$$

The derivatives occurring in Eq. (16) can be eliminated by making use of the operator identity:

$$P_1{}^t\varphi_{k_2,k_3,\cdots k_n}=\sum_{k_1=0}^{t}(-i\hbar)^{k_1}\binom{t}{k_1}\varphi_{k_1,k_2,\cdots k_n}P_1{}^{t-k_1}, \quad (18)$$

which can be proved by induction. Noting that

$$2^{\alpha-k_1}=(1+1)^{\alpha-k_1}=\sum_{l=0}^{\alpha-k_1}\binom{\alpha-k_1}{l},$$

we find that the term inside the innermost curly bracket in Eq. (16),

$$\sum_{k_1=0}^{\alpha}(-\tfrac{1}{2}i\hbar)^{k_1}\varphi_{k_1,\cdots k_n}\binom{\alpha}{k_1}P_1{}^{\alpha-k_1}$$
$$=2^{-\alpha}\sum_{k_1=0}^{\alpha}\sum_{l=0}^{\alpha-k_1}\binom{\alpha}{k_1}\binom{\alpha-k_1}{l}(-i\hbar)^{k_1}\varphi_{k_1,\cdots k_n}P_1{}^{\alpha-k_1}$$
$$=2^{-\alpha}\sum_{l=0}^{\alpha}\sum_{k_1=0}^{\alpha-l}(-i\hbar)^{k_1}\binom{\alpha}{l}\binom{\alpha-l}{k_1}\varphi_{k_1,\cdots k_n}P_1{}^{\alpha-l-k_1}P_1{}^l$$
$$=2^{-\alpha}\sum_{l=0}^{\alpha}\binom{\alpha}{l}P_1{}^{\alpha-l}\varphi_{k_2,\cdots k_n}P_1{}^l, \quad (19)$$

where in the last step we have made use of Eq. (18). After treating all the remaining sets of brackets in Eq. (16) in a similar manner, the expression for the operator $F(P_1,Q_1,\cdots P_n,Q_n)$ corresponding to $\varphi(q_1,\cdots,q_n)\times p_1{}^\alpha p_2{}^\beta\cdots p_n{}^\gamma$ becomes

$$F(P_1,Q_1,\cdots P_n,Q_n)$$
$$=2^{-\gamma}\sum_{s=0}^{\gamma}\binom{\gamma}{s}P_n{}^{\gamma-s}\Big\{\cdots\Big\{2^{-\beta}\sum_{m=0}^{\beta}\binom{\beta}{m}P_2{}^{\beta-m}\Big\{2^{-\alpha}\sum_{l=0}^{\alpha}\binom{\alpha}{l}$$
$$\times P_1{}^{\alpha-l}\varphi(Q_1,\cdots Q_n)P_1{}^l\Big\}P_2{}^m\Big\}\cdots\Big\}P_n{}^s. \quad (20)$$

Equation (20) can be expressed in an alternative form by making use of the anticommutator brackets. By induction, it is easy to prove that

$$\sum_{l=0}^{n}\binom{n}{l}P_1{}^{n-l}\varphi P_1{}^l$$
$$=[[\cdots[[\varphi,P_1]_+,P_1]_+\cdots]_+,P_1]_+=[\varphi,P_1]_{+n}, \quad (21)$$

where $[\varphi,P_1]_+=\varphi P_1+P_1\varphi$, and the subscript $+n$ indicates that n successive anticommutator brackets are to be taken with operator P_1. Substituting Eq. (21) into Eq. (20), we obtain the following expression for the quantum-mechanical operator:

$$F(P_1,Q_1,\cdots P_n,Q_n)$$
$$=2^{-(\alpha+\beta+\cdots\gamma)}[[\cdots[[\varphi,P_1]_{+\alpha},P_2]_{+\beta},\cdots],P_n]_{+\gamma}. \quad (22)$$

Equation (22) is still valid when two or more of the P's are the same operator.

