Long-Range Electromagnetic Forces on Neutral Particles*

G. FEINBERG
Department of Physics, Columbia University, New York, New York

AND

J. SUCHER†
Department of Physics and Astronomy, University of Maryland, College Park, Maryland

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We consider the long-range forces, i.e., those falling off as a power of the distance, which may act between pairs of particles, one of which is neutral and spinless. It is shown that these forces may easily be calculated from the discontinuity function in the momentum transfer of the scattering amplitude for the two particles. In particular, we have investigated the two-photon exchange force between two neutral, spinless systems, and the three-photon exchange force between a charged and a neutral, spinless system. In the former case, we find that the potential behaves as r^{-7} for large r, in contradiction to the London expression for the Van der Waals force, and in agreement with the result of Casimir and Polder. For the latter case, the potential is odd under charge conjugation and hence can convert a K_2 meson to a K_1 meson. We find again that the potential behaves as r^{-7}. It is found that such long-range electromagnetic interactions are presently unobservable in particle physics.

I. INTRODUCTION

SOME of the forces between elementary particles fall off, at large distances, as an inverse power of the distance r between the particles, rather than exponentially. Such forces we call long-range forces. In quantum theory, long-range forces may result from the exchange of massless quanta between the particles. The most familiar example of such a force is the inverse-square Coulomb interaction between charged particles, coming from the exchange of a single photon.

If one of the particles is neutral, single-photon exchange can still give rise to a long-range force, provided that the spin of the neutral particle is $\geq\frac{1}{2}$ and it possess some nonvanishing electromagnetic moment. An example is provided by the magnetic force between neutron and proton, which goes as $1/r^4$. The same is true for the magnetic force between two neutrons, so that neither particle need be charged in order that a long-range force may arise from one-photon exchange.

If the spin of the neutral particle is zero, then although one-photon exchange may still be possible, it will not give rise to a long-range force. One example of this is the interaction of neutral K mesons with charged particles. One-photon exchange can then occur through the charge form factor of the K^0 meson, but this gives rise only to a contact interaction between the K^0 meson and the charged particle.[1] If the spin of the neutral particle is not zero, but all its electromagnetic moments vanish anyway, there will still be no long-range force from one-photon exchange. An example of this is given by the two-component neutrino.

It is of interest to ask whether there are other mechanisms through which long-range forces could act on neutral spinless particles.

One long-range force which presumably exists between all particles is the inverse-square gravitational force. However, this is too weak to be of direct interest in particle physics, and we do not consider it here.

We might expect intuitively that the exchange of two or more massless quanta rather than a single quantum could also give rise to long-range forces. We shall see that this is in fact the case. The only particles in nature believed to be massless are photons and neutrinos. Forces due to neutrino pair exchange will be considered elsewhere. The purpose of this paper is to study the forces on neutral spinless particles arising from the exchange of two or three photons.

Neutral spinless particles may be divided into two classes. There are those like the π^0, which are their own antiparticle, and those like the K^0, which are distinct from their antiparticle. Since the photon is odd under charge conjugation, the π^0 cannot emit an odd number of photons, real or virtual. There is no such restriction for K^0 mesons. However, the amplitude for the emission of an even number of photons is the same for K^0 and $\bar{K}^0$. It is easy to see that a force which is opposite for K^0 and $\bar{K}^0$ will induce transitions between the "true" particle states $K_1{}^0$ and $K_2{}^0$. Thus a long-range force from three-photon exchange is of interest in connection with experiments in which transitions from $K_2{}^0$ to $K_1{}^0$ are observed in the presence of matter.[2]

It should be noted that two-photon exchange forces between two neutral particles have been known for some time; for the case of neutral molecules they are the well-known Van der Waals forces.[3] In this context, the two-photon exchange has been reconsidered by

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[1] G. Feinberg, Phys. Rev. **109**, 1381 (1958).

[2] L. B. Leipuner *et al.*, Phys. Rev. **132**, 2285 (1963). F. Eisler *et al.*, in Proceedings of the International Conference on Fundamental Aspects of Weak Interactions, Brookhaven National Laboratory Report, 1963, p. 82 (unpublished).

[3] F. London, Z. Phys. **63**, 245 (1930).

Casimir and Polder[4] who, using the techniques of field theory, found an r^{-7} behavior for the potential between two neutral atoms, each of angular momentum zero. This differed from the result of London, who, using only electrostatic effects, obtained a potential behaving as r^{-6}. We shall see that the result of Casimir and Polder is independent of any atomic model, and in fact holds for the force between any two neutral spinless systems coming from two-photon exchange.

We now outline the contents of the following sections. In Sec. II we discuss how the long-range force, if any, between two particles can be obtained from the properties of the discontinuity function in a spectral representation of the scattering amplitude, considered as a function of momentum transfer. In Sec. III we consider two-photon exchange between a neutral spinless particle and a charged particle (e.g., K^0 meson and proton) and find the long-range part of the force. In doing this, we obtain first the two-photon form factor of a neutral spinless particle, and use it to derive the above-mentioned generalization of the Casimir-Polder result, in a rather simple way. Also in this section we describe a simpler way to obtain the long-range part of the two-photon exchange force between a spinless neutral and a charged particle. This method is used in Sec. IV to obtain the three-photon exchange force between a K^0 meson and a proton. In Sec. V we consider the possibility of experimental detection of this force. The final section, VI, is devoted to a summary of our conclusions.

The Appendix contains a field-theoretic discussion of the one- and two-photon vertex function of a neutral particle.

II. PARTICLE FORCES AT LARGE DISTANCES AND SPECTRAL FUNCTION: GENERAL CONSIDERATIONS

Consider the elastic scattering of spinless particles "1" and "2," with masses M_1 and M_2, symbolized by

$$1+2\rightarrow 1'+2'. \tag{2.1}$$

We denote the four-momenta of the particles by p_i and p_i' in the initial and final state, respectively, and as usual define invariants s, t, and u by

$$s=(p_1+p_2)^2,\quad t=(p_1-p_1')^2,\quad u=(p_1-p_2')^2,$$

with

$$p_1+p_2=p_1'+p_2'.$$

The metric is chosen so that on the mass shell, $p_i^2=p_i'^2=M_i^2$ ($i=1, 2$) and

$$s+t+u=2M_1^2+2M_2^2.$$

[4] H. B. G. Casimir and D. Polder, Phys. Rev. **73**, 360 (1948). For more recent work, see I. E. Dzyaloshinskii, E. M. Lifshitz, and L. P. Pitaevski, Advan. Phys. **10**, 165 (1961); B. V. Derjaguin, Sci. Am. **203**, No. 1, 471 (1960).

