APPENDIX B: SPIN TRACE FOR THE TARGET-PROTON MOMENTUM TERM

The calculations involved are completely similar to those in Appendix A. For convenience we will consider separately the contributions for the g_A and g_V terms.

(1) g_A term: The contribution to the triplet state of two neutrons can be written as

$$(2g_A/M)\,\mathrm{Tr}[\mathfrak{M}\rho_{1/2}(\boldsymbol{\sigma}_1\cdot\mathbf{p}_1)P_t] = (2g_A/3M)(G_P-2G_A+G_V)(\hat{p}\cdot\mathbf{p}_1);$$

likewise, the corresponding term for the two-neutron singlet state is

$$(2g_A/M)\,\mathrm{Tr}[\mathfrak{M}\rho_{1/2}(\boldsymbol{\sigma}_1\cdot\mathbf{p}_1)P_s] = (g_A/M)(\tfrac{1}{3}G_P-G_A)(\hat{p}\cdot\mathbf{p}_1).$$

(2) g_V term: Only the two-neutron triplet state contributes:

$$(2g_V/M)\,\mathrm{Tr}[\mathfrak{M}\rho_{1/2}L(\boldsymbol{\sigma}\cdot\mathbf{P}_1)P_t] = (g_V/M)(\tfrac{1}{3}G_P-G_A-G_V)(\hat{p}\cdot\mathbf{p}_1).$$

In the above, M = proton mass, $L=\frac{1}{2}(1-\boldsymbol{\sigma}\cdot\hat{p})$, and

$$\mathfrak{M}=\tfrac{1}{2}(1-\boldsymbol{\sigma}\cdot\hat{p})[G_V+G_A(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}_1)+G_P(\hat{p}\cdot\boldsymbol{\sigma}_1)].$$

PHYSICAL REVIEW VOLUME 139, NUMBER 6B 20 SEPTEMBER 1965

Study of the "Breakup" Channels of Muon Capture by He^3†

I-T. Wang*
Columbia University, New York, New York

(Received 5 April 1965)

Neutron energy spectra are obtained for the processes of muon capture by He^3 into the "breakup" channels: $\mu+He^3\to\nu+n+d$ and $\mu+He^3\to\nu+2n+p$. The chief approximations are the use of plane waves for the relative motions between the final nuclear particles and a Gaussian-type wave function for the He^3 nucleus. The rates obtained for the two "breakup" channels are, respectively, 988 and 272 sec^{-1}.

I. INTRODUCTION

THE process of muon capture by He^3 into H^3 and neutrino has been the object of a great deal of experimental and theoretical interest. This is due, in part, to the relatively simple and fundamental nature of the He^3 nucleus and in part to the unique feature of the recoil triton that lends itself to refined experimental observations. Much less, however, has been done about the competing channels of muon capture by He^3. In this paper we wish to describe a brief effort to study these "breakup" channels:

$$\mu+He^3\to\nu+n+d,$$
$$\to\nu+2n+p.$$

It is not difficult to recognize the complexity of the nuclear physics involved in these reactions. Some of the problems are still far from being thoroughly understood. Consequently, the calculations we have made can only be said to be preliminary in nature, but we believe they help to reveal a few interesting features of the problem. In the meantime it is possible, with the help of the calculated neutron energy spectra, to make a comparison between the theory and the observed neutron rate. Such a comparison is reported in a separate paper.[1]

Aside from the practical aspects of the problem, there is an intrinsic theoretical interest in such calculations. Since the nuclear system involved is a composite of only three nucleons, it is still possible to compute the partial capture rates to all the final nuclear states without using the closure approximation. It would be interesting to see how the total capture rate so obtained compares with the closure approximation results, especially in view of the difficulty to provide a direct theoretical justification for the closure approximation.[2]

II. A COMPARISON BETWEEN THE "BOUND" AND "UNBOUND" CAPTURES

To understand the process of muon capture by the He^3 nucleus into a continuum state of neutron and deuteron, we find it helpful to illustrate it against the background of well-known theories of muon capture with bound final nuclear states.[3] In these latter cases, particularly muon captures by light nuclei, there can be stated some general physical principles. Among them, the following are of special interest to us. There can be no mixing between different orbital-angular-momentum states of the emitted neutrino waves (assuming no polarization in the initial muonic atom), a principle that follows from the absence of a physical mechanism

† Work supported in part by the Office of U. S. Naval Research under Contract No. ONR-266 (72).

* Present address: Carnegie Institute of Technology, Pittsburgh, Pennsylvania.