IV. QUANTIZATION BY THE GENERALIZED WEYL-McCOY METHOD

In the previous section we have demonstrated how transition can be made from a classical function to a quantum-mechanical operator by employing an extension to Weyl's,[2] and McCoy's[3] quantization procedures [Eqs. (15) and (22)]. We shall now write down the quantum-mechanical Hamiltonian corresponding to the classical Hamiltonian pertaining to a physical problem, using the generalized form of McCoy's method. In order to have a check on this procedure, we shall, in the following section, indicate an alternative approach for obtaining a quantized Hamiltonian.

The physical problem, we consider, is the interaction of two spinless charges e_a and e_b, having masses m_a and m_b, via retarded fields,[10] and confine ourselves to terms up to order c^{-2}. This is the so-called Darwin interaction. The steps involved in our derivation[4,5] of the classical Hamiltonian are summarized here in some detail since this was done in an unfamiliar manner and also since the quantization procedure discussed in Sec. V follows an analogous pattern. The derivation starts by finding a constant of motion for the system E which can be identified as the energy, given by

$$E=m_ac^2[1-(\mathbf{v}_a{}^2/c^2)]^{-1/2}+m_bc^2[1-(\mathbf{v}_b{}^2/c^2)]^{-1/2}+e_ae_bg, \quad (23)$$

where

$$g=(1/r)+(1/2c^2)[2\dot{\mathbf{x}}_a\cdot\dot{\mathbf{x}}_b(1/r)+D_aD_b\mathbf{r}]+(1/c^4)[\cdots]+\cdots, \quad (24)$$

$$\mathbf{r}=\mathbf{x}_a(t)-\mathbf{x}_b(t), \quad \mathbf{v}_a=\dot{\mathbf{x}}_a, \quad \mathbf{v}_b=\dot{\mathbf{x}}_b. \quad (25)$$

[10] To order c^{-2}, the prescription of Wheeler and Feynman (Ref. 6) of using half-retarded + half-advanced fields between pairs of interacting charges gives the interaction energy identical with Darwin's result obtained by using retarded fields only (see Ref. 4).

The operators D_a and D_b were introduced[4,5] in finding the constant of motion E. D_a (and D_b) denotes time differentiation treating $\mathbf{x}_a(t)$ [and $\mathbf{x}_b(t)$] as being the only time-dependent coordinate. After operating with D_a and D_b the expression for g becomes a function of the ordinary time derivatives. The higher order terms in g involve higher order time derivatives. Kerner[4] has suggested a general procedure for obtaining the classical joint Hamiltonian h which is time-independent and equal to the energy. His procedure is as follows. Assume that the Hamiltonian h can be expanded in powers of the coupling constant $\epsilon=e_ae_b$ so that

$$h(\mathbf{x}_a,\mathbf{x}_b,\mathbf{p}_a,\mathbf{p}_b)=h_0+\epsilon h_1+\epsilon^2h_2+\cdots. \tag{26}$$

The Hamiltonian formalism provides a method for obtaining time derivatives of the x's as functions of the x's and p's, and these derivatives can be written as follows in terms of Poisson brackets.

$$\begin{aligned}&\dot{\mathbf{x}}_a=(\mathbf{x}_a,h),\\ &\ddot{\mathbf{x}}_a=(\dot{\mathbf{x}}_a,h)=((\mathbf{x}_a,h),h),\ \text{etc.},\end{aligned} \tag{27}$$

where

$$(f,h)=\sum_{i=1}^{3}\left[\left(\frac{\partial f}{\partial x_{ai}}\frac{\partial h}{\partial p_{ai}}-\frac{\partial f}{\partial p_{ai}}\frac{\partial h}{\partial x_{ai}}\right)+\left(\frac{\partial f}{\partial x_{bi}}\frac{\partial h}{\partial p_{bi}}-\frac{\partial f}{\partial p_{bi}}\frac{\partial h}{\partial x_{bi}}\right)\right], \tag{28}$$

and Eq. (26) is substituted for h in each of the brackets of Eq. (27). Then the requirement that the Hamiltonian should be equal to the energy can be written in the form,

$$\begin{aligned}h&=h_0+\epsilon h_1+\epsilon^2h_2+\cdots\\ &=E\{\epsilon,r,\mathbf{r}/r,\dot{\mathbf{x}}_a,\dot{\mathbf{x}}_b,\ddot{\mathbf{x}}_a,\ddot{\mathbf{x}}_b,\cdots\}\\ &=E\{\epsilon,r,\mathbf{r}/r,(\mathbf{x}_a,h),(\mathbf{x}_b,h),((\mathbf{x}_a,h),h),\cdots\}.\end{aligned} \tag{29}$$

