Errors in Electromagnetic Cascade Measurements Due to the Transition Effect*

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The transition of an electromagnetic cascade which passes from one material to another having a different critical energy is discussed using Rossi and Greisen's approximation B. It is shown that an infinitely rapid change in the total number of particles takes place at the boundary layer. This leads in certain geometries to a strong dependence of the signal measured on the transition effect. A few of these arrangements are discussed and it is shown that they cause difficulties if measurements are made on cascades in ionization spectrometers or air-shower arrays. As a result, it appears to be strongly advisable to have no appreciable change in the critical energy over dimensions of more than several times 10^{-3} radiation length prior to the depth at which the cascade is measured.

INTRODUCTION

THE measurement of electromagnetic cascades requires that certain types of detectors (scintillators, ionization chambers, nuclear emulsions, etc.) are exposed to the shower particles. Very often, the development of the cascade takes place in a dense medium, like lead or iron (as in an emulsion chamber or ionization spectrometer), and the detectors are used to sample the cascade development. In other instances (air-shower measurements), the cascade has to pass before measurement through a protective cover of iron or aluminum, while cascade development and measurement take place in essentially similar materials, air and scintillator.

In all these cases, a transition effect must occur. Take the example of the extreme cases of cascades developing in lead and air, respectively, according to approximation B of Rossi and Greisen.[1] For the same primary energy, the total number of electrons of a cascade developing in lead is about one order of magnitude larger than if the cascade developed in air, since the critical energies[1] ($\epsilon_{Pb}=8$ MeV and $\epsilon_{air}=84$ MeV) have a ratio of 10. Plastic scintillator material, like air, has a large critical energy. A cascade emerging from a layer of lead and propagating into scintillation material must, therefore, exhibit a drastic reduction in number of particles and, therefore, energy loss and corresponding light signal.

It appears that experimenters involved with this problem have argued that the cascade will be influenced little if the layer of material with differing critical energy is sufficiently thin compared to the radiation length. It will be shown, however, that according to approximation B, the differential quotient $\partial\Pi/\partial t$ of the total number of electrons Π is $+\infty$ or $-\infty$, according to whether the critical energy of the second medium is smaller or larger than that of the first medium. Speaking within the framework of approximation B, therefore, no layer can be made "thin enough" so as not to influence the cascade development. The theoretical treatment of this problem is, therefore, of considerable practical significance for the interpretation of measurements on cascade development. This problem has been realized before by Christy and Kusaka,[2] following a suggestion by Bethe. These authors discussed the modification of a cascade developing in lead by a layer of about 1.5 cm of steel, and they took account of the transition effect by using an intermediate critical energy of 13 MeV. However, their very approximate treatment does not display the behavior of cascades if much thinner layers of different material are traversed, and it does not show the great rapidity of the change, which is so very influential in many arrangements for cascade measurement. The transition effect discussed here is also of a somewhat different nature from that discussed and measured widely in the years between 1940 and 1950. Transition effects were discussed then which were related to the increase in the burst rate in ionization chambers if the amount of shielding material was increased. This is an effect partly due to nuclear interaction which is not considered here. Alternatively, the term "transition effect" has been used to describe the transition curve of showers created in lead or iron absorbers by individual particles of the cosmic radiation. An account of these transition curves can be found in Rossi's[3] book. Again, this is not the effect which we wish to discuss here. We are specifically concerned with the change in the signal created by a single purely electromagnetic cascade if the entire cascade at a somewhat advanced stage of development passes through very thin layers of different critical energy, or if the cascade is measured in a medium with different critical energy. It will be shown then that layers as thin as 10^{-2} radiation length will produce a significant effect.

The use of approximation B must be justified here since it is known that this approximation gives a distorted picture of the cascade behavior at low energies.[4] At the energies considered here, which exceed the critical energy by many orders of magnitude, approximation B offers the only consistent theoretical framework from which one may obtain insight into the behavior of large

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[1] B. Rossi and K. Greisen, Rev. Mod. Phys. 13, 240 (1941).
[2] R. F. Christy and S. Kusaka, Phys. Rev. 59, 414 (1941).
[3] B. Rossi, *High Energy Particles* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1952).
[4] See Ref. 3, p. 262.

showers. In any case, the experimental applications of shower measurements in ionization spectrometers or air showers refer to cascades initiated by nuclear interactions, and their behavior also cannot be described by approximation B. In those cases, one wishes to obtain a measure of the primary (total) energy by sampling the energy loss of the cascade at various depths. The integral over the energy-loss curve must then give the primary energy. Since this follows from first principles, no complicated cascade theory enters into the result of the measurement. As described above, however, the sampling procedure is likely to disturb the cascade measurement to such a degree that a clear correlation between the measured energy loss and the primary energy cannot be established. Approximation B is used and taken seriously here only to exemplify this.

THEORETICAL CONSIDERATIONS

The nomenclature used is that of Rossi.[3] Our considerations will be carried out strictly within the framework of approximation B. Consider the differential energy spectra of electrons and photons at a distance $t>1$ from the origin. For an electron primary, these spectra are given by the expressions (5.13.8a), and (5.13.8b) of Rossi[3]:

$$\pi^{(\pi)}(E_0,E,t)dE=\frac{1}{(2\pi i)^2}\int_{\delta-i\infty}^{\delta+i\infty}ds\int_{-\delta-i\infty}^{-\delta+i\infty}dr\frac{\mu_0+\lambda_1(s)}{\lambda_1(s)-\lambda_2(s)}\frac{\Gamma(-r)\Gamma(s+r+1)}{\Gamma(s+1)}K_1(s,r)\left(\frac{\epsilon_0}{E}\right)^r\left(\frac{E_0}{E}\right)^s e^{\lambda_1(s)t}\frac{dE}{E}, \quad (1a)$$

$$\gamma^{(\pi)}(E_0,E,t)dE=\frac{1}{(2\pi i)^2}\int_{\delta-i\infty}^{\delta+i\infty}ds\int_{-\delta-i\infty}^{-\delta+i\infty}dr\frac{C(s+r)}{\lambda_1(s)-\lambda_2(s)}\frac{\Gamma(-r)\Gamma(s+r+1)}{\Gamma(s+1)}K_1(s,r)\left(\frac{\epsilon_0}{E}\right)^r\left(\frac{E_0}{E}\right)^s e^{\lambda_1(s)t}\frac{dE}{E}. \quad (1b)$$