[1] I-T. Wang, E. W. Anderson, E. J. Bleser, L. M. Lederman, S. L. Meyer, J. L. Rosen, and J. E. Rothberg, this issue, Phys. Rev. 139, B1528 (1965).

[2] R. Klein and L. Wolfenstein, Phys. Rev. Letters 9, 408 (1962); J. R. Luyten, H. P. C. Rood, and H. A. Tolhoek, Nucl. Phys. 41, 236 (1963).

[3] H. Primakoff, Rev. Mod. Phys. 31, 802 (1959); A. Fujii and H. Primakoff, Nuovo Cimento 12, 327 (1959).

that could correlate the direction of neutrino emission with a vector quantity of the nuclear system. From the parities and total angular momenta of the initial and final nuclei, one concludes that only neutrinos with even orbital angular momentum can be emitted; and in fact the S state is by far the most dominant. Such principles, however, do not hold in any strict sense in our present case of muon capture,

$$\mu^- + \mathrm{He}^3 \to \nu + n + d.$$

It is rather obvious that the conservation of momentum of the final three-body system automatically relates the directions of emission of neutrino and neutron for a given neutron energy. Furthermore, the parity of the nuclear system alone is not a constant throughout the process, it depends on the relative orbital angular momentum of the n-d unbound system which is in turn related to the neutrino orbital angular momentum. As a consequence, it is no longer advantageous to consider the emitted neutrino alone as a combination of different orbital-angular-momentum states, and it is necessary to relate its motion to the motion of the emitted nuclear particles in the over-all muon-capture process.

We like also to point out that this process has a rather pronounced Pauli exclusion principle effect. Since the isotopic spin of the final bound deuteron can only be zero, there is no antisymmetric spin state in the final nuclear system with respect to the nucleons in the deuteron. The overlap of the final nuclear state with the completely antisymmetric initial nuclear state, therefore, tends to suppress the vector coupling part and relatively enhance the effective-axial-vector and effective-induced-pseudoscalar parts in the transition rate.

In the following sections we will discuss in some detail the calculations on the rates of muon capture by He^3 into the "breakup" channels.

III. THE ISOTOPIC-SPIN FORMALISM OF THE THREE-FERMION SYSTEMS

It is well known that the totally antisymmetrized wave function of the ${}^{2,2}S_{1/2}$ state of He^3 takes the following general form:

$$\Psi = \varphi^S \xi^A - \varphi' \xi'' + \varphi'' \xi',$$

where φ represents the space part of the wave function, and ξ's contain the spin and isotopic-spin wave functions of the proper symmetries. (To be more specific, we have adopted the notation originally used by Verde[4]). The above form represents the most important part of the ground-state He^3 wave function. We will ignore the effect of the P state in our calculation, and make a further simplification by assuming no isotopic spin dependence and that the spin-dependent effect is weak. From all evidences, a good approximation of the S-state wave function can be represented by φ^S alone. Therefore we will approximate the He^3 ground-state wave function by a product of a totally symmetric space part φ^S and a spin–isotopic-spin part that is completely antisymmetric.

The final-state wave functions for the unbound n-d system can be written as

$$F_b\Psi_b = \exp(i\mathbf{q}\cdot\mathbf{r})\varphi_d(\mathbf{r}')\chi^S\zeta', \quad \text{for} \quad S=\tfrac{3}{2},$$

and

$$F_b\Psi_b = \exp(i\mathbf{q}\cdot\mathbf{r})\varphi_d(\mathbf{r}')\chi''\zeta', \quad \text{for} \quad S=\tfrac{1}{2},$$

where $\mathbf{r}'=\mathbf{r}_2-\mathbf{r}_3$. ζ' denotes the isotopic-spin wave function.

From the general formulas for muon capture with neutron emission obtained in the preceding paper,[5] we can write

$$N_1 = \int F_b^*\Psi_b^*[\tau_1^{(-)}\exp(-i\boldsymbol{\nu}\cdot\mathbf{r}_1)\varphi_\mu(\mathbf{r}_1) + 2\tau_2^{(-)}\exp(-i\boldsymbol{\nu}\cdot\mathbf{r}_2)\varphi_\mu(\mathbf{r}_2)]\Psi_a dV,$$

$$\mathbf{N}_2 = \int F_b^*\Psi_b^*[\tau_1^{(-)}\boldsymbol{\sigma}_1\exp(-i\boldsymbol{\nu}\cdot\mathbf{r}_1)\varphi_\mu(\mathbf{r}_1) + 2\tau_2^{(-)}\boldsymbol{\sigma}_2\exp(-i\boldsymbol{\nu}\cdot\mathbf{r}_2)\varphi_\mu(\mathbf{r}_2)]\Psi_a dV.$$

