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Possible Effects of CP Violation in $K^{\pm}\to 3\pi$ Decay*

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The effects of CP violation in $K\to 3\pi$ decay have been investigated. In a linear energy approximation for the decay matrix elements, one can prove, using the TCP theorem alone, that the partial decay rate of the τ (or τ') mode (and also θ modes) for the K^+ meson is nearly equal to that for the K^- meson, provided that one can neglect contributions from the final $I=3$ pion configuration. Also, effects of final pion-pion interactions have been computed by extending the dispersion-theoretical treatment of Khuri and Treiman.

I. INTRODUCTION AND SUMMARY OF RESULTS

THE existence of the decay mode $K_2^0\to 2\pi$ has been reported by Christenson, Cronin, Fitch, and Turlay,[1] and this has been subsequently confirmed by others.[2] Although several possible explanations[3] of this phenomenon have been advocated, the simplest is undoubtedly that of assuming CP violation[4] in the weak interaction. In this paper, we shall adopt this viewpoint and will investigate its consequences for the $K^{\pm}\to 3\pi$ decays:

$$K^+\to \pi^+ + \pi^+ + \pi^-, \tag{1a}$$

$$K^+\to \pi^0 + \pi^0 + \pi^+, \tag{1b}$$

$$K^-\to \pi^- + \pi^- + \pi^+, \tag{1c}$$

$$K^-\to \pi^0 + \pi^0 + \pi^-. \tag{1d}$$

If CP invariance holds true, then the decay modes Eq. (1a) and Eq. (1c) must have the same pion energy spectrum and the same decay rate. The same is true for decays Eqs. (1b) and (1d). Since we do not assume CP invariance in this paper, these are no longer valid, although the CPT theorem[5] demands that the sum of the decay rates of Eqs. (1a) and (1b) must be equal to that of Eqs. (1c) and (1d), as we shall see in the next section. However, we shall show also in the next section that the decay rate of Eq. (1a) must be equal to that of Eq. (1c), if we use the linear-energy approximation for the decay matrix element and, furthermore, if we can neglect contributions from the final $I=3$ pion state.

The second assumption essentially corresponds to the usual hypothesis of assuming the $\Delta I=\frac{1}{2}$, $\frac{3}{2}$ selection rule but not $\Delta I=\frac{5}{2}$. Thus, the experimental near-equality of partial rates of Eqs. (1a) and (1c) does not necessarily require CP invariance. To test that, we have to look at the energy spectrum of the final pions. We have performed a detailed computation of this problem in Secs. III and IV using a dispersion-theoretic technique which is a certain generalization of the method used by Khuri and Treiman. It is pointed out that the present experimental data may be consistent with the possibility of maximal CP violation in $K\to 3\pi$ decays.

II. PHENOMENOLOGICAL ANALYSIS

It is well known[6,7] that the CPT theorem requires a particle and its antiparticle to have the same lifetime. However, for partial decay rates, this may not be true[7] unless C or CP invariance holds, if we have strong interactions in its final state.

Now the unitarity condition gives us[8]

$$S^{\dagger}S=1. \tag{2a}$$

If we set

$$S=1+R, \tag{2b}$$

then we can rewrite Eq. (2a) as

$$S^{\dagger}+S^{\dagger}R=1. \tag{2c}$$

Taking the matrix element of Eq. (2c) for $K\to 3\pi$, and $\bar{K}\to 3\pi$ decay, one finds

$$\langle 3\pi|S^{\dagger}|K\rangle+\textstyle\sum_n \langle 3\pi|S^{\dagger}|n\rangle\langle n|R|K\rangle=0, \tag{3a}$$

$$\langle 3\pi|S^{\dagger}|K\rangle+\textstyle\sum_n \langle 3\pi|S^{\dagger}|n\rangle\langle n|R|\bar{K}\rangle=0. \tag{3b}$$

The intermediate state $|n\rangle$ on the left-hand side of Eqs. (3a) and (3b) must include all possible decay

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[1] J. H. Christenson, J. W. Cronin, V. L. Fitch, and R. Turlay, Phys. Rev. Letters **13**, 138 (1964).

[2] A. Abashian, R. J. Abrams, D. W. Carpenter, G. P. Fisher, B. M. K. Nefkens, and J. H. Smith, Phys. Rev. Letters **13**, 243 (1964); W. Galbraith, G. Manning, A. E. Taylor, B. D. Jones, J. Malos, A. Astbury, N. H. Lipman, and T. G. Walker, *ibid.* **14**, 383 (1965).

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[6] G. Lüders and B. Zumino, Phys. Rev. **106**, 385 (1957); T. D. Lee, R. Oehme, and C. N. Yang, *ibid.* **106**, 340 (1957).

[7] S. Okubo, Phys. Rev. **109**, 984 (1958).

[8] This method is originally due to Chou Kuang Chao, Nucl. Phys. **9**, 652 (1958/59). Although some readers may wonder whether unitarity is applicable to decay problems, the final result, Eq. (6), turns out to be the same as that obtained from the conventional method described in Ref. 7.

modes of $K^{\pm}$. Now, since we are considering a weak process, we can neglect higher order effects with respect to the weak interaction. Thus, the intermediate states $|n\rangle$ can be restricted to only 2π or 3π states, if we neglect electromagnetic decay modes.

Furthermore, a $2\pi\to 3\pi$ transition corresponding to $\langle 3\pi|S^\dagger|2\pi\rangle$ cannot proceed by strong interaction, because both the 2π and 3π states must have zero angular momentum and parity must be conserved. (Or, we may alternatively use G conjugation.) Therefore, we can restrict the summation in Eq. (3) to only the 3π intermediate states.

When one sets

$$\langle f|S|i\rangle=\delta_{fi}-2\pi i\delta(E_i-E_f)M(i\to f) \tag{4}$$

and notes that the CPT theorem gives

$$\langle f|S^\dagger|i\rangle=(\langle \bar{f}|S|\bar{i}\rangle)^*, \tag{5}$$

then Eq. (3) can be written as[7]

$$\begin{aligned} M^*(\bar{K}\to\bar{f})&=\textstyle\sum_n S^*(f\to n)M(K\to n)\,,\\ M^*(K\to f)&=\textstyle\sum_n S^*(\bar{f}\to n)M(\bar{K}\to n)\,, \end{aligned} \tag{6}$$

where f is the 3π final state of K decay and $\bar{f}$ is its CPT conjugate state. The summation in Eq. (6) is restricted to 3π states as we noted in the above, and $S(f\to n)$ designates the S-matrix element for strong interactions alone.

Now, the total decay rates for $K\to 3\pi$ decay are given by

$$\begin{aligned} w(K\to 3\pi)&=2\pi\sum_{f=3\pi}|M(K\to f)|^2\delta(E_k-E_f)\,,\\ w(\bar{K}\to 3\pi)&=2\pi\sum_{f=3\pi}|M(\bar{K}\to f)|^2\delta(E_k-E_f)\,. \end{aligned} \tag{7}$$

Then, from Eq. (6) and the unitarity of the S matrix, we find

$$w(K\to 3\pi)=w(\bar{K}\to 3\pi) \tag{8}$$

or, explicitly, that

$$\begin{aligned} &w(K^+\to\pi^+\pi^+\pi^-)+w(K^+\to\pi^0+\pi^0+\pi^+)\\ &\quad=w(K^-\to\pi^-+\pi^-+\pi^+)+w(K^-\to\pi^0+\pi^0+\pi^-)\,. \end{aligned} \tag{9}$$