In the c.m. system of reaction (2.1) we may put

$$p_1=(\omega_1,\mathbf{p}),\qquad p_2=(\omega_2,-\mathbf{p}),$$
$$p_1'=(\omega_1',\mathbf{p}'),\quad p_2'=(\omega_2',-\mathbf{p}').$$

Here

$$\omega_i=(M_i^2+\mathbf{p}^2)^{1/2}=\omega_i'$$

since

$$\mathbf{p}'^2=\mathbf{p}^2$$

on the energy shell. Then

$$s=(\omega_1+\omega_2)^2,\quad t=-2\mathbf{p}^2(1-\cos\Theta),$$
$$u=(\omega_1-\omega_2)^2-2\mathbf{p}^2(1+\cos\Theta), \tag{2.2}$$

where Θ is the scattering angle, and the physical region for the process (2.1) is given by

$$s\geq(M_1+M_2)^2,\quad 0\geq t\geq -4\mathbf{p}^2 \tag{2.3}$$

with

$$\mathbf{p}^2=[s-(M_1+M_2)^2][s-(M_1-M_2)^2]/4s. \tag{2.4}$$

Note also that in this region

$$u\leq(\omega_1-\omega_2)^2=(M_1^2-M_2^2)^2/s. \tag{2.5}$$

Now let $F=F(s,t)$ denote the invariant Feynman amplitude for the process (2.1) and let $F_d=F_d(s,t)$ denote the contribution to F of some subset "D" of the set of all Feynman diagrams for this process. We assume that for fixed s, F_d is an analytic function of t and admits a spectral representation of the form

$$F_d(s,t)=\frac{1}{\pi}\int_{t_0}^{\infty}\frac{A_d(s,t')}{t'-t}dt'+\frac{1}{\pi}\int_{u_0}^{\infty}\frac{B_d(s,u')}{u'-u}du'. \tag{2.6}$$

Here the first integral corresponds to a branch point at $t=t_0$, with a cut extending from t_0 to ∞ and the second integral to a branch point at $t=\bar{t}_0$, with

$$\bar{t}_0=2M_1^2+2M_2^2-s-u_0$$

and a cut extending from $\bar{t}_0$ to $-\infty$. The requirement that $F_d(s,t)$ have no singularity in the physical region, defined by Eq. (2.3) implies that, at least when $s\geq(M_1+M_2)^2$,

$$t_0\geq 0,\quad u_0\geq(M_1^2-M_2^2)^2/s.$$

We have ignored any subtractions which may be necessary in Eq. (2.6) for F. If, for example,

$$F_d=\sum_{n=0}^{N}a_nt^n+O[1/|t|](|t|\rightarrow\infty),$$

Eq. (2.6) remains valid *without changing the definition of A or B*, provided that the polynomial $\sum a_nt^n$ is added to the right-hand side of this equation. But such a term only introduces contact terms (delta functions and derivatives of delta functions) into V_d defined below, so that they are of no interest for large r. A similar conclusion may be reached even if the discontinuity of F across the branch line starting at t_0 ($\bar{t}_0$) does not vanish at $t=+\infty$ $(-\infty)$.

It is convenient to introduce "potentials" V_d and U_d, associated with F_d, in the following way. We note that with s, t, and u in the physical region and with $t'>t_0$, $u'>u_0$, we may write, using Eq. (2.2)

$$(t'-t)^{-1}=\int (4\pi r)^{-1}e^{-i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{r}}e^{-(\sqrt{t'})r}d\mathbf{r}$$

and, using also Eq. (2.5),

$$(u'-u)^{-1}=\int (4\pi r)^{-1}e^{-i(\mathbf{p}'+\mathbf{p})\cdot\mathbf{r}} \times\exp\{-[u'-(M_1^2-M_2^2)^2s^{-1}]^{1/2}r\}d\mathbf{r},$$

where $r=|\mathbf{r}|$ may be interpreted as the distance between particles 1 and 2. On defining an exchange operator, P_{ex}, acting on any function $\varphi(\mathbf{r})$ via

$$P_{\text{ex}}\varphi(\mathbf{r})=\varphi(-\mathbf{r}),$$

we may write

$$F_d(s,t)=\langle\mathbf{p}'|V_d+U_dP_{\text{ex}}|\mathbf{p}\rangle, \quad (2.7)$$

where V_d and U_d are (energy-dependent) "potentials," defined by

$$V_d=(4\pi r)^{-1}\int_{t_0}^{\infty}A_d(s,t)e^{-t^{1/2}r}dt \quad (2.8)$$

and

$$U_d=(4\pi r)^{-1}\int_{u_0}^{\infty}B_d(s,u) \times\exp\{-[u-(M_1^2-M_2^2)^2s^{-1}]^{1/2}r\}du. \quad (2.9)$$

We shall refer to V_d and U_d as the formal potentials (direct and exchange potentials, respectively) associated with the set D of Feynman diagrams under consideration. They have been defined to reproduce exactly the amplitude F_d when the matrix element of the operator $V_d+U_dP_{\text{ex}}$ is taken between plane wave states, $|\mathbf{p}\rangle$ and $|\mathbf{p}'\rangle$.

In the cases of interest in this paper the set D will generally contain only irreducible graphs. For such sets, the use of V_d in a Bethe-Salpeter equation will approximately include the contributions to the scattering amplitude of diagrams obtainable from those of D by iteration. (There is an approximation involved because V_d has been defined as the Fourier transform of an on-shell quantity whereas the iterations involve off-shell extensions of the graphs in D.) Furthermore, if V_d is used to obtain an effective potential to be inserted in a Schrödinger equation, suitable for describing scattering at low energies, one expects that the asymptotic form of any such effective potential will coincide with that of V_d. We will therefore be justified in referring to the results obtained by analysis of $V_d=V_d(r;s)$ as providing "forces" between the particles, which can act repeatedly. We note also that even if F_d, or equivalently the absorptive parts A_d and B_d, are "known," the analysis of the asymptotic form of V_d and U_d is not without interest, since an understanding of the nature of the interaction between pairs of particles at large distances, and at not too high energies, in terms of potentials has value both from the conceptual point of view and from the point of view of computation.

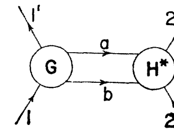

FIG. 1. Form of Feynman diagrams corresponding to the exchange of a pair of particles (a,b) between particles 1 and 2.

Although the "exchange potential" U_d may also have a long-range part, for our purpose the quantity of primary interest is the "direct potential" V_d. For the sake of orientation consider a set D of diagrams which involve the exchange of a pair of spinless particles, "a" and "b," between "1" and "2," as symbolized by Fig. 1. In this figure, the symbols G and H^* denote functions which, on analytic continuation in the variables p_1' and p_2, represent contributions to the scattering amplitudes for the processes $1+\bar{1}\to a+b$ and $a+b\to 2+\bar{2}$, respectively. The continuation is to values such that $\bar{p}_1$ and $\bar{p}_2$, defined by

$$\bar{p}_1=-p_1', \quad \bar{p}_2=-p_2,$$

are physical momenta for the antiparticles $\bar{1}$ and $\bar{2}$. Corresponding to Fig. 1 there is a contribution to $F(s,t)$ proportional to

$$\int H^*G\delta(Q-k_a-k_b)(k_a^2-m_a^2+i\epsilon)^{-1} \times(k_b^2-m_b^2+i\epsilon)^{-1}d^4k_ad^4k_b, \quad (2.10)$$

where m_a and m_b denote the masses of a and b and

$$Q=p_1+\bar{p}_1.$$

On continuation to the physical region of the crossed channel

$$1+\bar{1}'\to 2'+\bar{2}, \quad (2.11)$$

for which

$$t=Q^2$$

is the c.m. energy squared, the expression (2.10) represents a contribution to the amplitude for process (2.11), which will have a singularity at $t=t_0$, with

$$t_0=(m_a+m_b)^2,$$

in the absence of anomalous thresholds.