To evaluate the spin part of the matrix element and the summation over both the initial and the final nuclear spin states, we will employ a simple method of spin projection operators for a general three-fermion system. Since these operators are not found in the published literatures on the three-fermion systems, we will write out the explicit forms of the operators that correspond to the three symmetry classes in the isotopic spin formalism:

$$P^S = \tfrac{1}{6}[3 + \sum_{i<j=1}^{3} \boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j],$$

$$P' = \tfrac{1}{4}[1 - \boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_3],$$

$$P'' = \tfrac{1}{6}[\tfrac{3}{2} + \tfrac{1}{2}\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_3 - \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_3].$$

They can be most easily obtained by squaring the operator $\sum_i \boldsymbol{\sigma}_i$ and observing the symmetry properties of χ' and χ''. One can demonstrate explicitly that these operators satisfy the indempotent property, and that the product of any pair of them vanishes identically. Using these operators (Appendix A), one obtains at once

$$\mathrm{Tr}(N_1N_1^\dagger) = |J_1|^2;$$
$$\mathrm{Tr}(N_{2i}N_{2j}^\dagger) = [|J_1|^2 + \tfrac{4}{3}|J_2|^2]\delta_{ij}, \qquad (1)$$

where J_1 is the space part of the matrix element for the neutron; J_2 is that for nucleon 2.

IV. NEUTRON ENERGY SPECTRA

In this paper we shall limit ourselves to a rather simple approach to the calculation of the $\mu + \mathrm{He}^3 \to \nu + n + d$

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[4] M. Verde, *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39, p. 144.

[5] I-T. Wang, this issue, Phys. Rev. 139, B1539 (1965).

process. We shall neglect both the target-proton velocity term M_{rel} in the transition matrix element (see Ref. 5) and the final-state interactions between neutron and deuteron. The contribution of M_{rel} to muon capture by He^3 has been calculated by Goulart *et al.*[6] to be very small. A proper inclusion of the final-state interactions is made very difficult by lack of knowledge of the unbound doublet state of neutron and deuteron (possibly distorted) in the low-energy region (below 20 MeV). Low-energy *n-d* doublet scattering has long been one of the puzzling basic problems in nuclear physics. Only recently it became possible to account for the *n-d* scattering lengths and the triton binding energy satisfactorily in theory.[7] It appears that some of the pertinent features cannot be dealt with by any simple potential theory. We feel at present it is justified not to include the final-state *n-d* scattering in our calculation. However, we belive that the energy spectrum obtained in the plane-wave approximation will at least give an order-of-magnitude estimate to the process and provide us a comparison with our experimental data.

In the previous section we have studied the general form of the He^3 wave function in the isotopic-spin formalism. We shall now turn our attention to the detailed functional form of the symmetric space wave function φ^S. From the experimentally measured charge distribution of the He^3 nucleus, as well as from the point of view of computational expediency, it appears reasonable to choose a simple Gaussian type of wave function:

$$\varphi^S = N \prod_{i=1}^{3} \exp(-\tfrac{1}{2}\mu r_i^2),$$

where r_i is the distance of the ith nucleon from the center of mass of the He^3 nucleus. The parameter μ is chosen to fit the recent data of $\langle r\rangle$ for a Gaussian charge density distribution measured by Collard and Hofstadter.[8] A consistent choice of the deuteron wave function is then the Christian-Gammel function[9]:

$$\varphi_d(r') = N' \sum_{i=1}^{3} A_i \exp(-\alpha_i r'^2).$$

The integration variables in J_1 and J_2 are naturally chosen to be $\mathbf{r}=\mathbf{r}_1-\frac{1}{2}(\mathbf{r}_2+\mathbf{r}_3)$, $\mathbf{r}'=\mathbf{r}_2-\mathbf{r}_3$. Let us next rewrite

$$\exp(-i\mathbf{v}\cdot\mathbf{r}_1-i\mathbf{q}\cdot\mathbf{r})=\exp(i\mathbf{p}\cdot\mathbf{r}),$$

where $\mathbf{q}$ is the momentum associated with the *n-d* relative motion, and $\mathbf{p}$ is the recoil deuteron momentum. It is then easy to see that J_1 can be calculated analytically:

$$[(4\pi)^2NN']^{-1}J_1=\sum_{i=1}^{3} A_i\int_0^\infty\int_0^\infty drdr'(rr')^2 j_0(pr) \times\exp[-\alpha_i r'^2-\mu(r^2/3+r'^2/4)]$$