Equating like powers of ϵ, we obtain

$$h_n=\frac{1}{n!}\left[\frac{\partial^nE}{\partial\epsilon^n}\right]_{\epsilon=0},\quad n=0,1,2,\cdots. \tag{30}$$

Equation (30) leads to a set of partial differential equations which must be solved successively; the equation for a given h_n will only involve other h_i's with $i<n$. The partial differential equations for h_0 and h_1 are as follows:

$$h_0=m_ac^2[1-(\mathbf{x}_a,h_0)^2/c^2]^{-1/2}+m_bc^2[1-(\mathbf{x}_b,h_0)^2/c^2]^{-1/2}, \tag{31}$$

$$\begin{aligned}h_1&=\left[\left(\frac{\mathbf{p}_a}{m_ac}\right)^2+1\right]\mathbf{p}_a\cdot\frac{\partial h_1}{\partial\mathbf{p}_a}-\left[\left(\frac{\mathbf{p}_b}{m_bc}\right)^2+1\right]\mathbf{p}_b\cdot\frac{\partial h_1}{\partial\mathbf{p}_b}\\ &=-\frac{1}{r}+\frac{1}{2r}\frac{(\mathbf{p}_a\cdot\mathbf{p}_b)}{p_{a4}p_{b4}}+\frac{1}{2r^3}\frac{(\mathbf{r}\cdot\mathbf{p}_a)(\mathbf{r}\cdot\mathbf{p}_b)}{p_{a4}p_{b4}}+O\left(\frac{1}{c^4}\right),\end{aligned} \tag{32}$$

where $p_{a4}=(\mathbf{p}_a^2+m_a^2c^2)^{1/2}$ and we have made use of

$$\left[\frac{\partial}{\partial\epsilon}\dot{\mathbf{x}}_a\right]_{\epsilon=0}=\left[\frac{\partial}{\partial\epsilon}(\mathbf{x}_a,h)\right]_{\epsilon=0}=\frac{\partial h_1}{\partial\mathbf{p}_a}$$

in Eq. (32). The particular solution of Eqs. (31) and (32) can be shown to be

$$h_0=c(\mathbf{p}_a^2+m_a^2c^2)^{1/2}+c(\mathbf{p}_b^2+m_b^2c^2)^{1/2}, \tag{31a}$$

$$h_1=-\frac{1}{r}\left\{1-\frac{1}{2}\frac{(\mathbf{p}_a\cdot\mathbf{p}_b)}{p_{a4}p_{b4}}\right\}-\frac{1}{2r^3}\frac{(\mathbf{r}\cdot\mathbf{p}_a)(\mathbf{r}\cdot\mathbf{p}_b)}{p_{a4}p_{b4}}+O(c^{-4}), \tag{32a}$$

which to terms of order c^{-2} give the result

$$\begin{aligned}h=\sum_{k=a,b}&\{m_kc^2+(1/2m_k)\mathbf{p}_k^2-(1/8m_k^3c^2)(\mathbf{p}_k^2)^2\}\\ &+\frac{\epsilon}{r}\left\{1-\frac{1}{2}\frac{(\mathbf{p}_a\cdot\mathbf{p}_b)}{m_am_bc^2}\right\}-\frac{\epsilon}{2r^3}\frac{(\mathbf{r}\cdot\mathbf{p}_a)(\mathbf{r}\cdot\mathbf{p}_b)}{m_am_bc^2},\end{aligned} \tag{33}$$

in agreement with Darwin.[7] The quantum-mechanical Hamiltonian corresponding to Eq. (33) can be written quite readily, using Eq. (22) and is given by

$$\begin{aligned}H=\sum_{k=a,b}&\{m_kc^2+(1/2m_k)P_{ki}^2-(1/8m_k^3c^2)(P_{ki}^2)^2\}\\ &+\epsilon\{(1/R)-(1/8m_am_bc^2)([[(1/R),P_{ai}]_+,P_{bi}]_+\\ &+[[(1/R^3)R_iR_j,P_{ai}]_+,P_{bj}]_+)\},\end{aligned} \tag{34}$$

where i and j run over the components 1, 2, 3 and capitals denote operators.