This result has been already noted by Lüders and Zumino.[6] However, we can sharpen Eq. (8) as follows. We note that the final pion states can have total isospin $I=0, 1, 2, 3$. Then, define $M_I(K\to 3\pi)$ as the decay matrix element for $K\to 3\pi$ with 3π total isotopic spin I. If we notice that the strong interactions conserve isotopic spin, then Eq. (6) can be written as

$$\begin{aligned} M_I^*(\bar{K}\to\bar{f})&=\textstyle\sum_n S_I^*(f\to n)M_I(K\to n)\,,\\ M_I^*(K\to f)&=\textstyle\sum_n S_I^*(\bar{f}\to n)M_I(\bar{K}\to n)\,, \end{aligned} \tag{10}$$

where S_I is the S matrix evaluated between states of isospin I. Again, when we define

$$\begin{aligned} w_I(K\to 3\pi)&=2\pi\sum_{f=3\pi}|M_I(K\to f)|^2\delta(E_k-E_f)\,,\\ w_I(\bar{K}\to 3\pi)&=2\pi\sum_{f=3\pi}|M_I(\bar{K}\to f)|^2\delta(E_k-E_f)\,, \end{aligned} \tag{11}$$

then we find

$$w_I(K\to 3\pi)=w_I(\bar{K}\to 3\pi)\quad (I=0, 1, 2, 3)\,. \tag{12}$$

Because

$$\begin{aligned} w(K\to 3\pi)&=\textstyle\sum_I w_I(K\to 3\pi)\,,\\ w(\bar{K}\to 3\pi)&=\textstyle\sum_I w(\bar{K}\to 3\pi)\,, \end{aligned} \tag{13}$$

Eq. (8) follows immediately.

Now, as an application of Eq. (12), let us first consider the case of $I=1$. Then we can set, as usual,[9]

$$M_1(K\to 3\pi)=\boldsymbol{\alpha}(\boldsymbol{\beta}\cdot\boldsymbol{\gamma})A+\boldsymbol{\beta}(\boldsymbol{\gamma}\cdot\boldsymbol{\alpha})B+\boldsymbol{\gamma}(\boldsymbol{\alpha}\cdot\boldsymbol{\beta})C\,, \tag{14}$$

where vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$ are the isospin of pions 1, 2, 3 and A, B, C are scalar functions of the kinetic energies T_1, T_2, T_3 of the three pions in the rest system of the kaon. Owing to the Bose statistics obeyed by the pions, we have

$$\begin{aligned} A(T_1,T_2,T_3)&=C(T_3,T_2,T_1)\,,\\ B(T_1,T_2,T_3)&=C(T_1,T_3,T_2)\,,\\ C(T_1,T_2,T_3)&=C(T_2,T_1,T_3)\,. \end{aligned} \tag{15}$$

Choosing particles 1 and 2 to be the identical ones, the matrix elements for K^+ decays are now given by

$$\begin{aligned} &M_1(K^+\to\pi^+(1)\pi^+(2)\pi^-(3))\\ &\qquad=A(T_1,T_2,T_3)+B(T_1,T_2,T_3)\,, \end{aligned} \tag{16}$$

$$M_1(K^+\to\pi^0(1)\pi^0(2)\pi^+(3))=C(T_1,T_2,T_3)\,. \tag{17}$$

The corresponding matrix elements for K^- decay may similarly be written as

$$\begin{aligned} &M_1(K^-\to\pi^-(1)\pi^-(2)\pi^+(3))\\ &\qquad=\bar{A}(T_1,T_2,T_3)+\bar{B}(T_1,T_2,T_3)\,, \end{aligned} \tag{18}$$

$$M_1(K^-\to\pi^0(1)\pi^0(2)\pi^-(3))=\bar{C}(T_1,T_2,T_3)\,. \tag{19}$$

It is more convenient to use the S_i, defined by

$$S_i=T_i-\tfrac{1}{3}Q\,, \tag{20a}$$

$$S_1+S_2+S_3=0\,. \tag{20b}$$

where Q is the Q value of $K\to 3\pi$ decay [i.e., $Q=m_K-3\mu$, where m_K (μ) is the kaon (pion) mass].

Taking into account the experimental fact that the Dalitz plot for $K^+\to 3\pi$ is almost isotropic and since the S_i are rather small quantities, we expand A, B, and C in S_i, retaining only the first-order terms[10,11]

$$C(S_1,S_2,S_3)=a+c(S_1+S_2)+dS_3\,,$$

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[9] For example, M. Gell-Mann and A. H. Rosenfeld, Ann. Rev. Nucl. Sci. 7, 407 (1957).

[10] S. Weingerg, Phys. Rev. Letters 4, 87 (1960).

[11] R. Sawyer and K. C. Wali, Nuovo Cimento 17, 938 (1960); and Phys. Rev. 119, 1429 (1960).

where Eq. (15) has been taken into account, and a, c, d are some constants. Using Eq. (20), we can write this in the following form.

$$C(S_1,S_2,S_3)=a+bS_3. \tag{21a}$$

Then by (15) we have

$$A(S_1,S_2,S_3)=a+bS_1, \tag{21b}$$

$$B(S_1,S_2,S_3)=a+bS_2. \tag{21c}$$

Similarly, the matrix elements for K^- decay are given by

$$\bar{C}=\bar{a}+\bar{b}S_3;\quad \bar{B}=\bar{a}+\bar{b}S_2;\quad \bar{A}=\bar{a}+\bar{b}S_1. \tag{22}$$

The absolute squares of the matrix elements are, neglecting quadratic terms, in S_i,

$$|M_1(K^+\to\pi^+\pi^+\pi^-)|^2=4[|a|^2-\mathrm{Re}(a^*b)S_3], \tag{23}$$

$$|M_1(K^+\to\pi^0\pi^0\pi^+)|^2=|a|^2+2\,\mathrm{Re}(a^*b)S_3, \tag{24}$$

$$|M_1(K^-\to\pi^-\pi^-\pi^+)|^2=4[|\bar{a}|^2-\mathrm{Re}(\bar{a}^*\bar{b})S_3], \tag{25}$$

$$|M_1(K^-\to\pi^0\pi^0\pi^-)|^2=|\bar{a}|^2+2\,\mathrm{Re}(\bar{a}^*\bar{b})S_3. \tag{26}$$

Now we note the formula

$$\textstyle\sum_f S_3=\frac{1}{3}\sum_f (S_1+S_2+S_3)=0, \tag{27}$$

where the summation is taken over the phase space, and also we have neglected the mass difference between π^+ and π^0. Now, by virtue of Eqs. (11) and (12), this results in

$$\textstyle\sum_f|a|^2=\sum_f|\bar{a}|^2,$$

which gives

$$|a|^2=|\bar{a}|^2. \tag{28}$$

If we assume the $\Delta I=\frac{1}{2}$ rule, then we have only an $I=1$ final state. Therefore, we can replace M_1 by M and, we find equal partial-decay widths for Eqs. (7a) and (2a), etc.:

$$\textstyle\sum_f|M(K^+\to\pi^+\pi^+\pi^-)|^2=\sum_f|M(K^-\to\pi^-\pi^-\pi^+)|^2 \tag{29a}$$

or

$$w(K^+\to\pi^+\pi^+\pi^-)=w(K^-\to\pi^-\pi^-\pi^+), \tag{29b}$$

$$\textstyle\sum_f|M(K^+\to\pi^0\pi^0\pi^+)|^2=\sum_f|M(K^-\to\pi^0\pi^0\pi^-)|^2 \tag{30a}$$

or

$$w(K^+\to\pi^0\pi^0\pi^+)=w(K^-\to\pi^0\pi^0\pi^-) \tag{30b}$$

besides the well-known branching ratios[10–12]

$$w(K^+\to\pi^+\pi^+\pi^-)/w(K^+\to\pi^0\pi^0\pi^+)=4, \tag{31}$$

$$w(K^-\to\pi^-\pi^-\pi^+)/w(K^-\to\pi^0\pi^0\pi^-)=4. \tag{32}$$

The relations (29)–(32) hold true even if we take into account the $I=2$ term, because this term does not contain a symmetric space part in the linear approximation. We note that these relations were obtained without using CP invariance.