The discontinuity across this singularity is obtained from (2.10) by unitarity, or generalized unitarity[5] if $m_a+m_b<2M_1$, i.e., by replacing the propagators in (2.10) by delta functions. Thus, on setting

$$k_a=(\omega_a,\mathbf{k}), \quad k_b=(\omega_b,-\mathbf{k}),$$

with $\omega_{a,b}=(m_{a,b}^2+\mathbf{k}^2)^{1/2}$ in the c.m. system of reaction

[5] R. Cutkosky, J. Math. Phys. 1, 429 (1960).

(2.11), we get a contribution proportional to

$$I(t)\int H^*Gd\hat{\mathbf{k}}$$

to the absorptive part of F, for $t\geq t_0$. Here $I(t)$ is the phase-space integral

$$I(t)=\int d\mathbf{k}\delta(\sqrt{t}-\omega_a-\omega_b)/4\omega_a\omega_b.$$

On integration we get, using Eq. (2.4) with $s\to t$,

$$I(t)=(\pi/2)[t-(m_a+m_b)^2]^{1/2}\times[t-(m_a-m_b)^2]^{1/2}t^{-1}, \quad (2.12)$$

which, for

$$m_a\neq 0,\quad m_b\neq 0, \quad (2.13)$$

has a square-root type of branch point at $t=t_0>0$. Since the coefficient of $I(t)$ will in general be analytic at $t=t_0$, we are led to consider, when (2.13) holds, a spectral function of the form

$$A(s,t)=(t-t_0)^{1/2}\phi(s,t) \quad (2.14)$$

with $\phi(s,t)$ analytic in t in the neighborhood of $t=t_0$, say for $|t-t_0|<2T$. On substitution of (2.14) into Eq. (2.8) for V, the integral may be split into two parts, corresponding to integration over the intervals (t_0,T) and (T,∞). The second integral will contribute, at best, terms which for large r decrease exponentially like $\exp[-(t_0+T)r]$, perhaps multiplied by some power of r. For the interval (t_0,T) we put

$$\phi(s,t)=(t-t_0)^N\sum_{n=0}^{\infty}C_n(s)(t-t_0)^n,$$

extracting a possible zero of order $N\geq 0$ at $t=t_0$. We are thus led to consider integrals of the form

$$I_m=\int_{t_0}^{T}(t-t_0)^m e^{-(\sqrt{t})r}dt,$$

with $m=N+n+\frac{1}{2}$, for the contribution of the term in ϕ proportional to C_n. To obtain the leading term in I_m in an asymptotic expansion for large r, we may let $T\to\infty$ and introduce a new variable y via $t=(y+y_0)^2$, where

$$y_0=(t_0)^{1/2}.$$

Since for large r the major contribution to I_m comes from $t\sim t_0$, or $y\sim 0$, we may approximate

$$(t-t_0)^m=(y+2y_0)^m y^m\approx(2y_0)^m y^m,$$

and similarly, $dt\approx(2y_0)dy$, so that, on performing the now elementary integration on y, we get

$$I_m\sim(2y_0/r)^{m+1}m!e^{-y_0r}.$$

It follows that for large r, the V corresponding to Eq. (2.8) for A has the form

$$V\sim r^{-(N+5/2)}e^{-y_0r}.$$

For example, if $N=0$ ($\phi\neq 0$ at threshold t_0) and $m_a=m_b=\mu$ we get

$$V\sim e^{-2\mu r}/r^{5/2}.$$

This is in agreement with the spin-independent part of the asymptotic nucleon-nucleon force resulting from two pion exchange,[6] provided that μ is identified with the pion mass.

For the case $m_a=m_b=0$, of interest for this paper, the above discussion must be modified not only because $t_0=0$, so that there is no exponential, but also because in this case $I(t)$, the phase-space integral given by Eq. (2.12), reduces to a constant. Thus for example, with

$$A(s,t)=t^N\sum_{n=0}^{\infty}d_n(s)t^n \quad (d_0\neq 0) \quad (2.15)$$

in the interval $0\leq t\leq T$, the relevant integral is

$$J_m=\int_0^T t^m e^{-(\sqrt{t})r}dt$$

which for large r behaves as

$$J_m\sim 2(2m+1)!/r^{2m+2}.$$

It follows that

$$V\sim 1/r^{2N+3}, \quad (2.16)$$

a result which is valid whether or not N is an integer. For example, if $N=\frac{1}{2}$, so that A has a square-root type branch point, as in the case considered above, we have

$$V\sim 1/r^4.$$

On the other hand, if $N=0$

$$V\sim 1/r^3,$$

a type of force which is known to arise from two-photon exchange between charged particles.[7] In the next section we consider in some detail the nature of $A_d(s,t)$ for diagrams D corresponding to two-photon exchange between a pair of particles, at least one of which is neutral, and has spin zero.

III. TWO-PHOTON EXCHANGE FORCES

In part A of this section we consider the general form of the amplitude for the emission of a pair of photons by a spinless, neutral particle, designated as B^0. For completeness, and as an introduction to the two-photon case, we also consider the general form of the amplitude for the emission of a single photon by B^0. In part B, the results of part A are used to show,

[6] See, e.g., M. Lévy, Phys. Rev. **88**, 725 (1952).
[7] E. E. Salpeter, Phys. Rev. **84**, 328 (1952); T. Fulton and P. Martin, *ibid.* **95**, 811 (1954); J. Sucher, *ibid.* **109**, 1010 (1958) and Ph.D. thesis, Columbia University, 1957 (unpublished).

rather simply, that the long-range potential arising from two-photon exchange between any two neutral spinless systems behaves like r^{-7} for large r. In part C the results of part A are used to show that the two-photon exchange potential between B^0 and a charged particle behaves like r^{-4} for large r. This last result is rederived in a much simpler fashion in part D by consideration of a suitable phenomenological Lagrangian describing the interaction of the B^0 field and the electromagnetic field.

A. Amplitudes for One- and Two-Photon Emission

1. One-Photon Vertex Function Γ_μ.

Let Γ_μ denote the renormalized proper vertex function which describes the emission of a virtual photon of momentum $p-p'$ as B^0 makes a transition from a virtual state of momentum p to a virtual state of momentum p' (see Fig. 2). The most general form of the four-vector Γ_μ is a linear combination of the four-vectors p and p', or equivalently, of the four-vectors q and P defined by

$$q=p-p',\quad P=p+p'. \tag{3.1}$$

Thus, we may put

$$\Gamma_\mu=\alpha q_\mu+\beta P_\mu, \tag{3.2}$$

where α and β are functions of the invariants p^2, p'^2, and $p\cdot p'$, or equivalently, of the invariants q^2, P^2, and $q\cdot P$. If B^0 is on the mass shell, i.e., if

$$p^2=p'^2=m^2, \tag{3.3a}$$

where m is the B^0 mass, then

$$q\cdot P=0,\quad P^2=4m^2-q^2 \tag{3.3b}$$

and α and β become functions of q^2 only.