V. METHOD OF SOLVING DIRECTLY FOR A QUANTIZED HAMILTONIAN

We shall now consider a method of solving for a quantum-mechanical Hamiltonian. We make use of the correspondence between the classical Poisson brackets and the quantum-mechanical commutators. Equations (27) to (29) are replaced by the following equations:

$$\begin{aligned}&\partial\mathbf{X}_a/\partial t=(i\hbar)^{-1}[\mathbf{X}_a,H]=(i\hbar)^{-1}[\mathbf{X}_aH-H\mathbf{X}_a],\\ &\partial^2\mathbf{X}_a/\partial t^2=(i\hbar)^{-2}[[\mathbf{X}_a,H],H],\ \text{etc.},\end{aligned} \tag{35}$$

$$\begin{aligned}&H_0+\epsilon H_1+\epsilon^2H_2+\cdots\\ &=\sum_{k=a,b}\{m_kc^2+\tfrac{1}{2}m_k(\partial\mathbf{X}_k/\partial t)^2+(3/8c^2)m_k[(\partial\mathbf{X}_k/\partial t)^2]^2+\cdots\}+\epsilon G\\ &\simeq\sum_{k=a,b}\left\{m_kc^2+\frac{m_k}{2(i\hbar)^2}[\mathbf{X}_k,H_0+\epsilon H_1+\cdots]^2+\frac{3m_k}{8c^2(i\hbar)^4}[\mathbf{X}_k,H_0+\epsilon H_1+\cdots]^4\right\}+\epsilon G,\end{aligned} \tag{36}$$

where in deriving Eq. (36) we have assumed that the classical expression for the energy E, Eq. (23), is

interpretable as the quantum-mechanical Hamiltonian H after all the classical quantities occurring in Eq. (23) are replaced by operators. The operator G corresponds to g. Proceeding as in the classical case the equations determining $H_0, H_1, \cdots$ are obtained by equating like powers of ϵ in Eq. (36). The equation for H_0 has the following particular solution:

$$H_0=\sum_{k=a,b}[m_kc^2+(2m_k)^{-1}\mathbf{P}_k^2-(8m_k^3c^2)^{-1}(\mathbf{P}_k^2)^2] \quad (37)$$

as can readily be verified by direct substitution. The equation for H_1, obtained from the coefficients of ϵ, takes the form

$$\begin{aligned}H_1-\sum_{k=a,b}&[(2(i\hbar))^{-1}\{(\mathbf{P}_k-(2m_k^2c^2)^{-1}\mathbf{P}_k^2\mathbf{P}_k)\cdot[\mathbf{X}_k,H_1]\\&+[\mathbf{X}_k,H_1]\cdot(\mathbf{P}_k-(2m_k^2c^2)^{-1}\mathbf{P}_k^2\mathbf{P}_k)\}\\&+3(8m_k^2c^2(i\hbar))^{-1}\{\mathbf{P}_k\cdot\mathbf{P}_k^2[\mathbf{X}_k,H_1]+\mathbf{P}_k^2[\mathbf{X}_k,H_1]\cdot\mathbf{P}_k\\&+\mathbf{P}_k\cdot[\mathbf{X}_k,H_1]\mathbf{P}_k^2+[\mathbf{X}_k,H_1]\mathbf{P}_k^2\cdot\mathbf{P}_k\}]=[G]_{\epsilon=0}, \quad (38)\end{aligned}$$