If an $I=3$ term exists ($\Delta I=\frac{5}{2}$ rule or $\Delta I=\frac{7}{2}$ rule), these relations (29)–(32) must be modified,[13] because of interference between the $I=1$ and $I=3$ symmetric final states. Experimentally, relation (29) seems to be well satisfied, indicating that $I=3$ is absent or very small. Also in the $K_{20}\to 3\pi$ decay, the $\Delta I=\frac{5}{2}$ interaction appears to be small experimentally. Therefore we do not consider the possibility of a nonzero $I=3$ term hereinafter. Thus, we conclude that Eqs. (29) and (30) hold if the linear matrix approximation is good and if we have no $I=3$ final state of the pions. Now we consider the energy spectrum of the unlike pion.

If CP invariance holds, (1a) [(1b)] has the same spectrum as (1c) [(1d)], respectively. However, if CP invariance and C invariance do not hold, the spectrum has in general different shapes and will satisfy the following relation:

$$\frac{|M_I(K^+\to\pi^+\pi^+\pi^-)|^2-|M_I(K^-\to\pi^-\pi^-\pi^+)|^2}{|M_I(K^+\to\pi^0\pi^0\pi^+)|^2-|M_I(K^-\to\pi^0\pi^0\pi^-)|^2}=-2\quad\text{for}\quad I=1$$
$$=+2\quad\text{for}\quad I=2. \tag{33}$$

We characterize the energy spectrum by defining a slope λ_I so that, for example, the spectrum of the unlike pion in decay mode (1a) is given by

$$\sim 1+\lambda_I[K^+\to\pi^+\pi^+\pi^-]S_3. \tag{34}$$

Then the above relation can be expressed in terms of λ_I:

$$\frac{\lambda_I[K^+\to\pi^+\pi^+\pi^-]-\lambda_I[K^-\to\pi^-\pi^-\pi^+]}{\lambda_I[K^+\to\pi^0\pi^0\pi^+]-\lambda_I[K^-\to\pi^0\pi^0\pi^-]}=-2\ \text{for}\ I=1$$
$$=+2\ \text{for}\ I=2. \tag{35}$$

Similarly, we have the following relation:

$$\frac{\lambda_I[(K^+\to\pi^+\pi^+\pi^-)+(K^-\to\pi^-\pi^-\pi^+)]}{\lambda_I[(K^+\to\pi^0\pi^0\pi^+)+(K^-\to\pi^0\pi^0\pi^-)]}=-\tfrac{1}{2}\ \text{for}\ I=1$$
$$=+\tfrac{1}{2}\ \text{for}\ I=2. \tag{36}$$

[12] C. Zemach, Phys. Rev. **133**, B1201 (1964).

[13] In this case, we have

$$\textstyle\sum_f|M(K^+\to\pi^+\pi^+\pi^-)|^2=\sum_f|2a_1+a_3|^2,$$
$$\textstyle\sum_f|M(K^+\to\pi^0\pi^0\pi^+)|^2=\sum_f|a_1-2a_3|^2,$$
$$\textstyle\sum_f|M(K^-\to\pi^-\pi^-\pi^+)|^2=\sum_f|2\bar{a}_1+\bar{a}_3|^2$$
$$\textstyle\sum_f|M(K^-\to\pi^0\pi^0\pi^-)|^2=\sum_f|\bar{a}_1-2\bar{a}_3|^2$$

with $|a_1|^2=|\bar{a}_1|^2$, $|a_3|^2=|\bar{a}_3|^2$, where a_3 ($\bar{a}_3$) is a constant term which comes from $I=3$ final-state amplitudes.

Finally, we remark that the same method is also applicable to $K_0 \to 3\pi$ decays.[14] Restricting ourselves to the $I=1$ state alone, we may expand

$$M_1(K_0 \to \pi^+(1)\pi^-(2)\pi^0(3)) = a' + b'S_3, \quad (37a)$$

$$M_1(K_0 \to \pi^0(1)\pi^0(2)\pi^0(3)) = 3a', \quad (37b)$$

$$M_1(\bar{K}_0 \to \pi^-(1)\pi^+(2)\pi^0(3)) = \bar{a}' + \bar{b}'S_3, \quad (38a)$$

$$M_1(\bar{K}_0 \to \pi^0(1)\pi^0(2)\pi^0(3)) = 3\bar{a}', \quad (38b)$$

which are the analogs of Eqs. (21a)–(21c) and (22). Again, Eq. (12) results in

$$|\bar{a}'| = |a'|. \quad (39)$$

If we assume the $\Delta I=\frac{1}{2}$ rule furthermore, we get

$$\begin{aligned} a' &= a/\sqrt{2}, \quad b' = b/\sqrt{2}, \\ \bar{a}' &= \bar{a}/\sqrt{2}, \quad \bar{b}' = \bar{b}/\sqrt{2}. \end{aligned} \quad (40)$$

Hence, when we write

$$\begin{aligned} |K_1^0\rangle &= p|K_0\rangle - q|\bar{K}_0\rangle, \\ |K_2^0\rangle &= p|K_0\rangle + q|\bar{K}_0\rangle, \quad |p|^2+|q|^2=1, \end{aligned} \quad (41)$$

we find

$$\begin{aligned} M_1(K_2^0 &\to \pi^+(1)\pi^-(2)\pi^0(3)) \\ &= (\sqrt{2})^{-1}(pa+q\bar{a}) + (\sqrt{2})^{-1}(pb+q\bar{b})S_3, \end{aligned} \quad (42a)$$

$$M_1(K_2^0 \to \pi^0(1)\pi^0(2)\pi^0(3)) = (3/\sqrt{2})(pa+q\bar{a}), \quad (42b)$$

$$\begin{aligned} M_1(K_1^0 &\to \pi^+(1)\pi^-(2)\pi^0(3)) \\ &= (\sqrt{2})^{-1}(pa-q\bar{a}) + (\sqrt{2})^{-1}(pb-q\bar{b})S_3, \end{aligned} \quad (43a)$$

$$M_1(K_1^0 \to \pi^0(1)\pi^0(2)\pi^0(3)) = (3/\sqrt{2})(pa-q\bar{a}). \quad (43b)$$

From Eqs. (42a) and (43a), we notice that in our approximation π^+ and π^- in the K_1^0 and K_2^0 decays have symmetrical energy spectra. Hence, if one finds an asymmetrical energy spectrum for π^+ and π^-, this means that we must have an $I=2$ final pion state. This conclusion is independent of the $\Delta I=\frac{1}{2}$ rule, and also of the presence of $I=0$ and $I=3$ pion states. In the future, we hope to be able to check some of these predictions.

III. DERIVATION OF THE MODIFIED KHURI-TREIMAN RELATION

In this section, we shall discuss the effect of the final-state interactions on the energy spectrum of $K^\pm \to 3\pi$ decays when CP invariance does not hold. Here we shall restrict our discussion to the $I=1$ final state and write M instead of M_1. This would be justified if the $\Delta I=\frac{1}{2}$ rule holds.