$$=\frac{\pi}{2}\sum_{i=1}^{3} A_i\left[\frac{2\mu}{3}\left(2\alpha_i+\frac{\mu}{2}\right)\right]^{-3/2} \times\exp\left(-\frac{3}{2\mu}M_d(E-\nu-E_1)\right),$$

where E, E_1 are defined as in Ref. 5, and M_d= deuteron mass. In a completely similar manner we can calculate J_2, using the relation

$$\exp[-i\mathbf{q}\cdot\mathbf{r}-i\mathbf{v}\cdot\mathbf{r}_2]=\exp[-i\mathbf{p}_1\cdot\mathbf{r}-\tfrac{1}{2}i\mathbf{v}\cdot\mathbf{r}'].$$

Thus,

$$[(4\pi)^2NN']^{-1}J_2=(\pi/2)\sum_{i=1}^{3} A_i[(2\mu/3)(2\alpha_i+\mu/2)]^{-3/2} \times\exp(-3M_nE_1/2\mu)\exp[-\nu^2/4(\mu+4\alpha_i)].$$

If we now define

$$I_i=\int_{\nu_{min}}^{\nu_{max}}|J_i|^2\nu\, d\nu,$$

the calculation involved in I_1 and I_2 becomes elementary. To obtain the neutron energy spectrum, the remaining step is a straightforward one of averaging the transition matrix element $|M|^2$ over the initial muon spin states and summing over the neutrino spin states. For the unpolarized muon, we obtain, using the results in Eq. (1), the transition probability as

$$d\omega_1=[2M_nN_d|\varphi_\mu(0)|^2/(2\pi)^3(2J_a+1)] \times\{\tfrac{1}{2}G_V^2I_1+[2G_A^2+(G_A-G_P)^2] \times(\tfrac{1}{2}I_1+\tfrac{2}{3}I_2)\}dE_1, \quad (2)$$

where it is understood that the effective coupling constants have been averaged properly over the range of neutrino energy for a given value of E_1. We obtain[10] $\omega_1=988\ \text{sec}^{-1}$.

The least likely of all three modes of muon capture by the He^3 nucleus involves a total breakup of the three-nucleon system. The calculation for this process is more complicated than the other two modes, not only because of the additional degrees of freedom in the phase space, but also because of the larger multitudes of final-spin and isotopic-spin states. There are allowed three sets of

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[6] B. Goulart, G. Goulart, and H. Primakoff, Phys. Rev. **133**, 186 (1964).

[7] R. Aaron, R. D. Amado, and Y. Y. Yam, Phys. Rev. Letters **13**, 574 (1964).

[8] H. Collard and R. Hofstadter, Phys. Rev. **131**, 416 (1963); H. Collard, R. Hofstadter, A. Johnson, R. Parks, M. Ryneveld, A. Walker, M. R. Yearian, R. B. Day, and R. T. Wagner, Phys. Rev. Letters **11**, 132 (1963).

[9] R. A. Christian and J. L. Gammel, Phys. Rev. **91**, 100 (1953).

[10] The He^3 capture rates presented here have included the following effects: (1) a 5% depreciation due to the finite size of the He^3 nucleus; (2) a 3% depreciation due to the D state of either the He^3 nucleus or the deuteron.

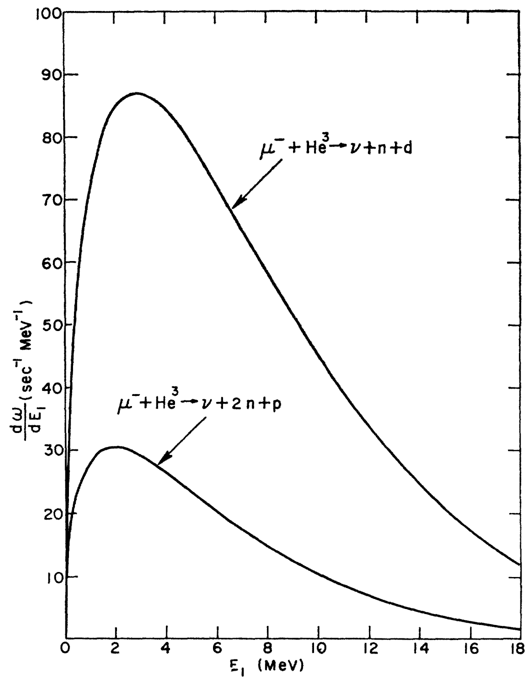

FIG. 1. Neutron energy spectra for the processes: $\mu^- + He^3 \rightarrow \nu + n + d$, and $\mu^- + He^3 \rightarrow \nu + 2n + p$.