where use has been made of Eqs. (1) and (37). Since Eq. (38) is linear, the particular solutions corresponding to individual terms in G can be obtained separately and the results superimposed. It can be seen that there are no ambiguities introduced in obtaining the quantized form in Eq. (37) or on the left-hand side of Eq. (38). In order that a particular solution of Eq. (38) gives an unambiguous expression for H_1, the operator on the right-hand side of this equation must also be obtained unambiguously from the classical g. A possible scheme for accomplishing this would involve expressing each term in g by a combination of classical D_a and D_b time derivatives operating on a function of $\mathbf{r}$. Then if D_a and D_b go over into commuting quantum-mechanical operators, the quantization of g would be unambiguous.

We consider the following term of Eq. (24)

$$\begin{aligned}\Delta g&=(1/2c^2)D_aD_br\\&=-(1/2c^2r)(\dot{\mathbf{x}}_a\cdot\dot{\mathbf{x}}_b)+(1/2c^2r^3)(\dot{\mathbf{x}}_a\cdot\mathbf{r})(\dot{\mathbf{x}}_b\cdot\mathbf{r}). \quad (39)\end{aligned}$$

It is to be noted that the expression Δg is already in the desired classical form. In the following discussion we will retain only this term although it is by no means clear whether all terms in the classical g can be expressed in the postulated form.

Before proceeding to solve the operator Eq. (38) with $G\rightarrow\Delta G$, we must determine the quantum-mechanical form $[\Delta G]_{\epsilon=0}$. According to Eq. (35), in going to quantum mechanics $[(D_a+D_b)f]_{\epsilon=0}\rightarrow(i\hbar)^{-1}[F,H_0]$. The quantum-mechanical operators corresponding to the individual time derivatives D_a and D_b should imply $[D_af]_{\epsilon=0}=(i\hbar)^{-1}[f,H_0^{(a)}]$, $[D_bf]_{\epsilon=0}=(i\hbar)^{-1}[f,H_0^{(b)}]$ if the Hamiltonian separates into two parts $H_0^{(a)}$ and $H_0^{(b)}$ associated with particles a and b respectively. From Eq. (37), it is clear that $H_0^{(a)}=m_ac^2+(2m_a)^{-1}\mathbf{P}_a^2-(8m_a^3c^2)^{-1}(\mathbf{P}_a^2)^2$ and a similar expression holds for $H_0^{(b)}$. It is then possible to determine the meaning of the operator $[\Delta G]_{\epsilon=0}$ as follows:

$$\begin{aligned}[\Delta G]_{\epsilon=0}&=(2c^2)^{-1}[D_aD_bR]_{\epsilon=0}\\&=(2c^2(i\hbar)^2)^{-1}[[R,(2m_b)^{-1}\mathbf{P}_b^2-(8m_b^3c^2)^{-1}\\&\quad\times(\mathbf{P}_b^2)^2],(2m_a)^{-1}\mathbf{P}_a^2-(8m_a^3c^2)^{-1}(\mathbf{P}_a^2)^2]. \quad (40)\end{aligned}$$

Noting that

$$[R,P^2]=[R,P]P+P[R,P], \quad [R,\mathbf{P}_b]=(i\hbar/R)\mathbf{R},$$

and similar relations, the right-hand side of Eq. (40) can be simplified, giving the following result to terms of order c^{-2},

$$\begin{aligned}[\Delta G]_{\epsilon=0}=&-(8m_am_bc^2)^{-1}\{R^{-1}\mathbf{P}_a\cdot\mathbf{P}_b+\mathbf{P}_a\cdot R^{-1}\mathbf{P}_b\\&+\mathbf{P}_b\cdot R^{-1}\mathbf{P}_a+\mathbf{P}_a\cdot\mathbf{P}_bR^{-1}-R^{-3}\mathbf{R}(\mathbf{R}\cdot\mathbf{P}_b)\cdot\mathbf{P}_a\\&-\mathbf{P}_a\cdot R^{-3}\mathbf{R}(\mathbf{R}\cdot\mathbf{P}_b)-(\mathbf{P}_b\cdot\mathbf{R})R^{-3}(\mathbf{R}\cdot\mathbf{P}_a)\\&-\mathbf{P}_a\cdot(\mathbf{P}_b\cdot\mathbf{R})R^{-3}\mathbf{R}\}. \quad (41)\end{aligned}$$