Following Khuri and Treiman,[15] we may use the following artifice to simplify some of the writing. Since we suppose the pions to come out in an $I=1$ state we can, as a matter of convenience, pretend that the K meson has isotopic spin unity and that isospin is conserved in the decay. In this sense, we denote by ρ the charge state of the K meson. Invariant matrix elements $M_{\rho\alpha\beta\gamma}$ and $\bar{M}_{\rho\alpha\beta\gamma}$ are now introduced by

$$\begin{aligned} (16\omega_1\omega_2\omega_3E_K)^{1/2}S(K^{(\rho)} &\to \pi^{(\alpha)}(1)\pi^{(\beta)}(2)\pi^{(\gamma)}(3)) \\ &= -(2\pi)^4 i\delta^4(p_K-p_1-p_2-p_3)M_{\rho\alpha\beta\gamma}, \end{aligned} \quad (44a)$$

$$\begin{aligned} (16\omega_1\omega_2\omega_3E_K)^{1/2}S(\bar{K}^{(\rho)} &\to \pi^{(\alpha)}(1)\pi^{(\beta)}(2)\pi^{(\gamma)}(3)) \\ &= -(2\pi)^4 i\delta^4(p_K-p_1-p_2-p_3)\bar{M}_{\rho\alpha\beta\gamma}, \end{aligned} \quad (44b)$$

where

$$M_{\rho\alpha\beta\gamma} = A\delta_{\rho\alpha}\delta_{\beta\gamma} + B\delta_{\rho\beta}\delta_{\gamma\alpha} + C\delta_{\rho\gamma}\delta_{\alpha\beta}, \quad (45a)$$

$$\bar{M}_{\rho\alpha\beta\gamma} = \bar{A}\delta_{\rho\alpha}\delta_{\beta\gamma} + \bar{B}\delta_{\rho\beta}\delta_{\gamma\alpha} + \bar{C}\delta_{\rho\gamma}\delta_{\alpha\beta}. \quad (45b)$$

Actually, our A, B, C, $\bar{A}$, $\bar{B}$, and $\bar{C}$ are essentially the same quantities as those introduced in the previous section, apart from a common multiplicative factor, and this is the reason we use the same notation.

Now, let us introduce the usual Mandelstam variables:

$$\begin{aligned} s &= -(p_K-p_1)^2, \\ t &= -(p_K-p_2)^2, \\ u &= -(p_K-p_3)^2. \end{aligned} \quad (46a)$$

Then we have

$$s+t+u = m_K^2+3\mu^2, \quad (46b)$$

where μ is the pion mass. In the rest system of the K meson, s, t, and u are related to the kinetic energy T_i of the pions by

$$\begin{aligned} s &= (m_K-\mu)^2 - 2m_KT_1, \\ t &= (m_K-\mu)^2 - 2m_KT_2, \\ u &= (m_K-\mu)^2 - 2m_KT_3. \end{aligned} \quad (46c)$$

Hence, A, B, C, $\bar{A}$, $\bar{B}$, and $\bar{C}$ can be regarded as functions of s, t, and u.

Similarly, we may define the T matrix for 3π-3π scattering by

$$(S-1)_{\alpha'\beta'\gamma',\alpha\beta\gamma} = -(2\pi)^4 i\delta^{(4)}(p_1+p_2+p_3-p_1'-p_2'-p_3')T_{\alpha'\beta'\gamma'\alpha\beta\gamma}. \quad (47)$$

Then, we can rewrite Eq. (6) as

$$\begin{aligned} &\bar{M}_{\rho\alpha\beta\gamma}(s,t,u) - M_{\rho\alpha\beta\gamma}^*(s,t,u) \\ &= -(2\pi)^4 i\frac{(\omega_1\omega_2\omega_3)^{1/2}}{3!}\sum_{\alpha'\beta'\gamma'}\int\frac{d^3p_1'd^3p_2'd^3p_3'}{(2\pi)^9}\frac{1}{(\omega_1'\omega_2'\omega_3')^{1/2}}\delta^4(p_1'+p_2'+p_3'-p_K)M_{\rho\alpha'\beta'\gamma'}^*(s't'u')T_{\alpha'\beta'\gamma',\alpha\beta\gamma}, \end{aligned} \quad (\mathrm{I})$$

[14] There is an experimental datum on $K_1^0 \to \pi^+\pi^-\pi^0$, which is compatible with CP conservation [J. A. Anderson, F. S. Crawford, R. L. Golden, D. Stern, T. O. Binford, and V. G. Lind, Phys. Rev. Letters 14, 475 (1965)].

[15] N. N. Khuri and S. B. Treiman, Phys. Rev. 119, 1115 (1960).

$$M_{\rho\alpha\beta\gamma}(s,t,u)-\bar{M}_{\rho\alpha\beta\gamma}{}^*(s,t,u)$$
$$=-(2\pi)^4 i\frac{(\omega_1\omega_2\omega_3)^{1/2}}{3!}\sum_{\alpha'\beta'\gamma'}\int\frac{d^3p_1'd^3p_2'd^3p_3'}{(2\pi)^9}\frac{1}{(\omega_1'\omega_2'\omega_3')^{1/2}}\delta^4(p_1'+p_2'+p_3'-p_K)\bar{M}_{\rho\alpha'\beta'\gamma'}{}^*(s't'u')T_{\alpha'\beta'\gamma',\alpha\beta\gamma}. \quad \text{(II)}$$

For the 3π-3π scattering amplitude T, we make the approximation that only one pair of pions interact at a time. This is expected to be correct for the relatively weak final-state interactions. Suppose first that pion 2 and pion 3 interact. We have

$$\bar{M}_{\rho\alpha\beta\gamma}(s,t,u)-M_{\rho\alpha\beta\gamma}{}^*(s,t,u)=(-i)(2\pi)^4(\omega_1\omega_2\omega_3)^{1/2}\frac{1}{2!}\int\frac{d^3p_1'd^3p_2'd^3p_3'}{(2\pi)^9}\delta^4(p_1+p_2+p_3-p_1'-p_2'-p_3')$$
$$\times\sum_{\beta'\gamma'}\frac{M_{\rho\alpha\beta'\gamma'}{}^*}{(\omega_1'\omega_2'\omega_3')^{1/2}}T_{\beta'\gamma',\beta\gamma}(2\pi)^3\delta^3(p_1'-p_1), \quad (48)$$

where $T_{\beta'\gamma',\beta\gamma}$ is the π-π scattering amplitude.

Since the Q value of the $K^+\to 3\pi$ decay is rather low, we assume that the dominant contribution comes from the pion-pion S-wave scattering and neglect the contributions from intermediate states with high mass. We take the point of view[15] that the high-mass contributions may affect the total decay rate, but not the spectrum shape which concerns us here. $T_{\beta'\gamma',\beta\gamma}$ can be expressed in terms of the π-π scattering phase shifts in the center-of-mass system as follows:

$$\begin{aligned}T_{\beta'\gamma',\beta\gamma}(s)&=(1/s)b_{\beta'\gamma',\beta\gamma}(s).\\ b_{\beta'\gamma',\beta\gamma}&=\tfrac{1}{3}(b^{(0)}-b^{(2)})\delta_{\beta'\gamma'}\delta_{\beta\gamma}+\tfrac{1}{2}b^{(2)}(\delta_{\beta'\beta}\delta_{\gamma'\gamma}+\delta_{\beta'\gamma}\delta_{\beta\gamma'}),\\ b^{(I)}(s)&=-32\pi[s/(s-4\mu^2)]^{1/2}e^{i\delta^{(I)}(s)}\sin\delta^{(I)}(s),\end{aligned} \quad (49)$$

where $\delta^{(I)}$ is the pion-pion S-wave scattering phase shift in the isospin state I.

Substituting these formulas into Eq. (48) and performing the isospin summation and momentum integration, we obtain

$$\bar{M}_{\rho\alpha\beta\gamma}-M_{\rho\alpha\beta\gamma}{}^*=2i\{\delta_{\rho\alpha}\delta_{\beta\gamma}[A_S{}^*(s)f^{(0)}(s)+(B_S{}^*(s)+C_S{}^*(s))\tfrac{1}{3}(f^{(0)}(s)-f^{(2)}(s))]$$
$$+\delta_{\rho\beta}\delta_{\alpha\gamma}[\tfrac{1}{2}f^{(2)}(s)(B_S{}^*(s)+C_S{}^*(s))]+\delta_{\rho\gamma}\delta_{\alpha\beta}[\tfrac{1}{2}f^{(2)}(s)(B_S{}^*(s)+C_S{}^*(s))]\},$$

where $f^{(I)}(s)=e^{i\delta^{(I)}(s)}\sin\delta^{(I)}(s)$ and $A_S(s)$, $B_S(s)$, and $C_S(s)$ are the S-wave amplitudes of A, B, and C, respectively, in the c.m. system of pion 2 and pion 3.