Inserting these expressions into the diffusion equations (5.5.11a)

$$[\partial\pi(E,t)]/\partial t=\mathfrak{A}\pi+\mathfrak{B}\gamma+[\partial(\epsilon\pi)]/\partial E. \quad (2)$$

We may investigate the behavior of $[\partial\pi(E,t)]/\partial t$ at the border of a new medium with critical energy ϵ instead of ϵ_0. That is, after travelling for t radiation lengths in a medium with critical energy ϵ_0 the cascade is assumed to enter another medium with critical energy ϵ.

We now obtain

$$\begin{aligned}\frac{\partial\pi^{(\pi)}(E,t)}{\partial t}=&\frac{1}{(2\pi i)^2}\int_{\delta+i\infty}^{\delta+i\infty}ds\int_{-\delta-i\infty}^{-\delta+i\infty}dr\frac{\mu_0+\lambda_1(s)}{\lambda_1(s)-\lambda_2(s)}\frac{\Gamma(-r)\Gamma(s+r+1)}{\Gamma(s+1)}K_1(s,r)\left(\frac{\epsilon_0}{E}\right)^r\left(\frac{E_0}{E}\right)^s\frac{1}{E}e^{\lambda_1(s)t}A(s+r)\\&+\frac{1}{(2\pi i)^2}\int_{\delta-i\infty}^{\delta+i\infty}ds\int_{-\delta-i\infty}^{-\delta+i\infty}dr\frac{C(s+r)}{\lambda_1(s)-\lambda_2(s)}\frac{\Gamma(-r)\Gamma(s+r+1)}{\Gamma(s+1)}K_1(s,r)\left(\frac{\epsilon_0}{E}\right)^r\left(\frac{E_0}{E}\right)^s\frac{1}{E}e^{\lambda_1(s)t}B(s+r)\\&-\frac{1}{(2\pi i)^2}\int_{\delta-i\infty}^{\delta+i\infty}ds\int_{-\delta-i\infty}^{-\delta+i\infty}dr\frac{\mu_0+\lambda_1(s)}{\lambda_1(s)-\lambda_2(s)}\frac{\Gamma(-r)\Gamma(s+r+2)}{\Gamma(s+1)}K_1(s,r)\left(\frac{\epsilon_0}{E}\right)^r\left(\frac{E_0}{E}\right)^s\frac{1}{E}\frac{\epsilon}{E}e^{\lambda_1(s)t}. \quad (3)\end{aligned}$$

However, we wish to investigate the variation in the total number of particles:

$$[\partial\Pi^{(\pi)}(E_0,0,t)]/\partial t, \quad (4)$$

where

$$\Pi^{(\pi)}(E_0,0,t)=\lim_{E\to 0}\int_E^\infty\pi^{(\pi)}(E,t)dE. \quad (5)$$

The first term of Eq. (3) yields zero, since integration over E leads to the terms

$$(s+r)^{-1}(\epsilon_0/E)^r(E_0/E)^s$$

and the contour of the integration over r can be closed to the left, yielding one contribution only from the pole $s=-r$ as $E\to 0$. However, $A(0)=0$, and thus the first term vanishes. Physically, this means that no change in the total number of particles occurs through radiation-energy losses of the electrons. This must be so since approximation B allows only for a loss in the total number of electrons by ionization.

Integration over the second and third terms on the right-hand side of Eq. (3) leads to the following formula:

$$\begin{aligned}\frac{\partial\Pi^{(\pi)}(E_0,0,t)}{\partial t}=&\lim_{E\to 0}\frac{1}{(2\pi i)^2}\int_{\delta-i\infty}^{\delta+i\infty}ds\int_{-\delta-i\infty}^{-\delta+i\infty}dr\frac{C(s+r)}{\lambda_1(s)-\lambda_2(s)}\frac{\Gamma(-r)\Gamma(s+r)}{\Gamma(s+1)}K_1(s,r)\left(\frac{\epsilon_0}{E}\right)^r\left(\frac{E_0}{E}\right)^s B(s+r)e^{\lambda_1(s)t}\\&-\lim_{E\to 0}\frac{1}{(2\pi i)^2}\int_{\delta-i\infty}^{\delta+i\infty}ds\int_{-\delta-i\infty}^{-\delta+i\infty}dr\frac{\mu_0+\lambda_1(s)}{\lambda_1(s)-\lambda_2(s)}\frac{\Gamma(-r)\Gamma(s+r+1)}{\Gamma(s+1)}K_1(s,r)\left(\frac{\epsilon_0}{E}\right)^r\left(\frac{E_0}{E}\right)^s\frac{\epsilon}{E}e^{\lambda_1(s)t}. \quad (6)\end{aligned}$$

Again, the contour of the integrations over r can be closed to the left, reducing, for sufficiently small E, to the contributions at the poles closest to the left of the integration path through $r=-\delta$. Furthermore, we may substitute in the first integral

$$B(s+r)C(s+r)K_1(s,r)=-rK_1(s,r-1)(\mu_0+\lambda_1(s))+(\mu_0+\lambda_1(s))(\lambda_1(s)+A(s+r))K_1(s,r)$$

from Eq. (5.13.13). This then leads to

$$\frac{\partial\Pi^{(\pi)}(E_0,0,t)}{\partial t}=\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}ds\frac{\mu_0+\lambda_1(s)}{\lambda_1(s)-\lambda_2(s)}\frac{1}{s}K_1(s,-s)\left(\frac{E_0}{\epsilon_0}\right)^s\lambda_1(s)e^{\lambda_1(s)t}+\lim_{E\to 0}\pi^{(\pi)}(E_0,E,t)(\epsilon_0-\epsilon). \tag{7}$$