A particular solution of the operator Eq. (38) with $[\Delta G]_{\epsilon=0}$ [i.e., Eq. (41)] substituted for the right-hand side is

$$\Delta H_1=-[\Delta G]_{\epsilon=0}. \quad (42)$$

In order to check the agreement of Eq. (42) with the result obtained by the generalized McCoy's method, Eq. (22), we must replace g by Δg in Eq. (23), obtaining Δh_1 instead of h_1, Eq. (32a), for the classical Hamiltonian, where

$$\Delta h_1=(2m_am_bc^2)^{-1}\{r^{-1}\mathbf{p}_a\cdot\mathbf{p}_b-r^{-3}(\mathbf{r}\cdot\mathbf{p}_a)(\mathbf{r}\cdot\mathbf{p}_b)\}. \quad (43)$$

The quantum-mechanical Hamiltonian operator ΔH_1 corresponding to Eq. (43), resulting from Eq. (22), is found to be in agreement with Eqs. (42) and (41), thus exhibiting, in this particular example, the equivalence of our generalization of the methods of Weyl and McCoy with the method of direct solution for the quantum-mechanical operator.

It is to be noted that the results given in Eqs. (42) and (41) cannot be obtained by the combination of all possible Hermitian arrangements of the variables. For example, the combination

$$(\mathbf{R}\cdot\mathbf{P}_aR^{-1}\mathbf{R}\cdot\mathbf{P}_bR^{-2}+R^{-2}\mathbf{P}_b\cdot\mathbf{R}R^{-1}\mathbf{P}_a\cdot\mathbf{R})$$

does not appear and is not equivalent to the Hermitian pairs which are present.

VI. CONCLUSION

The methods of Weyl[2] and McCoy[3] for carrying a classical function $f(p,q)$ of canonical variables p and q, to the quantum-mechanical operator function $F(P,Q)$ corresponding to it, have been extended to terms involving more than one pair of conjugate variables. Such generalization of Weyl's[2] and McCoy's[3] result is of particular usefulness if one is interested in transcribing complicated functions[4,5] (Hamiltonian) into quantum mechanics. The method is illustrated by applying it to the physical problem of two interacting charges which have zero spins; terms to order c^{-2} in the interaction

have been retained. The classical Hamiltonian for this problem is transformed into quantum mechanics, using the general procedure of quantization developed here. The validity of this result is indicated by a method of solving directly for a quantized Hamiltonian from equations, which are derived by equating the classical energy E to the Hamiltonian operator H after classical quantities occurring in E have been replaced by operators.

Another method of quantizing a classical function is derived in the Appendix, which can be reduced to the prescription given by Born and Jordan[8] for quantizing p^rq^s. This method is not in general equivalent to the generalized Weyl-McCoy method but gives the same quantum-mechanical Hamiltonian for the example of two interacting charges considered here.

In[11] a subsequent paper[5] the method developed here is applied to quantize the classical Hamiltonian for two spin-zero charged particles, interacting through $\frac{1}{2}$ retarded $+\frac{1}{2}$ advanced potentials. The Hamiltonian is derived by an extension of the method used by Kerner[4] and the calculations are carried out to include terms to order e^4 and c^{-4}. However, the problem of physical interest is the interaction of two spin-$\frac{1}{2}$ charged particles for which the Hamiltonian generally used is made up of the Coulomb and the Breit operators in addition to the Dirac free-particle terms; the Breit interaction being specified to terms of order e^2 and $(v/c)^2$. The problem of determining the character of the higher order terms in the spin-dependent Hamiltonian is investigated[5] by introducing spin phenomenologically into the Hamiltonian derived to order e^4 and c^{-4} for two spin-zero charged particles. Several *ad hoc* assumptions are found necessary in carrying out this process. The guiding principle utilized in developing a set of rules which enable the introduction of spin is that the classical (relativistic) one particle Hamiltonian corresponds to the Dirac Hamiltonian. It is then possible to show, using these rules, that the classical (relativistic) Hamiltonian of a particle interacting with an external potential, when expanded in powers of e, and after replacing Poisson brackets by commutators, yields the corresponding Dirac Hamiltonian. Also to orders e^2 and $(v/c)^2$, the spin-dependent Hamiltonian derived phenomenologically from the classical Hamiltonian is found to be in agreement with the Breit interaction.