Now we include the interaction between particle 1 and particle 3 and the one between particle 2 and particle 1. This is easily done if we notice the boson property of pions.

Combining, we have

$$\begin{aligned}\bar{M}_{\rho\alpha\beta\gamma}(s,t,u)-M_{\rho\alpha\beta\gamma}{}^*(s,t,u)&\\ =2i\{&\delta_{\rho\alpha}\delta_{\beta\gamma}[U(s)+V(t)+V(u)]\\ &+\delta_{\rho\beta}\delta_{\alpha\gamma}[V(s)+U(t)+V(u)]\\ &+\delta_{\rho\gamma}\delta_{\alpha\beta}[V(s)+V(t)+U(u)]\},\end{aligned} \quad (50)$$

where

$$U(x)=\{A_S{}^*(x)+\tfrac{1}{3}(B_S{}^*+C_S{}^*)\}f^{(0)}(x)-\tfrac{1}{3}(B_S{}^*+C_S{}^*)f^{(2)}(x), \quad \text{(50a)}$$

$$V(x)=\tfrac{1}{2}(B_S{}^*(x)+C_S{}^*(x))f^{(2)}(x)\quad (x=s,\,t,\,u). \quad \text{(50b)}$$

We can write this in the following form:

$$\bar{A}(s,t,u)-A^*(s,t,u)=2i[U(s)+V(t)+V(u)], \quad \text{(Ia)}$$
$$\bar{B}(s,t,u)-B^*(s,t,u)=2i[V(s)+U(t)+V(u)], \quad \text{(Ib)}$$
$$\bar{C}(s,t,u)-C^*(s,t,u)=2i[V(s)+V(t)+U(u)]. \quad \text{(Ic)}$$

However, these relations are not independent since we have

$$A(s,t,u)=C(u,t,s),\quad B(s,t,u)=C(s,u,t),\ \text{etc.} \quad (51)$$

(Ia) and (Ib) are automatically satisfied if (Ic) is, so we consider only the relation (Ic) hereinafter.

(Ic) can be written

$$\bar{C}(s,t,u)-C^*(s,t,u)=2i\sum_{I=0,2}[V^{(I)}(s)+V^{(I)}(t)+U^{(I)}(u)], \quad (52)$$

where

$$\begin{aligned}U^{(0)}&=[A_S{}^*+\tfrac{1}{3}(B_S{}^*+C_S{}^*)]e^{i\delta^{(0)}}\sin\delta^{(0)},\\ U^{(2)}&=-\tfrac{1}{3}(B_S{}^*+C_S{}^*)e^{i\delta^{(2)}}\sin\delta^{(2)},\\ V^{(0)}&=0,\\ V^{(2)}&=\tfrac{1}{2}(B_S{}^*+C_S{}^*)e^{i\delta^{(2)}}\sin\delta^{(2)}\quad(\equiv-\tfrac{3}{2}U^{(2)}).\end{aligned} \quad (53)$$

Now if we start from relation II instead of I, we get

$$C(s,t,u)-\bar{C}^*(s,t,u)=2i\sum_{I=0,2}[\bar{V}^{(I)}+\bar{V}^{(I)}(t)+\bar{U}^{(I)}(u)], \quad (54)$$

where

$$\begin{aligned}\bar{U}^{(0)}&=\{\bar{A}_S{}^*+\tfrac{1}{3}(\bar{B}_S{}^*+\bar{C}_S{}^*)\}e^{i\delta^{(0)}}\sin\delta^{(0)},\\ \bar{U}^{(2)}&=-\tfrac{1}{3}(\bar{B}_S{}^*+\bar{C}_S{}^*)e^{i\delta^{(2)}}\sin\delta^{(2)},\\ \bar{V}^{(0)}&=0,\\ \bar{V}^{(2)}&=\tfrac{1}{2}(\bar{B}_S{}^*+\bar{C}_S{}^*)e^{i\delta(2)}\sin\delta^{(2)}.\end{aligned} \quad (55)$$

Comparing Eqs. (52) and (54), we find

$$g(s)+g(t)+f(u)=0\,, \tag{56}$$

where

$$g(s)\equiv \sum_{I=0,2}[V^{(I)}(s)-(\bar{V}^{(I)}(s))^*]\,,$$
$$f(u)\equiv \sum_{I=0,2}[U^{(I)}(u)-(\bar{U}^{(I)}(u))^*]\,. \tag{57}$$

Noting Eq. (46b), i.e., $s+t+u=m_K^2+3\mu^2$, we see that Eq. (56) is a function equation which holds for arbitrary values of s and t within a certain range of kinematically allowed values. If we assume that $g(s)$ and $f(u)$ are differentiable with respect to s and u, respectively, then we can easily solve this function equation with the following general solution:

$$f(u)=a+bu\,,$$
$$g(s)=-\tfrac{1}{2}a+b[s-\tfrac{1}{2}(m_K^2+3\mu^2)]\,, \tag{58}$$

where a and b are arbitrary constants. Now note that the pion-pion phase shifts $\delta(s)^{(I)}(I=0,\,2)$ go to zero in the zero-energy limit, $s\to 4\mu^2$. Then, from Eqs. (53) and (55), we see that $g(s)\to 0$ and $f(u)\to 0$ for $s\to 4\mu^2$, $u\to 4\mu^2$, provided that A or A^* remains finite in that limit, which is reasonable experimentally. Therefore, we conclude from Eq. (58) that $a\equiv b\equiv 0$. Thus again $f(u)\equiv g(s)\equiv 0$ by Eq. (58). Hence, Eq. (57) now reads:

$$\bar{A}_S+\tfrac{1}{3}(\bar{B}_S+\bar{C}_S)=[A_S^*+\tfrac{1}{3}(B_S^*+C_S^*)]e^{2i\delta^{(0)}}\,, \tag{58}$$

$$\bar{B}_S+\bar{C}_S=[B_S^*+C_S^*]e^{2i\delta^{(2)}}\,. \tag{59}$$

When we introduce two real functions $X^{(I)}(x)$ and $\varphi_I(x)$ by

$$A_S(x)+\tfrac{1}{3}[B_S(x)+C_S(x)]=X^{(0)}(x)e^{i\varphi_0(x)}e^{i\delta^{(0)}(x)}\,, \tag{60}$$

$$B_S(x)+C_S(x)=X^{(2)}(x)e^{i\varphi_2(x)}e^{i\delta^{(2)}(x)}\,, \tag{61}$$

it follows from Eqs. (58) and (59) that

$$\bar{A}_S(x)+\tfrac{1}{3}[\bar{B}_S(x)+\bar{C}_S(x)]=X^{(0)}(x)e^{-i\varphi_0(x)}e^{i\delta^{(0)}(x)}\,, \tag{60a}$$

$$\bar{B}_S(x)+\bar{C}_S(x)=X^{(2)}(x)e^{-i\varphi_2(x)}e^{i\delta^{(2)}(x)}\,. \tag{61a}$$

Comparing these with Eq. (53), we have

$$U^{(0)}=X^{(0)}\sin\delta^{(0)}e^{-i\varphi_0}\,,\quad V^{(0)}=0\,, \tag{62a}$$

$$U^{(2)}=-\tfrac{2}{3}V^{(2)}=-\tfrac{1}{3}X^{(2)}\sin\delta^{(2)}e^{-i\varphi_2}\,. \tag{62b}$$

If CP invariance (or C invariance) holds, then we have $\varphi_0=\varphi_2=0$.