For small E, the differential spectrum $\pi^{(\pi)}(E_0,E,t)$ is given by Eq. (45) of Snyder,[5] i.e., in transcription to Rossi's notation by

$$\pi^{(\pi)}(E_0,E,t)=\frac{2\mu_0(\frac{4}{3}+2b)}{\epsilon_0[\lambda_1(s)+\mu_0]}\Pi^{(\pi)}(E_0,0,t)\left(\ln\frac{\epsilon_0}{E}+f(s)-1\right), \tag{8}$$

where $f(s)$ is tabulated in Table I, reproduced from Snyder.[5]

Thus, as $E\to 0$, $\pi^{(\pi)}(E_0,E,t)\to+\infty$, and

$$\begin{aligned}\partial\Pi^{(\pi)}(E_0,0,t)/\partial t&=+\infty \quad \text{if} \quad \epsilon_0>\epsilon\\ &=-\infty \quad \text{if} \quad \epsilon_0<\epsilon.\end{aligned} \tag{9}$$

This can be understood in the following way: The differential energy spectrum of γ rays diverges as $1/E$ due to the similar radiation spectrum. This leads to a logarithmic divergence of the differential electron spectrum at low energies, Eq. (8). As a cascade propagates through matter, infinitely many low-energy electrons are absorbed by ionization loss, but they are replaced by production of new electrons from the infinite number of low-energy γ rays. Both infinities cancel each other as has been shown. If, however, thec ascade moves into a medium with, say, higher critical energy ϵ, the absorption process of electrons is stepped up suddenly, while the replenishment by γ rays takes place at the old rate, since the replenishment by γ rays takes place at a rate determined by the radiation length and not by the critical energy.

To obtain numerical values for the transition of the cascade in the new medium we assume that the change from ϵ_0 to ϵ occurs at t, and we indicate ϵ_0 or ϵ, respectively, amongst the variables for the spectra. That is, $\gamma^{(\pi)}(E_0,E,\epsilon_0,t)$ is, for example, the differential energy spectrum of γ rays produced by a primary electron, at a distance of t radiation lengths from the origin, in a medium with critical energy ϵ_0. We then obtain for the number of electrons at a distance Δt beyond the boundary layer

$$\Pi^{(\pi)}(E_0,0,\epsilon,t+\Delta t)=\int_0^\infty dE\pi^{(\pi)}(E_0,E,\epsilon_0,t)\Pi^{(\pi)}(E,0,\epsilon,\Delta t)+\int_0^\infty dE\gamma^{(\pi)}(E_0,E,\epsilon_0,t)\Pi^{(\gamma)}(E,0,\epsilon,\Delta t). \tag{10a}$$

Let us first consider the case $\Delta t>1$. Since in all practical applications $t>1$ also, the approximate expression of Snyder and Serber quoted by Rossi[3] may be used, i.e.,

$$\pi^{(\pi)}(E_0,E,\epsilon_0,t)=\frac{1}{(2\pi i)^2}\int_{\delta-i\infty}^{\delta+i\infty}ds\int_{-\delta-i\infty}^{-\delta+i\infty}dr\frac{\mu_0+\lambda_1(s)}{\lambda_1(s)-\lambda_2(s)}\frac{\Gamma(-r)\Gamma(s+r+1)}{\Gamma(s+1)}K_1(s,r)\left(\frac{\epsilon_0}{E}\right)^r\left(\frac{E_0}{E}\right)^s\frac{1}{E}e^{\lambda_1(s)t}, \tag{1a}$$

$$\gamma^{(\pi)}(E_0,E,\epsilon_0,t)=\frac{1}{(2\pi i)^2}\int_{\delta-i\infty}^{\delta+i\infty}ds\int_{-\delta-i\infty}^{-\delta+i\infty}dr\frac{C(s+r)}{\lambda_1(s)-\lambda_2(s)}\frac{\Gamma(-r)\Gamma(s+r+1)}{\Gamma(s+1)}K_1(s,r)\left(\frac{\epsilon_0}{E}\right)^r\left(\frac{E_0}{E}\right)^s\frac{1}{E}e^{\lambda_1(s)t}, \tag{1b}$$

$$\Pi^{(\pi)}(E,0,\epsilon,\Delta t)=\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}dp\frac{\mu_0+\lambda_1(p)}{\lambda_1(p)-\lambda_2(p)}\frac{1}{p}K_1(p,-p)\left(\frac{E}{\epsilon}\right)^p e^{\lambda_1(p)\Delta t}, \tag{11a}$$

$$\Pi^{(\gamma)}(E,0,\epsilon,\Delta t)=\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}dp\frac{[\mu_0+\lambda_1(p)][-\mu_0-\lambda_2(p)]}{C(p)(\lambda_1(p)-\lambda_2(p))}\frac{1}{p}K_1(p,-p)\left(\frac{E}{\epsilon}\right)^p e^{\lambda_1(p)\Delta t}. \tag{11b}$$

[5] H. S. Snyder, Phys. Rev. **76**, 1563 (1949).