It is a consequence of the conservation of the electromagnetic current and the assumed neutrality of B^0 that (see the Appendix)

$$q^\mu\Gamma_\mu=0, \tag{3.4}$$

whether or not B^0 is on the mass shell. It follows from Eqs. (3.2) and (3.4) that

$$\alpha q^2+\beta q\cdot P=0 \tag{3.5}$$

for all values of q^2, P^2, and $q\cdot P$. If we now assume that $\alpha(q^2, P^2, q\cdot P)$ does not have a pole at $q^2=0$, with P^2 and $q\cdot P$ fixed, as is indicated by perturbation theory, we are led to put

$$\beta=q^2 g(q^2, P^2, q\cdot P),$$

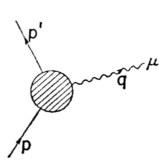

FIG. 2. Symbolic representation of the one-photon vertex function Γ_μ.

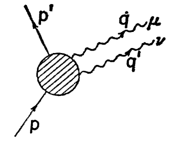

FIG. 3. Symbolic representation of the two-photon vertex function $\Gamma_{\mu\nu}$.

with g nonsingular at $q^2=0$. Then

$$-\alpha=q\cdot Pg,$$

and Eq. (3.2) becomes

$$\Gamma_\mu=g(q^2, P^2, q\cdot P)[-q\cdot Pq_\mu+q^2P_\mu], \tag{3.6}$$

which is the general form of Γ_μ, consistent with Eq. (3.4). From Eq. (3.3) it follows that on the B^0 mass shell

$$\Gamma_\mu=g_0(q^2)q^2P_\mu, \tag{3.7}$$

where $g_0(q^2)=g(q^2, 4m^2-q^2, 0)$. Equation (3.7) may be compared with the vertex factor for emission of a photon by a spin-zero particle with charge e: eP_μ, to lowest order in e. In general, one expects that for $q^2\sim 0$, $g_0\sim e/M_0^2$ where M_0^{-1} is an inverse mass characteristic of the range of the strong interactions of B^0.

We note further that invariance under charge conjugation implies that (see Appendix)

$$\Gamma_\mu(p',p)=-\Gamma_\mu(-p,-p'), \tag{3.8}$$

which yields

$$g(q^2, P^2, q\cdot P)=g(q^2, P^2, -q\cdot P).$$

[If B^0 is identical with its antiparticle, e.g., if $B^0=\pi^0$, then one also has

$$\Gamma_\mu(p',p)=\Gamma_\mu(-p,-p')$$

which, when combined with Eq. (3.8), implies that $\Gamma_\mu\equiv 0$.]

2. Two-Photon Vertex Function $\Gamma_{\mu\nu}$.

$\Gamma_{\mu\nu}$ is defined, analogous to Γ_μ, as the "proper" or "truncated" amplitude for the emission of two virtual photons by B^0, as symbolized by Fig. 3. The second-rank tensor $\Gamma_{\mu\nu}$ may be written as a linear combination of the metric tensor $g_{\mu\nu}$ and the $3\times 3=9$ independent tensors which may be constructed from the three linearly independent four-vectors available. Since

$$p-p'=q+q',$$

it is convenient to choose the photon momenta q, q' together with

$$P=p+p',$$

as for the one-photon vertex. We may then write

$$\Gamma_{\mu\nu}=\Gamma_{\mu\nu}'+\Gamma_{\mu\nu}'' \tag{3.9}$$

with

$$\Gamma_{\mu\nu}'=Ag_{\mu\nu}+BP_\mu P_\nu+Cq_\nu q_\mu'+Dq_\nu P_\mu+D'q_\mu'P_\nu \tag{3.10a}$$

and

$$\Gamma_{\mu\nu}''=Eq_\mu q_\nu'+Fq_\mu q_\nu+F'q_\mu'q_\nu'+Gq_\mu P_\nu+G'q_\nu'P_\mu. \quad (3.10b)$$

For later convenience we have separated $\Gamma_{\mu\nu}$ into two parts, $\Gamma_{\mu\nu}'$ and $\Gamma_{\mu\nu}''$ in such a way that $\Gamma_{\mu\nu}''$ is the sum of those terms which contain either a factor q_μ or a factor q_ν' (or both). Here, μ and ν are the indices associated with the photons of momentum q and q', respectively. The quantities $A, B, C\cdots$, are Lorentz-invariant functions of the four-vectors q, q', and P and so depend only on the values of the six scalar products

$$q^2,\, q'^2,\, P^2,\, P\cdot q,\, P\cdot q',\, q\cdot q'. \quad (3.11)$$

It can be shown that the conservation of the electromagnetic current and the neutrality of B^0 imply that (see Appendix)

$$q^\mu\Gamma_{\mu\nu}=0, \quad (3.12a)$$

and

$$q'^\nu\Gamma_{\mu\nu}=0 \quad (3.12b)$$

analogous to Eq. (3.4) for the one-photon vertex Γ_μ. Furthermore, since the emitted photons are identical particles, if we write $\Gamma_{\mu\nu}$ in the form

$$\Gamma_{\mu\nu}=\Gamma_{\mu\nu}(q,q';P),$$

we have

$$\Gamma_{\mu\nu}(q,q';P)=\Gamma_{\nu\mu}(q',q;P). \quad (3.13)$$

It follows that if Eq. (3.13) is satisfied then Eq. (3.12a) implies Eq. (3.12b). If we use the symbol X to denote any of the invariants A, B, C, E and write

$$X=X(P^2,\, q\cdot q',\, P\cdot q,\, P\cdot q',\, q^2,\, q'^2)$$

then Eq. (3.13) implies that

$$X=X|_{q\to q',\, q'\to q} \quad (3.14a)$$

since A, B, C, and E are coefficients of tensors which are invariant under the transformation $\mu\leftrightarrow\nu$, $q\leftrightarrow q'$. On the other hand, if Y denotes one of the invariants D, F, or G, and Y' the corresponding invariant D', F', or G', Eq. (3.13) implies that

$$Y'=Y|_{q\to q',\, q'\to q}. \quad (3.14b)$$

If B^0 is on the mass shell, we have

$$P^2=4m^2-(q^2+2q\cdot q'+q'^2)$$

and

$$P\cdot q=-P\cdot q',$$

so that the number of independent scalar products reduces to four. If these are taken to be

$$q^2,\, q'^2,\, q\cdot q'$$

and

$$\zeta=P\cdot q=-P\cdot q',$$

and we define

$$X_0(q^2,\, q'^2,\, q\cdot q',\, \zeta) \equiv X(4m^2-q^2-2q\cdot q'-q'^2,\, q\cdot q',\, \zeta,\, -\zeta,\, q^2,\, q'^2),$$

the symmetry condition (3.14a) reduces to

$$X_0(q^2,\, q'^2,\, q\cdot q',\, \zeta)=X_0(q^2,\, q'^2,\, q\cdot q',\, -\zeta). \quad (3.15a)$$