Since the right-hand sides of Eqs. (52) and (54) consist of the contributions from the $I=0$ and $I=2$ 2π states, we may separate C and $\bar{C}$ by

$$C=C^{(0)}+C^{(2)}\,,$$
$$\bar{C}=\bar{C}^{(0)}+\bar{C}^{(2)}\,, \tag{63}$$

with the condition

$$\bar{C}^{(I)}-C^{(I)*}=2i[V^{(I)}(s)+V^{(I)}(t)+U^{(I)}(u)] \quad (I=0,\,2)\,, \tag{64}$$

so that Eq. (54) is automatically satisfied. Of course, the decomposition Eq. (63) is not unique. In fact, for an arbitrary complex number α, we may change $C^{(I)}$ and $\bar{C}^{(I)}$ as

$$C^{(0)}\to C^{(0)}+\alpha\,,$$
$$C^{(2)}\to C^{(2)}-\alpha\,,$$
$$\bar{C}^{(0)}\to \bar{C}^{(0)}+\alpha^*\,,$$
$$\bar{C}^{(2)}\to \bar{C}^{(2)}-\alpha^*\,,$$

without affecting Eqs. (63) and (64). Taking advantage of this freedom, we can normalize $C^{(I)}$ and $\bar{C}^{(I)}$ so as to satisfy the following additional condition:

$$|\bar{C}^{(I)}|=|C^{(I)}| \quad (I=0,\,2) \tag{65}$$

provided that we have

$$\mathrm{Im}\{[V^{(0)}(s)+V^{(0)}(t)+U^{(0)}(u)]^* \times[V^{(2)}(s)+V^{(2)}(t)+U^{(2)}(u)]\}\neq 0\,. \tag{66}$$

Noting Eqs. (62), this condition is equivalent to

$$\sin(\varphi_0-\varphi_2)\neq 0\,. \tag{67}$$

If CP invariance is violated, then Eq. (67) would in general hold. If CP is conserved, then Eq. (67) is not true. But in that case, we can use the result of the calculation of Khuri and Treiman[15] and there is no problem. At any rate, the case $\sin(\varphi_0-\varphi_2)\equiv 0$ may be regarded as the limiting case of Eq. (67). Indeed, our final formula agrees with that of Khuri and Treiman in the limit of $\varphi_0-\varphi_2\to 0$. Therefore, we believe that we can assume Eq. (67) without having any difficulty. Then, we can set the additional condition Eq. (65) as has been noted in the above. Thus, $C^{(I)}$ and $\bar{C}^{(I)}$ may be written as

$$C^{(I)}=e^{i\theta_I}\tilde{C}^{(I)}\,,\quad \bar{C}^{(I)}=e^{-i\theta_I}\tilde{C}^{(I)}\,,\quad (I=0,\,2)$$

where θ_I is a real number. Inserting these into Eq. (64) and using Eqs. (53) and (62), we find that we must have $\theta_I\equiv\varphi_I$. Thus we obtain

$$C^{(I)}=e^{i\varphi_I}\tilde{C}^{(I)}\,,$$
$$\bar{C}^{(I)}=e^{-i\varphi_I}\tilde{C}^{(I)}\,, \tag{68}$$

and then Eqs. (64) with Eq. (62) gives

$$\mathrm{Im}\tilde{C}^{(0)}=\sin\delta^{(0)}(u)X^{(0)}(u)\,, \tag{69}$$

$$\mathrm{Im}\tilde{C}^{(2)}=\tfrac{1}{2}\sin\delta^{(2)}(s)X^{(2)}(s)+\tfrac{1}{2}\sin\delta^{(2)}(t)X^{(2)}(t) -\tfrac{1}{3}\sin\delta^{(2)}(u)X^{(2)}(u)\,. \tag{69a}$$

Now, the real parts of the $C^{(I)}$ can be determined by using dispersion relations. We assume that the $\tilde{C}^{(I)}$ (but not the $C^{(I)}$) satisfies the once-subtracted dispersion relations of Cini and Fubini.[16]

We summarize the above procedure. We write C and $\bar{C}$ in the following form:

$$C=e^{i\varphi_0}\tilde{C}^{(0)}+e^{i\varphi_2}\tilde{C}^{(2)}\,, \tag{70}$$

$$\bar{C}=e^{-i\varphi_0}\tilde{C}^{(0)}+e^{-i\varphi_2}\tilde{C}^{(2)}\,. \tag{70a}$$

[16] M. Cini and S. Fubini, Ann. Phys. (N.Y.) 3, 352 (1960).

Then from the CPT theorem, we determine $\mathrm{Im}\tilde{C}^{(I)}$ by Eqs. (69) and (69a). $\mathrm{Re}\tilde{C}^{(I)}$ can be determined from the dispersion relation, so that finally we obtain

$$\tilde{C}^{(I)}(s,t,u)=D^{(I)}+W^{(I)}(s)+W^{(I)}(t)+Z^{(I)}(u), \quad (71)$$

where

$$W^{(0)}(x)=\frac{x-x_0}{\pi}\int_{4\mu^2}^{\infty}dx'\frac{\sin\delta^{(0)}(x')X^{(0)}(x')}{(x'-x_0)(x'-x-i\epsilon)}, \quad (72a)$$

$$W^{(2)}(x)=-\frac{1}{3}\frac{x-x_0}{\pi}\int_{4\mu^2}^{\infty}dx'\frac{\sin\delta^{(2)}(x')X^{(2)}(x')}{(x'-x_0)(x'-x-i\epsilon)}, \quad (72b)$$

$$Z^{(2)}(x)\equiv-\tfrac{3}{2}W^{(2)}(x), \quad (72c)$$

$$Z^{(0)}(x)=0, \quad (72d)$$

with

$$X^{(0)}=e^{-i\delta^{(0)}}[\tilde{A}_S+\tfrac{1}{3}(\tilde{B}_S+\tilde{C}_S)], \quad (73)$$

$$X^{(2)}=e^{-i\delta^{(2)}}[\tilde{B}_S+\tilde{C}_S]. \quad (73a)$$

On the derivation of Eqs. (73) and (73a), we have replaced $\bar{A}_S$, $\bar{B}_S$, and $\bar{C}_S$ by $\tilde{A}_S$, $\tilde{B}_S$, and $\tilde{C}_S$ in Eqs. (60a) and (61a). This is justifiable because the combination $\tilde{A}_S+\frac{1}{3}(\tilde{B}_S+\tilde{C}_S)$ and $\tilde{B}_S+\tilde{C}_S$ correspond to the pure $I=0$ and $I=2$ 2π states, respectively. $D^{(I)}$ in Eq. (71) is a subtraction constant which is chosen so that

$$\tilde{C}^{(I)}(x_0,x_0,x_0)=D^{(I)} \quad \text{at} \quad s=t=u=x_0. \quad (74)$$

In order to maintain a correspondence with the procedure used by Khuri and Treiman, it is more useful to consider the quantity $\tilde{C}$ defined by

$$\tilde{C}=\tilde{C}^{(0)}+\tilde{C}^{(2)} \quad (75)$$

and also to introduce a subtraction constant D by

$$\tilde{C}(x_0,x_0,x_0)=D \quad \text{at} \quad s=t=u=x_0. \quad (76)$$

Then, we have

$$D=D^{(0)}+D^{(2)}. \quad (77)$$

It can be easily seen that our expression for C reduces to the complex conjugate form of the expression C given by Khuri and Treiman if CP invariance holds. The integral equations are solved by the iteration method, replacing $\tilde{A}_S$, $\tilde{B}_S$, and $\tilde{C}_S$ by the constant D.[15,17] This corresponds to small final-state interactions which may be reasonable because of the experimental evidence that the matrix element of $K^+\to 3\pi$ is practically a constant.