Carrying out the integration according to (10a) yields

$$\Pi^{(\pi)}(E_0, 0, \epsilon, t+\Delta t)=\frac{1}{(2\pi i)^2}\int_{\delta-i\infty}^{\delta+i\infty} ds \int_{\delta-i\infty}^{\delta+i\infty} dp \left(\frac{E_0}{\epsilon_0}\right)^s \left(\frac{\epsilon_0}{\epsilon}\right)^p \frac{\Gamma(s-p)\Gamma(p)}{\Gamma(s+1)}$$

$$\times K_1(s, p-s)K_1(p, -p)\frac{[\mu_0+\lambda_1(p)][\lambda_1(s)-\lambda_2(p)]}{[\lambda_1(s)-\lambda_2(s)][\lambda_1(p)-\lambda_2(p)]}e^{\lambda_1(s)t}e^{\lambda_1(p)\Delta t}. \quad (10b)$$

The integral in Eq. (10b) can be solved by a double saddle point method; cf., Bhabha and Chakrabarty,[6] where the saddle point is determined by simultaneously solving the following equations:

$$\ln(E_0/\epsilon_0)+\Psi(s-p-1)-\Psi(s)+\lambda_1'(s)t=0, \quad (12a)$$

$$\ln(\epsilon_0/\epsilon)-\Psi(s-p-1)+\Psi(p-1)+\lambda_1'(p)\Delta t=0. \quad (12b)$$

Here, the logarithmic derivatives of the functions $K_1(s, p-s)$ and $K_1(p, -p)$ with respect to s and p, respectively, and the logarithmic derivatives of

$$\frac{[\mu_0+\lambda_1(p)][\lambda_1(s)-\lambda_2(p)]}{[\lambda_1(s)-\lambda_2(s)][\lambda_1(p)-\lambda_2(p)]}$$

have been neglected as they are rather unimportant compared with the other terms. $\Psi(x)=(d/dx)\ln\Gamma(x+1)$ is tabulated by Jahnke and Emde.[7] Values are given in Table I, and the values given there can be extended using the relation $\Psi(x)=\Psi(x-1)+x^{-1}$ which shows that $\Psi(x)$ approaches $-\infty$ as $x\to -1$. If $\epsilon=\epsilon_0$ in Eq. (10b), then this expression must coincide with the well-known expression

$$\Pi^{(\pi)}(E_0, 0, \epsilon_0, t+\Delta t)$$

of the total number of particles under approximation B in a homogeneous medium without boundary layer. In that case, the term $(\epsilon_0/\epsilon)^p$ equals 1 and the double complex integral of Eq. (10b) is therefore known. The solution of our problem for arbitrary values of the critical energies is, therefore, given by

$$\Pi^{(\pi)}(E_0, 0, \epsilon, t+\Delta t)=\left(\frac{\epsilon_0}{\epsilon}\right)^p \Pi^{(\pi)}(E_0, 0, \epsilon_0, t+\Delta t), \quad (13)$$

where $\Pi^{(\pi)}(E_0, 0, \epsilon_0, t+\Delta t)$ can, for example, be obtained from the graphs (5.13.2) and (5.13.3) of Rossi,[3] and p is a function of E_0, ϵ_0, ϵ, t, and Δt according to Eqs. (12a) and (12b). In practice, for any given problem, E_0, ϵ_0, ϵ, and t are known. Inserting values for s in Eq. (12a), a correlation can be established between s and p using a graph of the function $\Psi(s-p-1)$. Then, for a given pair of s and p, Δt can be calculated from Eq. (12b). The required values of $\Pi^{(\pi)}(E_0, 0, \epsilon, t+\Delta t)$ can then be plotted at $t+\Delta t$ by using Eq. (13).

[6] H. J. Bhabha and S. K. Chakrabarty, Phys. Rev. **74**, 1352 (1948).

[7] E. Jahnke and F. Emke, *Tables of Functions* (B. G. Teubner, Leipzig, 1938).

However, Eqs. (11a) and (11b) hold only if $\Delta t>1$, and $E>\epsilon_0$, since the boundary conditions of a single incident electron or photon at $\Delta t=0$ are not accurately fulfilled by these expressions. If $\Delta t>1$, however, any error in the vicinity of $E\sim\epsilon_0$ matters little.[8] For $\Delta t<1$ therefore, Eq. (13) is of no use. This is also indicated by the fact that Eq. (12b) does not lead to a solution for $p\to 0$, i.e., no saddle point exists then.

In fact, solutions of electron spectra have been given by Scott[9] and Snyder[5] which accurately take account of the boundary conditions. The use of these functions would, however, complicate greatly the numerical computation of the transition effect for small Δt and, even if carried out for certain cases, might not easily be repeated by an experimental physicist who might like to study a special situation created by his equipment. It is therefore preferred to proceed using a few simple approximate calculations which are based on an appraisal of the physical situation.

To study the transition problem at small Δt, an accurate knowledge of the electron and γ-ray energy spectra is required. These spectra have been calculated

TABLE I. Numerical values of various constants and functions used in the formulas; from Rossi (Ref. 3) and Snyder (Ref. 5).

$\mu_0=0.773$

$2b=\dfrac{1}{9\ln(183z^{-1/3})}=0.024\cdots0.030$

s	$f(s)$	$\Psi(s)$	$\lambda_1'(s)$	$\lambda_1(s)$
0		−0.577	$-\infty$	$+\infty$
0.1	3.600	−0.424	−25.005	3.789
0.2	3.385	−0.289	−9.488	2.270
0.3	3.140	−0.169	−5.415	1.569
0.4	2.890	−0.061	−3.654	1.127
0.5	2.635	0.037	−2.693	0.813
0.6	2.375	0.126	−2.093	0.576
0.7	2.095	0.209	−1.685	0.389
0.8	1.705	0.285	−1.389	0.235
0.9	1.530	0.356	−1.166	0.108
1.0	1.284	0.423	−0.991	0.000
1.1	1.085	0.485	−0.850	−0.092
1.2	0.810	0.544	−0.733	−0.171
1.3	0.750	0.600	−0.636	−0.239
1.4	0.615	0.653	−0.553	−0.298
1.5	0.480	0.704	−0.483	−0.350
1.6	0.360	0.751	−0.421	−0.395
1.7	0.250	0.797	−0.369	−0.435
1.8	0.155	0.841	−0.324	−0.470
1.9	0.055	0.882	−0.284	−0.500
2.0	−0.034	0.923	−0.250	−0.526

[8] See Ref. 3, p. 256.