Similarly, if Y_0 is defined in terms of Y in the same way as X_0 is defined from X, Eq. (3.14b) reduces to

$$Y_0'(q^2,\, q'^2,\, q\cdot q',\, \zeta)=Y_0(q^2,\, q'^2,\, q\cdot q',\, -\zeta). \quad (3.15b)$$

We now consider the implications of Eq. (3.12a) for the invariants $A, B, \cdots G'$. On substitution of Eq. (3.9) into Eq. (3.12a) the coefficients of the four-vectors q_ν, q_ν' and P_ν may be set equal to zero, so that we obtain

$$A+q\cdot q'C+q\cdot PD+q^2F=0, \quad (3.16a)$$

$$q^2E+q\cdot q'F'+q\cdot PG'=0, \quad (3.16b)$$

$$q\cdot PB+q\cdot q'D'+q^2G=0. \quad (3.16c)$$

As in the one-photon case, we assume that none of the invariants $A, B, \cdots G'$ have poles at zero values of the independent variables listed in Eq. (3.11). To avoid a pole in D' at $q\cdot q'=0$, we put

$$B=bq\cdot q',$$

$$G=gq\cdot q',$$

with b and g regular at $q\cdot q'=0$, and solve Eq. (3.16c) for D':

$$D'=-q\cdot Pb-q^2g. \quad (3.17a)$$

Similarly, to avoid a pole in F' at $q\cdot q'=0$, we also set

$$E=eq\cdot q',$$

$$G'=g'q\cdot q',$$

and solve Eq. (3.16b) for F':

$$F'=-q^2e-q\cdot Pg'. \quad (3.17b)$$

On interchanging q and q' in Eqs. (3.17a) and (3.17b) and noting that the symmetry conditions imply that under the transformation

$$q\to q', \quad q'\to q$$

we have

$$b\to b, \quad e\to e, \quad g'\to g, \quad D'\to D, \quad F'\to F,$$

we get

$$D=-q'\cdot Pb-q'^2g', \quad (3.18a)$$

$$F=-q'^2e-q'\cdot Pg. \quad (3.18b)$$

On substitution of Eqs. (3.18a) and (3.18b) into Eq. (3.16a) we get, on solving for A,

$$A=-q\cdot q'C+q\cdot Pq'\cdot Pb+q^2q'^2e+q^2q'\cdot Pg+q'^2q\cdot Pg'.$$

From our hypothesis concerning the absence of poles it follows in particular that b is regular at $q\cdot P=0$ or $q'\cdot P=0$, C at $q\cdot q'=0$, etc. Thus, *on introducing a superscript zero to indicate a quantity evaluated on both the B^0 mass shell and the mass shell of the photons,*

$$q^2=0, \quad q'^2=0$$

we may write

$$\begin{aligned} &A^0=-q\cdot q'c^0-\zeta^2b^0, \quad E^0=q\cdot q'e^0, \\ &B^0=q\cdot q'b^0, \qquad F^0=\zeta g^0, \\ &C^0=c^0, \qquad F'^0=-\zeta g'^0, \\ &D^0=-D'^0=\zeta b^0, \quad G^0=q\cdot q'g^0, \quad G'^0=q\cdot q'g'^0. \end{aligned} \tag{3.19}$$

Here, b^0, c^0, and e^0 may be regarded as functions of $q\cdot q'$ and ζ, even under the transformation

$$\zeta\rightarrow-\zeta$$

and g^0, g'^0 are related by

$$g'^0(q\cdot q',\zeta)=g^0(q\cdot q',-\zeta).$$

Equation (3.19) constitutes the principal result of this subsection.

We note in passing that on the mass shell of all particles the variable ζ is simply related to the invariant squared momentum transfers σ, σ' defined by

$$\sigma=(p_1-q)^2, \quad \sigma'=(p_1-q')^2.$$

Thus, with

$$t=(q+q')^2=2q\cdot q',$$

we have

$$\sigma=2m^2-\tfrac{1}{2}t-\zeta, \quad \sigma'=2m^2-\tfrac{1}{2}t+\zeta,$$

so that

$$\zeta=(\sigma-\sigma')/2,$$

and, of course,

$$\sigma+\sigma'+t=2m^2.$$

We consider finally the consequence of C invariance for $\Gamma_{\mu\nu}$. On writing

$$\Gamma_{\mu\nu}=\Gamma_{\mu\nu}(q,q';P),$$

it may be shown that C invariance implies that (see the Appendix)

$$\Gamma_{\mu\nu}(q,q';P)=\Gamma_{\mu\nu}(q,q';-P). \tag{3.20}$$

It follows that

$$Z=\pm Z|_{P\rightarrow -P},$$

with the plus sign holding for

$$Z=A,\ B,\ C,\ E,\ F, \text{ or } F',$$

and the minus sign for

$$Z=D,\ D',\ G, \text{ or } G'.$$

Hence, on the B^0 mass shell

$$Z_0(q^2,q'^2;q\cdot q',\zeta)=\pm Z_0(q^2,q'^2;q\cdot q',-\zeta).$$

Thus, there is no additional condition imposed on A_0, B_0, C_0, or E_0, and, *a fortiori*, on A^0, B^0, C^0, or E^0. On the other hand, we now have also

$$D_0'=-D_0 \quad G_0'=-G_0, \quad F_0'=-F_0. \tag{3.21}$$

Thus, if B^0 is on the mass shell,

$$\begin{aligned} \Gamma_{\mu\nu}' &\rightarrow A_0g_{\mu\nu}+B_0P_\mu P_\nu+C_0q_\nu q_\mu+D_0(q_\nu P_\mu-q_\mu'P_\nu), \\ \Gamma_{\mu\nu}'' &\rightarrow E_0q_\mu q_\nu'+F_0(q_\mu q_\nu+q_\mu'q_\nu') \\ &\qquad +G_0(q_\mu P_\nu-q_\nu'P_\mu), \end{aligned} \tag{3.22}$$

with A_0, B_0, C_0, E_0, and F_0 even functions of ζ and D_0, G_0 odd in ζ. The only new consequence when the photons are also on the mass shell is, on inspection of Eq. (3.10) and (3.21), that

$$g^0(q\cdot q',-\zeta)=-g^0(q\cdot q',\zeta), \tag{3.23}$$

since we have already concluded, without C invariance, that b^0, c^0, and e^0 are even functions of ζ.