Then we have

$$\tilde{C}^{(0)}=D^{(0)}+(5/3)DI^{0}(u), \quad (78)$$

$$\tilde{C}^{(2)}=D^{(2)}+D[I^{(2)}(s)+I^{(2)}(t)-\tfrac{2}{3}I^{(2)}(u)], \quad (78a)$$

where

$$I^{(I)}(x)=\frac{x-x_0}{\pi}\int_{4\mu^2}^{\infty}dx'\frac{e^{-i\delta^{(I)}(x')}\sin\delta^{(I)}(x')}{(x'-x_0)(x'-x-i\epsilon)}\,(x=s,\,t,\,u). \quad (79)$$

From Eq. (51), we can write down A, B, $\bar{A}$, and $\bar{B}$ immediately;

$$A=e^{i\varphi_0}\tilde{A}^{(0)}+e^{i\varphi_2}\tilde{A}^{(2)}, \quad (80)$$

$$B=e^{i\varphi_0}\tilde{B}^{(0)}+e^{i\varphi_2}\tilde{B}^{(2)}, \quad (81)$$

$$\bar{A}=e^{-i\varphi_0}\tilde{A}^{(0)}+e^{-i\varphi_2}\tilde{A}^{(2)}, \quad (80a)$$

$$\bar{B}=e^{-i\varphi_0}\tilde{B}^{(0)}+e^{-i\varphi_2}\tilde{B}^{(2)}, \quad (81a)$$

with

$$\tilde{A}^{(0)}=D^{(0)}+(5/3)DI^{(0)}(s), \quad (82)$$

$$\tilde{A}^{(2)}=D^{(2)}+D[-\tfrac{2}{3}I^{(2)}(s)+I^{(2)}(t)+I^{(2)}(u)], \quad (82a)$$

$$\tilde{B}^{(0)}=D^{(0)}+(5/3)DI^{(0)}(t), \quad (83)$$

$$\tilde{B}^{(2)}=D^{(2)}+D[I^{(2)}(s)-\tfrac{2}{3}I^{(2)}(t)+I^{(2)}(u)]. \quad (83a)$$

Now the absolute square of the matrix element for $K^+\to 2\pi^0+\pi^+$ is

$$|C|^2=|\tilde{C}^{(0)}|^2+|\tilde{C}^{(2)}|^2+2\cos(\varphi_2-\varphi_0)\times\mathrm{Re}[\tilde{C}^{(2)}\tilde{C}^{0*}]-2\sin(\varphi_2-\varphi_0)\,\mathrm{Im}[\tilde{C}^{(2)}\tilde{C}^{0*}], \quad (84)$$

while that of $K^-\to 2\pi^0+\pi^-$ is

$$|\bar{C}|^2=|\tilde{C}^{(0)}|^2+|\tilde{C}^{(2)}|^2+2\cos(\varphi_2-\varphi_0)\times\mathrm{Re}[\tilde{C}^{(2)}\tilde{C}^{0*}]-2\sin(\varphi_2-\varphi_0)\,\mathrm{Im}[\tilde{C}^{(2)}\tilde{C}^{0*}], \quad (85)$$

where $\tilde{C}^{(I)}$ is given by Eqs. (78), (78a), and (79). Similar expressions are obtained for $|A+B|^2$ and $|\bar{A}+\bar{B}|^2$. At first we check the CPT theorem, i.e.,

$$\textstyle\sum_f[|C|^2+|A+B|^2]=\sum_f[|\bar{C}|^2+|\bar{A}+\bar{B}|^2]. \quad (86)$$

It is equivalent to show that

$$\textstyle\sum_f\{\mathrm{Im}[\tilde{C}^{(2)}\tilde{C}^{(0)*}]+\mathrm{Im}[(\tilde{A}^{(2)}+\tilde{B}^{(2)})(\tilde{A}^{(0)}+\tilde{B}^{(0)})^*]\}=0.$$

Had we used the nonsubtracted dispersion relations for $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, this could be easily shown. If we use the once-subtracted dispersion relation, the above identity will not hold in general due to the arbitrariness of the subtraction constants $D^{(0)}$ and $D^{(2)}$, although these are connected by $D=(D^{(0)}+D^{(2)})$. However, it is reasonable to choose $D^{(0)}$ and $D^{(2)}$ so that the above identity holds even after a subtraction. In the scattering-length approximation for the pion-pion scattering, $D^{(2)}$ and $D^{(0)}$ are given in this way by

$$D^{(0)}=[5a_0/(5a_0+4a_2)]D, \quad (87)$$

$$D^{(2)}=[4a_2/(5a_0+4a_2)]D, \quad (87a)$$

where a_I is the scattering length. [See Eq. (88).]

Our model shows then that

$$\textstyle\sum_f|C|^2=\sum_f|\bar{C}|^2, \quad (30)$$

$$\textstyle\sum_f|A+B|^2=\sum_f|\bar{A}+\bar{B}|^2. \quad (29)$$

[17] The Khuri and Treiman equation has been investigated extensively. See, for instance, I. J. R. Aitchison, Phys. Rev. 137, B1070 (1965). Earlier references will be found in this article.

Also from Eqs. (91) and (96) we have (see the next section)

$$\sum_f |A+B|^2 = 4\sum_f |C|^2. \qquad (31)$$

IV. SCATTERING-LENGTH APPROXIMATION AND COMPARISON WITH EXPERIMENT

Since our model is closely related to Khuri and Treiman's[15] approach, we follow their approximation in the following. The scattering phase shift for S-wave pion-pion scattering is expressed in terms of the dimensionless scattering length as

$$[(x-4\mu^2)/x]^{1/2}\cot\delta^{(I)}(x) = 1/a_I, \qquad (88)$$

which accords well with the S-wave dominant solution of the pion-pion problem obtained by Chew and Mandelstam.[18] With this form we have

$$e^{-i\delta^{(I)}}\sin\delta^{(I)} \approx [(x-4\mu^2)/x]^{1/2} a_I, \qquad (89)$$

which is consistent with the weak pion-pion interaction. Under this approximation, $I^{(I)}(x)$ is evaluated to be

$$I^{(I)}(x) = J^{(I)}(x) - J^{(I)}(x_0), \qquad (90)$$

with

$$J^{(I)}(x) = (a_I/x^{1/2})[(x-4\mu^2)^{1/2} i - (x-4\mu^2)/\pi\mu]. \qquad (90a)$$

To calculate further, we neglect the quadratic terms of the final state interactions assuming that the final-state interaction is small. After some calculations, we find

$$|C|^2 = |D|^2\{[1-(80\xi/(5+4\xi)^2)\sin^2(\tfrac{1}{2}y)] - (20/3\pi)a_0 h(u)[1-\xi+(2\xi/(5+4\xi))\sin^2(\tfrac{1}{2}y)] - (10\xi a_0 \sin y/(5+4\xi))[n(s)+n(t)-2n(u)]\}, \qquad (91)$$

$$|\bar{C}|^2 = \text{Eq. (91)} \quad \text{with} \quad y\to -y, \qquad (91a)$$

where

$$y = \varphi_2 - \varphi_0, \qquad (92)$$

$$\xi = a_2/a_0, \qquad (93)$$

$$h(x) = (x-x_0)/2\mu x_0^{1/2}, \qquad (94)$$

$$n(x) = [(x-4\mu^2)^{1/2} - (x_0-4\mu^2)^{1/2}]/x_0^{1/2}, \quad (x=s,\,t,\,u). \qquad (95)$$

Similarly, the square of the matrix element for $K^+\to\pi^+\pi^+\pi^-$ decay is given by

$$|A+B|^2 = 4|D|^2\{[1-(80\xi/(5+4\xi)^2)\sin^2(\tfrac{1}{2}y)] + (10/3\pi)a_0 h(u)[1-\xi+(2\xi/(5+4\xi))\sin^2(\tfrac{1}{2}y)] + (5\xi a_0\sin y/(5+4\xi))[n(s)+n(t)-2n(u)]\}, \qquad (96)$$

$$|\bar{A}+\bar{B}|^2 = \text{Eq. (96)} \quad \text{with} \quad y\to -y. \qquad (96a)$$