[9] W. T. Scott, Phys. Rev. **80**, 611 (1950).

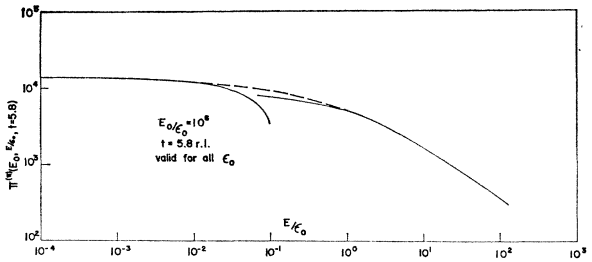

FIG. 1. Energy spectrum of electrons. $E_0/\epsilon_0=10^6$, $t=5.8$ r.l.

by Snyder,[5] and they are, in the notation used here,

$$\Pi^{(\pi)}(E_0,E,t)=\Pi^{(\pi)}(E_0,0,t)\left\{1-2\mu_0\frac{\frac{4}{3}+2b}{\lambda_1(s)+\mu_0}\times\left(\frac{E}{\epsilon_0}\right)\left(\ln\frac{\epsilon_0}{E}+f(s)\right)\right\}, \quad (14)$$

$$\pi^{(\pi)}(E_0,E,t)=\Pi^{(\pi)}(E_0,0,t)2\mu_0\frac{\frac{4}{3}+2b}{\lambda_1(s)+\mu_0}\frac{1}{\epsilon_0}\times\left\{\ln\frac{\epsilon_0}{E}+f(s)-1\right\}, \quad (15)$$

$$\gamma^{(\pi)}(E_0,E,t)=\Pi^{(\pi)}(E_0,0,t)\frac{\frac{4}{3}+2b}{\lambda_1(s)+\mu_0}\frac{1}{E}. \quad (16)$$

The correlation between s and t is given here by

$$\ln\frac{E_0}{\epsilon_0}-\frac{1}{s}+\lambda_1'(s)t=0. \quad (17a)$$

These formulas are expected to hold for $E\ll\epsilon_0$. For $E>\epsilon_0$, we may use the first term of Bhabha and Chakrabarty's[6] expansion [their Eq. (20)], which is in our notation.

$$\Pi^{(\pi)}(E_0,E,t)=\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}ds\frac{\mu_0+\lambda_1(s)}{\lambda_1(s)-\lambda_2(s)}\times\left[\frac{E_0}{E+\epsilon_0 g(s,t)}\right]^s s^{-1}e^{\lambda_1(s)t}$$
$$=\Pi_A^{(\pi)}(E_0,E+\epsilon_0 g(s,t),t), \quad (18)$$

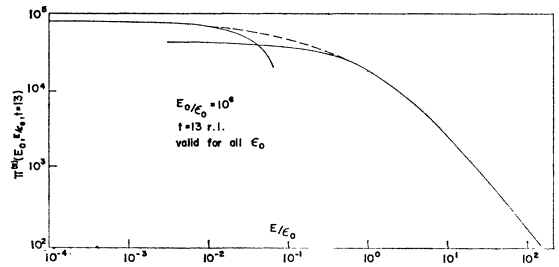

FIG. 2. Energy spectrum of electrons. $E_0/\epsilon_0=10^6$, $t=13$ r.l.

TABLE II. Values of the function $g(s,t)$ appearing in Eq. (18); from Bhabha and Chakrabarty (Ref. 6).

	$g(s,t)$								
t	$s=0.3$	0.4	0.5	0.8	1.0	1.3	1.5	1.8	2.0
0.25	0.210		0.221	0.228	0.231	0.234	0.235	0.237	0.238
0.5	0.320		0.367	0.403	0.417	0.430	0.437	0.444	0.448
1.0	0.379	0.439	0.489	0.596	0.631	0.699	0.726	0.757	0.774
2.0	0.390	0.462	0.527	0.687	0.771	0.877	0.934	1.006	1.048
4.0	0.392	0.467	0.535	0.709	0.801	0.905	0.956	1.010	1.037
6.0	0.392	0.467	0.536	0.715	0.812	0.926	0.974	1.023	1.041
8.0	0.392	0.467	0.536	0.717	0.816	0.933	0.989	1.043	1.063
10.0	0.392	0.467	0.536	0.717	0.819	0.939	0.999	1.059	1.083
15.0	0.392	0.467	0.536	0.717	0.819		1.009		1.112
20.0	0.392	0.467	0.536	0.717	0.819		1.011		1.126
∞	0.392	0.467	0.536	0.717	0.819	0.945	1.013	1.095	1.141

where $\Pi_A^{(\pi)}(E_0,E,t)$ is the integral electron spectrum obtained under approximation A of Rossi, and this function is tabulated in his Figs. 5.10.1 and 5.10.2. In Eq. (18) however, E has been substituted by $E+\epsilon_0 g(s,t)$. The function $g(s,t)$ is tabulated in Table II, and the value of s for any given E and t may be found solving the equation

$$\ln\frac{E_0}{E+\epsilon_0 g(s,t)}-\frac{1}{s}+\lambda_1'(s)t=0. \quad (17b)$$

As an example, electron spectra have been drawn in Figs. 1, 2, and 3 for a ratio $E_0/\epsilon_0=10^6$ at $s=0.6$, 1.0, and 1.4, respectively. These values correspond to $t=5.8$, 13.0, and 24.0, respectively. The dashed lines in Figs. 1, 2, and 3 represent a graphical interpolation between the full lines, which are valid at $E/\epsilon_0\ll 1$ and at $E/\epsilon_0\gtrsim 1$, respectively. Imagine for a moment that the cascade at depth t would proceed in the same medium with critical energy ϵ_0. In Δt, the total number of electrons would be reduced from $\Pi^{(\pi)}(E_0,0,t)$ to $\Pi^{(\pi)}(E_0,\epsilon_0\Delta t,t)$ since only the electrons with energy $\epsilon_0\Delta t$ manage to survive until Δt. This loss in number of electrons is replenished by converting γ rays. In fact, the number of electrons produced by the γ rays on Δt is then

$$\Delta\Pi^{(\gamma)}(E_0,0,\Delta t)=\Delta t([\partial\Pi^{(\pi)}(E_0,0,t)]/\partial t)-\Pi^{(\pi)}(E_0,\epsilon_0\Delta t,t)+\Pi^{(\pi)}(E_0,0,t). \quad (19)$$