B. Two-Photon Exchange Force between Two Neutral Spinless Particles

The results of the preceding subsection may be used to compute in a simple manner the asymptotic form of the two-photon exchange force between two neutral spinless particles or, more generally, "systems" B_1^0 and B_2^0. The general form Eq. (3.10a,b) applies to the two-photon vertex functions $\Gamma_{1\mu\nu}$ and $\Gamma_{2\mu\nu}$ of B_1^0 and B_2^0 regardless of whether B_1^0 or B_2^0 is an "elementary particle" or, say, an atom or a molecule.

On identifying B_1^0 and B_2^0 with particles "1" and "2" of Sec. II, we see that there will be a contribution to the absorptive part A of the scattering amplitude $F(s,t)$, in the "t" channel, proportional to

$$A_{(2\gamma)}=\int \mathcal{A}\delta(q^2)\delta(q'^2)\delta(Q-q-q')d^4qd^4q', \tag{3.24}$$

arising from two-photon exchange, in the region $t\geqslant 0$. Here

$$\mathcal{A}=\Gamma_1{}^{\mu\nu}(q,q';P_1)\Gamma_{2\mu\nu}{}^*(-q,-q';P_2)$$

where

$$P_1\equiv p_1+p_1', \quad P_2\equiv p_2+p_2'.$$

The minus signs before q and q' in $\Gamma_2{}^*$ correct for the fact that Γ_2 and Γ_1 were both defined as amplitudes for *emission* of photons of momentum q and q'.

It is convenient to consider $A_{(2\gamma)}$ first in the physical region of the crossed reaction (2.11) and to work in the c.m. system of this reaction. Then we may put

$$Q=(\surd t,0)$$

and

$$q=\tfrac{1}{2}(\surd t)(1,\hat{q}), \quad q'=\tfrac{1}{2}(\surd t)(1,-\hat{q}),$$

with $\hat{q}$ a unit vector. Equation (3.24) then reduces to

$$A_{(2\gamma)}=\frac{1}{8}\int \mathcal{A}^0 d\hat{q},$$

where $\mathcal{A}^0$ denotes the function $\mathcal{A}$ when all particles are on the mass shell. In the crossed channel we may also put

$$p_1=(\tfrac{1}{2}\sqrt{t},\mathbf{p}),\quad \bar{p}_1=-p_1'=(\tfrac{1}{2}\sqrt{t},-\mathbf{p})$$

and

$$p_2'=(\tfrac{1}{2}\sqrt{t},\mathbf{p}'),\quad \bar{p}_2=-p_2=(\tfrac{1}{2}\sqrt{t},-\mathbf{p}'),$$

with

$$\mathbf{p}^2=(t-4M_1^2)/4,\quad \mathbf{p}'^2=(t-4M_2^2)/4.$$

We then have

$$P_1=(0,2\mathbf{p}),\quad P_2=(0,2\mathbf{p}').$$

On defining

$$\cos\bar{\theta}=\hat{p}\cdot\hat{p}',\quad \cos\bar{\theta}_1=\hat{p}\cdot\hat{q},\quad \cos\bar{\theta}_2=\hat{p}'\cdot\hat{q}$$

we may write, for the various nonvanishing scalar products which occur in the computation of $\mathcal{A}^0$,

$$q\cdot q'=t/2,\quad P_1^2=4M_1^2-t,\quad P_2^2=4M_2^2-t,\qquad (3.25a)$$

and

$$\zeta_1\equiv q\cdot P_1=-q'\cdot P_1=-\tfrac{1}{2}[t(t-4M_1^2)]^{1/2}\cos\bar{\theta}_1,$$
$$\zeta_2\equiv q\cdot P_2=-q'\cdot P_2=-\tfrac{1}{2}[t(t-4M_2^2)]^{1/2}\cos\bar{\theta}_2,\qquad (3.25b)$$
$$P_1\cdot P_2=[(t-4M_1^2)(t-4M_2^2)]^{1/2}\cos\bar{\theta}.$$

On introducing a colon to indicate a contraction on the indices μ and ν, the quantity $\mathcal{A}$ may be written in the form

$$\mathcal{A}=\Gamma_1:\Gamma_2^*,$$

where it is to be understood that Γ_2 is evaluated at $-q$, $-q'$. We may decompose Γ_1 and Γ_2 into parts Γ_1', Γ_1'' and Γ_2', Γ_2'', respectively, obtained by letting $A, B, \cdots G'\to A_j, B_j\cdots G_j'$ $(j=1, 2)$ in Eqs. (3.10a,b) and (3.9). Since

$$q^\mu\Gamma_{j\mu\nu}=q'^\nu\Gamma_{j\mu\nu}=0,\quad (j=1,2),$$

it follows that

$$\Gamma_1'':[\Gamma_2'^*+\Gamma_2''^*]=0$$

and

$$[\Gamma_1'+\Gamma_2'']:\Gamma_2''^*=0.$$

Hence

$$\mathcal{A}=[\Gamma_1'+\Gamma_1'']:[\Gamma_2'+\Gamma_2'']^*$$

may also be written as

$$\mathcal{A}=\Gamma_1':\Gamma_2'^*-\Gamma_1'':\Gamma_2''^*.\qquad (3.26)$$

On the mass shell of all the particles we may write, using Eqs. (3.10a), (3.10b), and (3.19),

$$\Gamma_{j\mu\nu}'=A_j^0g_{\mu\nu}+B_j^0P_{j\mu}P_{j\nu}+C_j^0q_\nu q_\mu' +D_j^0(q_\nu P_{j\mu}-q_\mu'P_{j\nu})\qquad (3.27a)$$

and

$$\Gamma_{j\mu\nu}''=E_j^0q_\mu q_\nu'+F_j^0(q_\mu q_\nu+q_\mu'q_\nu') +G_j^0(q_\mu P_{j\nu}-q_\nu'P_{j\mu}),\qquad (3.27b)$$

with

$$A_j^0=-\tfrac{1}{2}c_j^0t-b_j^0\zeta_j^2,\quad B_j^0=\tfrac{1}{2}b_j^0t,\quad C_j^0=c_j^0,\quad D_j^0=b_j^0\zeta_j,\qquad (3.28a)$$

and

$$E_j^0=\tfrac{1}{2}e_j^0t,\quad F_j^0=g_j^0\zeta_j,\quad G_j^0=\tfrac{1}{2}g_j^0t,\qquad (3.28b)$$

where $j=1$ or 2.