$h(u)$ and $n(x)$ can be expressed in terms of the variables S_i introduced in Sec. I.

$$h(u) = -S_3\left\{\frac{3(m_K/\mu)^2}{(m_K^2+3\mu^2)}\right\}^{1/2}, \qquad (94a)$$

$$n(s)+n(t)-2n(u) = \left\{\frac{6m_K}{m_K^2+3\mu^2}\right\}^{1/2}[(\bar{S}-S_1)^{1/2}+(\bar{S}-S_2)^{1/2}-2(\bar{S}-S_3)^{1/2}] \qquad (97)$$

with $\bar{S}=(m_K^2-9\mu^2)/6m_K$. If we select experimental points so that $\bar{S}\gg S_1, S_2, S_3$, then we can expand this as

$$n(s)+n(t)-2n(u) \approx 9S_3\{m_K^2/(m_K^2+3\mu^2)(m_K^2-9\mu^2)\}^{1/2} \quad \text{for} \quad S_1, S_2, S_3 \ll \bar{S}. \qquad (97a)$$

Now we consider the following quantities:

$$|C|^2 - |\bar{C}|^2 = -|D|^2(20\xi a_0\sin y/(5+4\xi)) \times [n(s)+n(t)-2n(u)] \qquad (98)$$

and

$$|A+B|^2 - |\bar{A}+\bar{B}|^2 = |D|^2(40\xi a_0\sin y/(5+4\xi)) \times [n(s)+n(t)-2n(u)]. \qquad (99)$$

If CP invariance does not hold ($\sin y\neq 0$), the spectrum of $K^+\to 2\pi^0\pi^+$ decay and that of $K^-\to 2\pi^0\pi^-$ will have a different shape and the difference will be given by the above formula. In particular, the energy dependence of the difference will be proportional to S_3 in the region where all S_i are small compared with $\bar{S}$. The same thing is true for $K^+\to 2\pi^+\pi^-$ and $K^-\to 2\pi^-\pi^+$.

If CP invariance is violated, we get the "$\Delta I=\frac{1}{2}$ rule" prediction

$$(|A+B|^2 - |\bar{A}+\bar{B}|^2)/(|C|^2 - |\bar{C}|^2) = -2. \qquad (33a)$$

Other experimentally interesting quantities are $|C|^2+|\bar{C}|^2$ and $|A+B|^2+|\bar{A}+\bar{B}|^2$. The energy spectrum of the unlike pion in the $K^\pm\to\pi^0\pi^0\pi^\pm$ decay has the following form:

$$|C|^2+|\bar{C}|^2 \sim 1+\lambda[(K^+\to\pi^0\pi^0\pi^+) + (K^-\to\pi^0\pi^0\pi^-)]S_3, \qquad (100)$$

[18] J. Kirz, J. Schwarz, and R. D. Tripp, Phys. Rev. **126**, 763 (1962); G. F. Chew and S. Mandelstam, *ibid.* **119**, 467 (1960). B. P. Desai, Phys. Rev. Letters **6**, 497 (1961).

where the slope λ, introduced in Sec. II, is now expressed by

$$\lambda[(K^+\to 2\pi^0\pi^+)+(K^-\to 2\pi^0\pi^-)] = \frac{1-\xi+[2\xi/(5+4\xi)]\sin^2(y/2)}{1-[80\xi/(5+4\xi)^2]\sin^2(y/2)}\frac{20}{3\pi}\frac{\mu_0}{\mu}\left\{\frac{3m_K^2}{m_K^2+3\mu^2}\right\}^{1/2}. \tag{100a}$$

Similarly, the slope of the energy spectrum of the unlike pion in $K^{\pm}\to\pi^{\pm}\pi^{\pm}\pi^{\mp}$ decays is given by

$$\lambda[(K^+\to 2\pi^+\pi^-)+(K^-\to 2\pi^-\pi^+)] = -\frac{1-\xi+[2\xi/(5+4\xi)]\sin^2(y/2)}{1-[80\xi/(5+4\xi)^2]\sin^2(y/2)}\frac{10}{3\pi}\frac{\mu_0}{\mu}\left\{\frac{3m_K^2}{m_K^2+3\mu^2}\right\}^{1/2}. \tag{101}$$

The ratio of these slopes gives the "$\Delta I=\frac{1}{2}$" prediction

$$\frac{\lambda[(K^+\to 2\pi^+\pi^-)+(K^-\to 2\pi^-\pi^+)]}{\lambda[(K^+\to 2\pi^0\pi^+)+(K^-\to 2\pi^0\pi^-)]}=-\tfrac{1}{2}. \tag{101a}$$

Now, let us compare our theory with the experimental data on $K\to 3\pi$ decay. First of all, we note that the experimental decay rates of the K^+ meson for the θ, τ, and τ' modes are experimentally almost equal[19] to those of the K^- meson in the corresponding modes. The equality of the θ modes can be readily explained[6] by the *CPT* theorem alone, if we use the method explained in Sec. II. To explain the equality of the τ and τ' branching ratios of the $K^{\pm}$ mesons, we can appeal to the theorem of Sec. II. Therefore, the experimental near-equality of these $K^{\pm}$-meson decay rates can be understood by the *CPT* theorem without having recourse to *CP* invariance.

The energy spectra[20–22] of the unlike pions in the $K^{\pm}\to\pi^{\pm}\pi^{\pm}\pi^{\mp}$ decays, appear on the basis of present experimental error. The simplest explanationof this fact is certainly to assume *CP* invariance. However, there is also another possibility. As is seen from Eqs. (98) and (99), identical energy spectra of unlike pions for the $K^{\pm}\to\pi^{\pm}+\pi^{\pm}+\pi^{\mp}$ decay can be obtained if we have $\sin y=0$, i.e., $y=0$ or π. The first choice $y=0$ corresponds to the case of *CP* invariance, while the second one, $y=\pi$ may be said to correspond to the case of maximal *CP* violation. Then, we get

(i) $\frac{1}{4}|A+B|^2=1-1.645a_0(S_3/\mu)(1-\xi)$ for $y=0$ (*CP* invariance),

(ii) $\frac{1}{4}|A+B|^2=1-(80\xi/(5+4\xi)^2)-1.645a_0(S_3/\mu)[1-\xi+(2\xi/5+4\xi)]$ for $y=\pi$ (maximal *CP* violation).

From comparison with the data we obtain[15]

$$a_2-a_0=0.95$$

for case (i), while case (ii) is consistent with the spectrum if, for instance, we have $a_0\approx-1$ and $\xi\ll 1$, although the determination is not unique. Unfortunately, these values of a_0 and $a_2=a_0\xi$ determined from $K^{\pm}\to 3\pi$ decays are inconsistent with values from other sources.[18,22,23] For instance, Hamilton *et al.*[23] gives $a_0=1.3\pm 0.4$ with a_2 also positive but much smaller. Therefore, if this discrepancy is really essential, then our assumption of neglecting *P*-wave pion-pion interaction may be questionable. However, we must bear in mind that the experimental data for the K^- decay spectrum is far from unambiguous, especially near the origin of the Dalitz plot where the linear approximation we made is applicable. Hence, there is a possibility of having nonzero $\sin y$ giving rise to different spectra for $K^{\pm}\to 3\pi$ decays. In such a case, we must select experimental points only near the origin of the Dalitz plot so as to maintain the validity of the linear approximation for the function $n(x)$ [see Eq. (97)]. However, the existing experimental data are not accurate enough to permit such an analysis at the moment. For a different approach to $K\to 3\pi$, see the recent article by Gaillard.[24]

Note added in proof. After completion of this paper, it came to our notice that S. L. Glashow and S. Weinberg [Phys. Rev. Letters **14**, 835 (1965)] have arrived at a similar conclusion of maximal *CP* violation in $K_{1,2}{}^0$ decay.

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