Thus, if the cascade now moves into a layer with critical energy ϵ, we obtain for the change in number of electrons

$$\Delta\Pi^{(\pi)}(E_0,0,t+\Delta t)=\Delta t([\partial\Pi^{(\pi)}(E_0,0,t)]/\partial t)-\Pi^{(\pi)}(E_0,\epsilon_0\Delta t,t)+\Pi^{(\pi)}(E_0,\epsilon\Delta t,t). \quad (20)$$

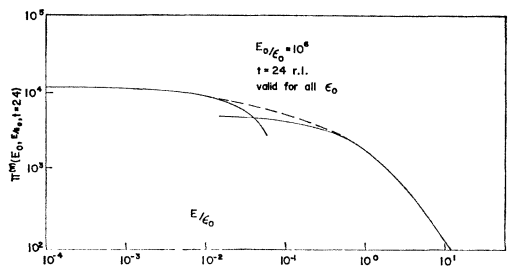

FIG. 3. Energy spectrum of electrons. $E_0/\epsilon_0=10^6$, $t=24$ r.l.

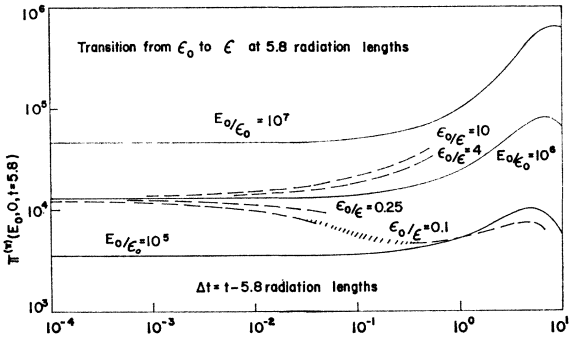

FIG. 4. Transition at $t=5.8$ r.l. from a medium with critical energy ϵ_0 to one with critical energy ϵ.

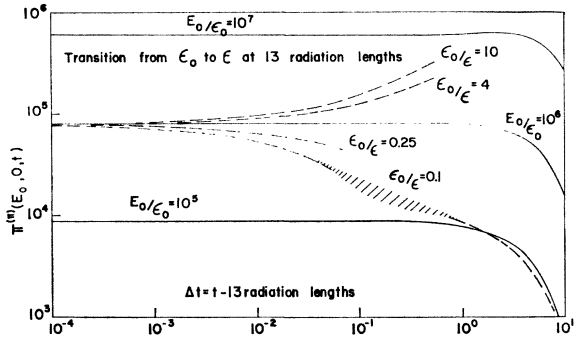

FIG. 5. Transition at $t=13$ r.l.

If $\epsilon>\epsilon_0$, the absorption of electrons will be larger than the replenishment by γ rays, and vice versa.

The estimate has not yet taken account of the fact that the new electrons, produced by the conversion of the γ rays, will have a range determined by $\Delta t=E/\epsilon$, while the calculation so far implicitly made use of a range $\Delta t=E/\epsilon_0$. This effect can be easily accounted for by making use of the low-energy γ-ray spectrum (16). Using this, the number of electrons produced in dy at y (expressed in radiation lengths) is

$$2\gamma^{(\pi)}(E_0,E,t)e^{-\mu_0 y}dE\mu_0 dy$$

$$=2\Pi^{(\pi)}(E_0,0,t)\frac{\frac{4}{3}+2b}{\lambda_1(s)+\mu_0}\mu_0\frac{dE}{E}e^{-\mu_0 y}dy \quad \text{for } E\lesssim\epsilon_0. \quad (21)$$

These electrons reach Δt only if their energy is larger than $\epsilon(\Delta t-y)$. Thus, the contribution towards the integral number of electrons at Δt from dy at y is

$$2\mu_0 dy e^{-\mu_0 y}\int_{\epsilon(\Delta t-y)}^{\infty}\gamma^{(\pi)}(E_0,E,t)dE,$$

and the number of electrons produced in Δt by γ rays and reaching Δt is

$$\int_0^{\Delta t}2\mu_0 dy e^{-\mu_0 y}\int_{\epsilon(\Delta t-y)}^{\infty}\gamma^{(\pi)}(E_0,E,t)dE. \quad (22)$$

The expression for $\gamma^{(\pi)}(\epsilon_0,\epsilon,t)$ is, in general, rather complicated. However, we already know from Eq. (19) the total contribution from γ rays if the critical energy were ϵ_0. That is, we known and have already taken account of the expression

$$\int_0^{\Delta t}2\mu_0 dy e^{-\mu_0 y}\int_{\epsilon_0(\Delta t-y)}^{\infty}\gamma^{(\pi)}(E_0,E,t)dE.$$