It is now a straightforward task to verify that, near $t=0$

$$\mathcal{A}^0\sim t^2,$$

where $\mathcal{A}^0$ denotes the value of $\mathcal{A}$ [Eq. (3.26)] when all particles are on the mass shell. For example, the contribution to $\mathcal{A}^0$ from the first two terms in $\Gamma_{1\mu\nu}'$, contracted with the first two terms in $\Gamma_{2\mu\nu}'^*$ is given by

$$(A_1^0g_{\mu\nu}+B_1^0P_{1\mu}P_{1\nu})(A_2^{0*}g^{\mu\nu}+B_2^{0*}P_2^\mu P_2^\nu)$$

which is equal to

$$4A_1^0A_2^{0*}+A_1^0B_2^{0*}P_2^2+A_2^{0*}B_1^0P_1^2 +B_1^0B_2^{0*}(P_1\cdot P_2)^2.\qquad (3.29)$$

From Eqs. (3.28a), (3.25a) and (3.25b) we see that

$$A_j^0\sim t,\quad B_j^0\sim t$$

since, e.g.,

$$b_j^0=b_j^0(\tfrac{1}{2}t,\zeta_j)$$

approaches the constant $b_j^0(0,0)$ as $t\to 0$, regardless of the value of $\cos\bar{\theta}$; and similarly, $C_j^0\to C_j^0(0,0)$. Since we also have, as $t\to 0$,

$$P_j^2\to 4M_j^2,\quad P_1\cdot P_2\to 4M_1M_2\cos\bar{\theta},$$

the expression (3.29) has the form, near $t=0$,

$$t^2\mathcal{P}(\cos\bar{\theta}_1,\cos\bar{\theta}_2,\cos\bar{\theta}),$$

where $\mathcal{P}$ is a simple polynomial in each of the indicated variables. Now for fixed s,

$$s=(p_1-\bar{p}_2)^2,$$

$\mathcal{P}$ still depends on t since $\cos\bar{\theta}$ does:

$$\cos\bar{\theta}=2(M_1^2+M_2^2-s-\tfrac{1}{2}t)/[(t-4M_1^2)(t-4M_2^2)]^{1/2}.$$

But this dependence may be neglected near $t=0$, where

$$\cos\bar{\theta}\to(M_1^2+M_2^2-s)/2M_1M_2.$$

It follows that

$$\int\mathcal{P}d\hat{q}\to\text{const}$$

as $t\to 0$, so that the contribution from (3.29) to $A_{(2\gamma)}$ is proportional to t^2.

The other terms in $\mathcal{A}^0$ may be handled in an entirely similar manner, so that we conclude that, for fixed s, and $t\gtrsim 0$

$$A_{(2\gamma)}(s,t)\sim t^2.\qquad (3.30)$$

On setting $N=2$ in Eq. (2.16) of Sec. II, it then follows immediately that $V_{2\gamma}^{(0)}(r)$, the two-photon potential between neutral spinless particles, has the property for $r\to\infty$,

$$V_{2\gamma}^{(0)}\sim 1/r^7,\qquad (3.31)$$

corresponding to a force which varies as $1/r^8$ for large r.

We have thus shown that the result of Casimir and Polder[4] concerning Van der Waals forces is quite general, being independent of any detailed dynamical models for the individual systems B_1^0 and B_2^0.

C. Two-Photon Exchange Force between a Spinless, Neutral Particle and a Charged Particle

Let $\Gamma_{\mu\nu}$ and $\Phi_{\mu\nu}$ denote the form factors for two-photon emission by a neutral spinless particle "1" and a charged spinless particle "2," respectively. It is convenient to change the notation slightly from that of the preceding subsection B and to write

$$\Gamma_{\mu\nu}=\Gamma_{\mu\nu}(q,q';P)$$

with

$$P=p_1+p_1'$$

as in part A, and

$$\Phi_{\mu\nu}=\Phi_{\mu\nu}(q,q';R)$$

with

$$R=p_2+p_2'.$$

Unlike $\Gamma_{\mu\nu}$, the form factor $\Phi_{\mu\nu}$ satisfies Eqs. (3.12a) and (3.12b) only when "2" is on the mass shell. Thus, we have

$$q^\mu\Phi_{\mu\nu}=0\,,\quad q'^\nu\Phi_{\mu\nu}=0\,,\tag{3.32}$$

only when

$$q\cdot R=-q'\cdot R\,,\quad R^2=4M_2^2-(q+q')^2.\tag{3.33}$$

Although we could write down the general form of $\Phi_{\mu\nu}$ consistent with Eqs. (3.32), and symmetry conditions, such as

$$\Phi_{\mu\nu}(q,q';R)=\Phi_{\nu\mu}(q',q;R)\,,\tag{3.34}$$

it is much simpler, and will suffice for our purpose, to approximate $\Phi_{\mu\nu}$ by $\Lambda_{\mu\nu}$, the value of $\Phi_{\mu\nu}$ as given by second-order perturbation theory. Corresponding to Figs. 4(a), (b), and (c) we have, omitting a factor e^2 and other constant factors,

$$\Lambda_{\mu\nu}=\frac{(p_2'+p_2-q')_\mu(2p_2'-q')_\nu}{(p_2-q')^2-M_2^2}+\frac{(p_2'+p_2-q)_\nu(2p_2-q)_\mu}{(p_2-q)^2-M_2^2}-2g_{\mu\nu}.\tag{3.35}$$

We shall only need $\Lambda_{\mu\nu}^0$, the value of $\Lambda_{\mu\nu}$ when all the particles are on the mass shell. On use of Eq. (3.33) and the relations

$$p_2=(R+q+q')/2\,,\quad p_2'=(R-q-q')/2$$

we readily obtain, after combining the first two terms in Eq. (3.35),

$$\Lambda_{\mu\nu}^0=(2N_{\mu\nu}/\mathfrak{D})-2g_{\mu\nu}\,,$$

where

$$N_{\mu\nu}=\tfrac{1}{2}(q_\mu'q_\nu-R_\mu R_\nu)t+(q_\mu'R_\nu-q_\nu R_\mu)\zeta_2$$

and

$$\mathfrak{D}=\tfrac{1}{4}t^2-\zeta_2^2\,,$$

with

$$\zeta_2=q\cdot R=-q'\cdot R\,,\quad t=2q'\cdot q.$$

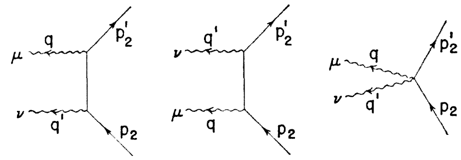

FIG. 4. Lowest order Feynman diagrams corresponding to the emission of two photons by a charged spinless particle.

We now note that $\Lambda_{\mu\nu}^0$ is a gauge-invariant approximation to $\Phi_{\mu\nu}$, i.e., $\Lambda_{\mu\nu}^0$ may replace $\Phi_{\mu\nu}$ in Eq. (3.32). Thus, on use of Eqs. (3.9) and (3.10a,b),

$$\Gamma^{\mu\nu}\Lambda_{\mu\nu}^0=\Gamma'^{\mu\nu}\Lambda_{\mu\nu}^0=(2\Gamma'^{\mu\nu}N_{\mu\nu}/\mathfrak{D})-2\Gamma'^{\mu\nu}g_{\mu\nu}.$$

The absorptive part $A_{(2\gamma)}^{(+)}$, analogous to $A_{(2\gamma)}$ of part B, is thus given, in the present approximation, by

$$A_{(2\gamma)}^{(+)}=2\int(\Gamma'^{\mu\nu}N_{\mu\nu}/\mathfrak{D})d\hat{q}-2\int\Gamma'^{\mu\nu}g_{\mu\nu}d\hat{q}\,,\tag{3.36a}$$

where $\Gamma'^{\mu\nu}$ is given by [see Eqs. (3.10a) and (3.19)]

$$\Gamma'^{\mu\nu}=(-\tfrac{1}{2}tc^0-\zeta_1^2b^0)g_{\mu\nu}+\tfrac{1}{2}tb^0P_\mu P_\nu+c^0q_\nu q_\mu'+\zeta_1b^0(q_\nu P_\mu-q_\mu'P_\nu).\tag{3.36b}$$