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[11] This paragraph was added in the revised manuscript (BPN).

APPENDIX

Alternative Method of Quantization

Consider the classical function

$$f(p_1,q_1,\cdots p_n,q_n)=\varphi(q_1,q_2,\cdots q_n)p_1{}^{\alpha}p_2{}^{\beta}\cdots p_n{}^{\gamma}. \quad \text{(A1)}$$

A function of this type can always be expressed as a succession of Poisson brackets as follows:

$$f(p_1,q_1,\cdots p_n,q_n)=[(\alpha+1)(\beta+1)\cdots]^{-1}\times(\cdots(\phi,p_1{}^{\alpha+1}),p_2{}^{\beta+1},\cdots p_n{}^{\gamma+1}), \quad \text{(A2)}$$

where

$$\phi=\int dq_1\cdots\int dq_n\varphi(q_1,q_2,\cdots q_n). \quad \text{(A3)}$$

Since the operators $P_1,\cdots P_n$ which correspond to $p_1,\cdots p_n$ commute, the transition to quantum mechanics can be made by replacing the Poisson brackets by commutator brackets giving,

$$F(P_1,Q_1,\cdots P_n,Q_n)=[(\alpha+1)\cdots(\gamma+1)]^{-1}(i\hbar)^n\times[\cdots[[\Phi,P_1{}^{\alpha+1}]_-,P_2{}^{\beta+1}]_-,\cdots P_n{}^{\gamma+1}]_-. \quad \text{(A4)}$$

Considering a single bracket

$$\begin{aligned}&[i\hbar(\alpha+1)]^{-1}[\Phi,P_1{}^{\alpha+1}]\\&=[i\hbar(\alpha+1)]^{-1}\{\Phi P_1{}^{\alpha+1}-P_1{}^{\alpha+1}\Phi\}\\&=[i\hbar(\alpha+1)]^{-1}\{(\Phi P_1-P_1\Phi)P_1{}^{\alpha}\\&\quad+P_1(\Phi P_1-P_1\Phi)P_1{}^{\alpha-1}+\cdots+P_1{}^{\alpha}(\Phi P_1-P_1\Phi)\}\\&=(\alpha+1)^{-1}\sum_{i=0}^{\alpha}P_1{}^{i}(\partial\Phi/\partial q_1)P_1{}^{\alpha-i}. \quad \text{(A5)}\end{aligned}$$

Proceeding in a similar manner with the remaining brackets, we obtain

$$\begin{aligned}&F(P_1,Q_1,\cdots P_n,Q_n)\\&=[(\alpha+1)\cdots(\gamma+1)]^{-1}\sum_{k=0}^{\gamma}\cdots\sum_{j=0}^{\beta}\sum_{i=0}^{\alpha}P_n{}^{k}\cdots\\&\qquad\times P_2{}^{j}P_1{}^{i}\Phi P_1{}^{\alpha-i}P_2{}^{\beta-j}\cdots P_n{}^{\gamma-k}, \quad \text{(A6)}\end{aligned}$$

which turns out to be a generalization of an expression given by Born and Jordan.[8] The operator $F(P_1,Q_1,\cdots P_n,Q_n)$ in Eq. (A6) is not in general the same as the generalized McCoy operator, Eq. (22), but the two expressions give identical results for the Darwin Hamiltonian. The values of operators obtained by the two methods differ by terms of order $\hbar^2$ in several cases considered. This difference has also been noted by McCoy,[3] namely, that his expression for the operator corresponding to the classical term p^rq^s would not be the same as the one given by Born and Jordan[8] if r and s are both greater than unity.