Therefore, we have to calculate only

$$\int_0^{\Delta t}2\mu_0 dy e^{-\mu_0 y}\int_{\epsilon(\Delta t-y)}^{\epsilon_0(\Delta t-y)}\gamma^{(\pi)}(E_0,E,t)dE,$$

and in this integral, $\epsilon_0(\Delta t-y)$; $\epsilon(\Delta t-y)<\epsilon_0$, so that the expression (21) may be used for $\gamma^{(\pi)}(\epsilon_0,E,t)$. We then obtain

$$\int_0^{\Delta t}2\mu_0 e^{-\mu_0 y}dy\int_{\epsilon(\Delta t-y)}^{\epsilon_0(\Delta t-y)}\Pi^{(\pi)}(E_0,0,t)\frac{\frac{4}{3}+2b}{\lambda_1(s)+\mu_0}\frac{1}{E}$$

$$=2\ln\frac{\epsilon_0}{\epsilon}\Pi^{(\pi)}(E_0,0,t)\frac{\frac{4}{3}+2b}{\lambda_1(s)+\mu_0}[1-e^{-\mu_0\Delta t}].$$

With this correction, we obtain finally

$$\Delta\Pi^{(\pi)}(E_0,0,t+\Delta t)$$

$$=\Delta t\left(\frac{\partial\Pi^{(\pi)}(E_0,0,t)}{\partial t}\right)-\Pi^{(\pi)}(E_0,\epsilon_0\Delta t,t)+\Pi^{(\pi)}(E_0,\epsilon\Delta t,t)$$

$$+2\ln\frac{\epsilon_0}{\epsilon}\Pi^{(\pi)}(E_0,0,t)\frac{\frac{4}{3}+2b}{\lambda_1(s)+\mu_0}[1-e^{-\mu_0\Delta t}]. \quad (23)$$

CALCULATIONS

Making use of the energy spectra shown in Figs. 1, 2, and 3, the transition effect has been calculated for a cascade of energy E_0, where $E_0/\epsilon_0=10^6$. This cascade is assumed to penetrate into an absorber with critical

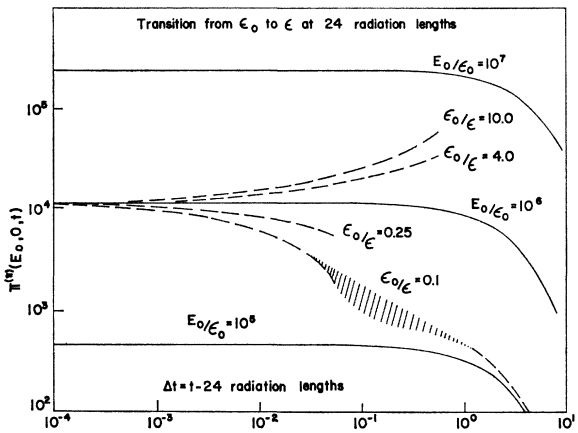

FIG. 6. Transition at $t=24$ r.l.

TABLE III. Density ρ, radiation length X_0, and critical energy ϵ_0 of typical scintillator and ionization-chamber materials.

	ρ[g/cm³]	X_0[g/cm²]	ϵ_0[MeV]
Lead	11.34	5.83	7.6
Iron	7.85	13.9	21
Carbon (graphite)	2.25	44.6	76
Copper	8.93	13.1	20
Scintillator (plastic)	1.0	42.5	70
Argon	1.78×10^{-3}	19.7	30
Air	1.28×10^{-3}	37.7	65

energy ϵ at $t=5.8$ radiation lengths (r.l.) ($s=0.6$), $t=13.0$ r.l. ($s=1.0$), and $t=24.0$ r.l. ($s=1.4$) as examples. Figures 4, 5, and 6 display the transition effects expected under approximation B.

Four cases have been calculated in each figure, namely, those corresponding to a ratio of $\epsilon_0/\epsilon=10.0$; 4.0; 0.25; and 0.1, respectively. The related functions $\Pi^{(\pi)}(E_0,0,t)$ for $E_0/\epsilon_0=10^7$ and 10^5 have also been drawn.

In all four cases of ϵ_0/ϵ, the transition effect according to Eq. (23) has been indicated. For $\epsilon_0/\epsilon=0.1$, the solution of Eq. (13) has also been shown. This solution is expected to be good for $\Delta t>1$.

The solutions Eq. (23) and Eq. (13) cannot be made to meet. This is due to the fact that, at $\Delta t\gtrsim10^{-1}$, it is no longer possible to describe adequately the behavior of electrons and photons by pair production and ionization loss only. At those distances, cascade multiplication of electrons and photons begins to play a role. A rough interpolation between the solutions of Eq. (13) and Eq. (23) has therefore been indicated in Figs. 4, 5, and 6 by the shaded area.

RESULTS

In typical cascade measurements, one may employ, for example, plastic or liquid[10] scintillator, or ionization chambers, filled with argon.[11] In Table III, a few typical material constants are listed.

An ionization chamber of 5 cm thickness filled with 5-atm argon thus has a depth of about 2×10^{-3} r.l., while 5 cm of scintillator offers 1.2×10^{-1} r.l.

Assume now an arrangement of lead absorber and 5-cm detection layers consisting of scintillator. Denote by E_1 the energy loss in the scintillator, by E_2 the energy loss on the same 1.2×10^{-1} r.l. in lead, and by E_3 the energy loss in scintillator if no decrease in the particle number occurred. The following table results:

Transition at	E_1 (MeV)	E_2 (MeV)	E_3 (MeV)	E_3/E_1
5.8 r.l.	6.0×10^4	1.23×10^4	1.14×10^5	1.9
13 r.l.	2.6×10^5	7.3×10^4	6.75×10^5	2.6
24 r.l.	2.4×10^4	1.08×10^4	1.0×10^5	4.2

[10] R. Raghavan, B. V. Sreekantan, A. Subramanian, and S. D. Verma, J. Phys. Soc. Japan **17**, Suppl. A III, 251 (1962).

[11] Proposal for an Ultra High Energy Physics Experiment Using Cosmic Rays, Midwestern Universities Research Association, 1965 (unpublished).