such states, corresponding to $S=\frac{3}{2}$, $T=\frac{1}{2}$; $S=\frac{1}{2}$, $T=\frac{1}{2}$; and $S=\frac{1}{2}$, $T=\frac{3}{2}$. The space part of the final three-nucleon system can be constructed in the plane-wave approximation by means of the symmetry operators due to Verde.[4] The basic form of the wave function used to generate the various space functions of desired symmetry properties can be simply written as $\exp(i\mathbf{p}\cdot\mathbf{r})\cos(\mathbf{q}\cdot\mathbf{r}')$, $[\mathbf{q}=\frac{1}{2}(\mathbf{p}_2-\mathbf{p}_3)]$, which is clearly symmetric with respect to particles 2 and 3. Pauli's exclusion principle is then rigorously fulfilled by constructing, for each set of S and T quantum numbers, the normalized antisymmetric wave function.

It is rather simple to write down the nuclear matrix element. The sight of it will be spared the reader. The actual evaluation cannot be done in a completely analytical manner. Here again the numerical integration has been carried out on the IBM-7094 computer. This involves the last two integration variables q and ν in the momentum space. As is obvious, the neutrino momentum for a given value of neutron energy E_1 always starts from zero to a maximum value allowed by the energy-momentum conservation, unlike the processes previously studied, whereby ν takes in general a nonvanishing minimum value, for a given E_1. The kinematically allowed range of q is given by $q_{\pm}=[m(E-\nu-E_1)-\frac{1}{4}(\nu\mp p_1)^2]^{1/2}$, where q_+ and q_- designate the upper and lower limits, respectively. The spectrum obtained is plotted in Fig. 1. The rate for this process ω_2 is then obtained by numerical integration of the neutron energy spectrum. We have[10] $\omega_2=272$ sec^{-1}. This is slightly over one-quarter the rate of the process $\mu^-+He^3\rightarrow\nu+n+d$ considered before. Incorporated with the known theoretical rate of $\omega_0=1450$ sec^{-1} for the H^3 mode,[11] the rates obtained above yield a total rate of muon capture by He^3, $\omega=2.71\times10^3$ sec^{-1}.

To sum up, we note that a qualitative understanding of the behavior of the neutron spectra (Fig. 1) can be realized on rather simple physical grounds: that the phase-space factor favors higher energies for the light outgoing particle (i.e., the neutrino); and that the low-energy final states of the three-nucleon system lead to the major portion of the contribution to the spectrum. The total rate obtained above by computing individually the partial rates of the breakup channels in the plane-wave approximation is in fair agreement with the closure approximation results of 2.5×10^3 sec^{-1} by Primakoff,[3] and $(2.4\times10^3\pm10\%)$ sec^{-1} by Goulart *et al.*[6] It is, however, significantly higher than the closure approximation result of 2.13×10^3 sec^{-1} calculated by Yano.[12]

ACKNOWLEDGMENTS

The author has benefited greatly from discussions with Professor L. M. Lederman, Professor R. Serber, Professor L. Wolfenstein, and Professor J. Rosen. Careful reading and clarifying comments on an earlier version of the paper by Professor R. Serber and Professor L. Wolfenstein are most gratefully acknowledged.

APPENDIX A: SPIN TRACE

The three-nucleon spin projection operators introduced in Sec. III can be used in the following way to evaluate the spin sum. For instance, in evaluating the trace involving $\mathbf{N}_2$, the following spin sum is obtained for the $|J_1|^2$ term:

$$\mathrm{Tr}[(P^s+P'')\sigma_{1i}P''\sigma_{1j}]=\mathrm{Tr}[\sigma_{1i}(1-P')P''\sigma_{1j}]=2\delta_{ij}.$$

Similarly, the $|J_2|^2$ term has a corresponding spin sum of the form

$$\mathrm{Tr}[(P^s+P'')\sigma_{2i}P'\sigma_{2j}]=\mathrm{Tr}[(\tfrac{3}{2}\sigma_{2i}+\tfrac{1}{2}\sigma_{3i})P'\sigma_{2j}]=2\delta_{ij}.$$

[11] The capture rate for this mode has been calculated by many authors. See, for example, the number calculated in Ref. 12 for $g_P\cong7.2g_A$, and the papers quoted therein. A recent attempt [R. J. Oakes, Phys. Rev. 136, B1848 (1964)] to calculate this rate directly from the measured He³ form factors shows no significant deviation.

[12] A. F. Yano, Phys. Rev. Letters 12, 110 (1964).