Therefore, if the experimentalist had assumed that the 1.2×10^{-1} r.l. of scintillator would represent a sampling device which did not disturb the cascade, he would have made an error of a factor $E_3/E_1=1.9$ at 5.8 r.l. or, $E_3/E_1=4.26$ at 24 r.l.

On the other hand, 2×10^{-3} r.l. of argon result only in an error of $E_3/E_1=1.05$–1.1, and this can be neglected.

As a third example, it may be assumed that a cascade which had developed in a light material (air) is being measured by a scintillator of thickness 5 cm. This scintillator may be shielded from the air by 2 mm of iron (10^{-1} r.l.). It is then seen that the cascade loses an energy E_4 in the 10^{-1} r.l. of iron, while it would have lost an energy E_5 if it had passed through 10^{-1} r.l. of the light material. It may be assumed that this energy difference occurs as additional light signal in the 1.2×10^{-1} r.l. of scintillator on top of the expected one E_6 yielding a total of E_7. E_7/E_6 is the error which one is likely to make if one neglects the influence of 2-mm iron shielding in an otherwise uniform medium.

Transition at	E_4	E_5	E_6	E_7	E_7/E_6
5.8 r.l.	3.42×10^4	9.4×10^4	1.13×10^5	1.73×10^5	1.53
13.0 r.l.	2.32×10^5	5.6×10^5	6.72×10^5	1.0×10^6	1.5
24 r.l.	3.7×10^4	8.3×10^4	1.0×10^5	1.46×10^5	1.46

From these considerations, it is seen how important a sufficiently thin shielding is. Alternatively, if it is not possible to reduce the material above the scintillator, it appears to be advisable to place some 5 cm of plain Plexiglas on top of the scintillator inside the iron shielding, since the 10^{-1} r.l. of iron will distort the energy spectrum up to energies of some 10^{-1} ϵ_{iron}, and these particles are completely brought to rest within 5 cm of plastic material.

DISCUSSION

Within the framework of approximation B, it has been shown that the arrangement of the measuring devices has a decisive influence on the result. In particular, the measurements carried out using plastic scintillators together with other materials showed a very pronounced influence of the transition effect. This happens also if only a thin layer of material having different ϵ is traversed in an otherwise uniform medium. Unfortunately, the transition effect is also dependent upon the parameter s, i.e., on the shape of the energy spectra of electrons and photons, if the measuring layer is thicker than several times 10^{-3} r.l. Usually, the observer wishes to determine the energy loss of a cascade created by a nuclear interaction in order to measure its total energy by summing up the signals received from various layers of scintillator. However, due to secondary nuclear interactions, the energy spectra of electrons and photons are subject to large fluctuations in these cascades, and are further dependent upon the model of the interaction assumed for cascade calculations. The most convincing point about the ionization spectrometer,

namely, that one uses first principles without special assumptions, thus does not seem to hold if scintillator layers in lead or iron absorbers are used, or if for example, brass-walled ionization chambers are placed in graphite or lead absorbers, as may have been the case in some experiments.[12,13]

Any kind of measurement of the total energy contained in a cascade must thus use experimental equipment which disturbs the cascade very little irrespective of the energy spectrum. This can be done by (a) using very thin sampling devices, like ionization chambers, whose walls are made of material with a critical energy similar to the absorber material, or (b) by using scintillator together with absorber materials which have a value of $\epsilon_0 \sim 70$ MeV so that no transition effect occurs.

CONCLUSIONS

It has been pointed out that even if the transition effect had been calculated accurately, it is still dependent upon the energy spectra of electrons and photons, and these are expected to have particularly large fluctuations in nuclear cascades where secondary interactions close to the layer of measurement may give a large contribution to the signal. Thus, even if the transition effect had been calculated accurately, or the average behavior had been established experimentally, one would expect large fluctuations from the average behavior in individual cases. One of the most attractive features of ionization spectrometers, the determination of the primary energy from first principles, would then be lost. It has been shown using approximation B that the transition effect becomes sizeable if transition layers of more than several times 10^{-3} radiation lengths are involved, and that the effect then depends on the shower age s. This can be understood by recognizing the fact that in approximation B, 50% of the ionization produced, i.e., 50% of the signal in a recording device, comes from electrons of energies less than $4\times10^{-1}\epsilon_0$ at $s=0.6$, less than $1.7\times10^{-1}\ \epsilon_0$ at $s=1.0$, and less than $7\times10^{-2}\ \epsilon_0$ at $s=1.4$. It is clear that a change in the ionization loss per radiation length must introduce a rapid change in the signal. Of course approximation B then does not seem to be the right theoretical framework in which to discuss such a question. However, the situation cannot be expected to be improved by taking into account the correct cross sections at low energies, and other processes like the Compton effect and multiple scattering. (For example, multiple scattering tends to increase the effective layer traversed by the particles and thus tends to increase the transition effect.)

The general conclusions which should be drawn from the results of this discussion are that great care must be applied in the interpretation of signals received from scintillators or ionization chambers if inhomogeneities in ϵ_0 over thicknesses of more than 10^{-3} radiation lengths were encountered prior to measurements. This (a) necessitates the choice of particular absorber materials, if plastic scintillators are to be used in ionization spectrometers, (b) makes it advisable to use ionization chambers built of the same material as the absorber in ionization spectrometers, and (c) requires care in the construction of housing for plastic scintillators in extensive air-shower experiments.

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[12] V. V. Guseva, N. A. Dobrotin, N. G. Zelevinskata, K. A. Kotelnikov, A. M. Levedev, and S. A. Slavatinsky, J. Phys. Soc. Japan **17**, Suppl. A III, 375 (1962).

[13] H. D. Babayan, N. G. Bejadjan, Ya. Sa. Babecki, Z. A. Buja, H. L. Grigorov, J. Loskiewicz, J. Massalski, A. Oles, C. A. Tretyakova, and V. Ya. Schestoerov, J. Phys. Soc. Japan **17**, Suppl. A III, 383 (1962).