It is easily seen that the integrand of the second term in Eq. (3.36) is $\sim t$, for small t, so that the integral is also $\sim t$. However, although

$$\Gamma'^{\mu\nu}N_{\mu\nu}\sim t^2\mathcal{P}(\cos\bar{\theta}_1,\,\cos\bar{\theta}_2,\,\cos\bar{\theta})$$

for small t, with $\mathcal{P}$ a polynomial in each of the indicated variables, and

$$\mathfrak{D}\sim t\cos^2\theta_2\tag{3.37}$$

for small t, the first integral behaves not as t but as $t^{1/2}$ near $t=0$. This is related to the fact that if Eq. (3.37) is used for $\mathfrak{D}$ this integral is divergent. Since

$$\mathfrak{D}=\tfrac{1}{4}t^2-\tfrac{1}{4}t(t-4M_2^2)\cos^2\bar{\theta}_2\,,$$

we may approximate $\mathfrak{D}$ as

$$\mathfrak{D}\approx M_2^2t[(t/4M_2^2)+\cos^2\bar{\theta}_2]$$

so that the first term in Eq. (3.36a) is proportional to

$$t\int_0^{2\pi}d\varphi\int_{-1}^1dx\,\mathcal{P}/[(t/4M_2^2)+x^2].\tag{3.38}$$

Here we have chosen $\hat{p}$ as the polar axis for $\hat{q}$, and put $x=\hat{q}\cdot\hat{p}'=\cos\bar{\theta}_2$. Since

$$\int_0^1dx[a+x^2]^{-1}=a^{-1/2}\tan^{-1}(a^{-1/2})\,,$$

it follows that the expression (3.38) behaves as $t\times t^{-1/2}=t^{1/2}$ for small t. Hence, we may conclude that

$$A_{(2\gamma)}^{(+)}\sim t^{1/2}\tag{3.39}$$

for $t\gtrsim0$.

From Eq. (2.15), and Eq. (2.16) with $N=\frac{1}{2}$, we then obtain for $V_{2\gamma}^{(+)}$ the two-photon exchange potential between a neutral and a charged particle,

$$V_{2\gamma}^{(+)}\sim 1/r^4 \tag{3.40}$$

for $r\rightarrow\infty$.

Although we have assumed in the derivation that "2" has zero spin, it may be assumed that the result (3.40) also holds for the spin-independent part of the two-photon exchange potential acting between B^0 and a charged particle of arbitrary spin. For a spin-one-half particle "2," this may be verified explicitly be replacing $\Lambda_{\mu\nu}{}^0$ by $L_{\mu\nu}$, defined by

$$L_{\mu\nu}=\bar{u}(2')\left[\gamma_\mu\frac{1}{p_2-q'-m_2}\gamma_\nu+\gamma_\nu\frac{1}{p_2-q-m_2}\gamma_\mu\right]u(2). \tag{3.41}$$

We may write

$$L_{\mu\nu}=\mathcal{L}_{\mu\nu}\bar{u}(2')u(2),$$

where $\mathcal{L}_{\mu\nu}$ is a tensor function of its arguments,

$$\mathcal{L}_{\mu\nu}=\mathcal{L}_{\mu\nu}(p_2',\tau_2';p_2,\tau_2)$$

and τ_2', τ_2 are the covariant spin four-vectors associated with the states $u(2')$ and $u(2)$, respectively. For the case of no-spin flip, $\tau_2'=\tau_2=\tau$ say, we have

$$\begin{aligned}\bar{u}(p_2'\tau)\gamma_\mu u(p_2,\tau)&=(2M_2/R^2)R_\mu\bar{u}u,\\ \bar{u}(p_2',\tau)i\gamma_5\gamma_\mu u(p_2,\tau)&=-\tau_\mu\bar{u}u.\end{aligned} \tag{3.42}$$

On rationalization of the denominators in Eq. (3.41), and the use of the Dirac equation, together with Eq. (3.42) and the identity

$$\gamma_\lambda\gamma_\mu\gamma_\nu=g_{\lambda\mu}\gamma_\nu+g_{\mu\nu}\gamma_\lambda-g_{\lambda\nu}\gamma_\mu-\epsilon_{\lambda\mu\nu\rho}\gamma^\rho\gamma_5$$

the spin-independent part $\mathcal{L}_{\mu\nu}{}^{\text{s.i.}}$ of $\mathcal{L}_{\mu\nu}$, defined formally by

$$\mathcal{L}_{\mu\nu}{}^{\text{s.i.}}=\mathcal{L}_{\mu\nu}(p_2',0;p_2,0),$$

may be obtained. The result is

$$\begin{aligned}\mathcal{L}_{\mu\nu}{}^{\text{s.i.}}=&(2M_2/R^2)\\ &\times[(q\cdot Rg_{\mu\nu}+R_\mu R_\nu+q_\mu'R_\nu-q_\nu R_\mu)/(R\cdot q-q'\cdot q)\\ &+(q'\cdot Rg_{\mu\nu}+R_\mu R_\nu+q_\nu R_\mu-q_\mu'R_\nu)/\\ &(R\cdot q'-q\cdot q')].\end{aligned} \tag{3.43}$$

Since

$$q^\mu\mathcal{L}_{\mu\nu}{}^{\text{s.i.}}=q'^\nu\mathcal{L}_{\mu\nu}{}^{\text{s.i.}}=0,$$

we need only compute

$$\int\Gamma'^{\mu\nu}\mathcal{L}_{\mu\nu}{}^{\text{s.i.}}d\hat{q}.$$

On combining the two terms in Eq. (3.43) and on using the relation (3.36b), we then find, just as in the case of spin zero for "2", that

$$A_{(2\gamma)}{}^{(+)}\sim t^{1/2}$$

near $t=0$, so that again

$$V_{2\gamma}^{(+)}\sim 1/r^4 \tag{3.44}$$

for large r.

D. Alternative Approach to $V_{2\gamma}^{(+)}$

The result (3.40) for $V_{2\gamma}^{(+)}$ is of course not surprising, since it follows from an extremely simple argument using only classical electrostatics. Consider the neutral particle "1" and the particle "2," with charge e at relative rest and separated by a distance r. "2" creates an electric field

$$\mathbf{E}=e\hat{r}/r^2$$

at the position of "1," which induces a dipole moment

$$\mathbf{d}=\alpha_0\mathbf{E}$$

in "1," assumed to have polarizability α_0. The potential energy of the dipole $\mathbf{d}$ in the field $\mathbf{E}$ is, however,

$$-\mathbf{d}\cdot\mathbf{E}=-\alpha_0e^2/r^4$$

so that the $1/r^4$ behavior of Eq. (3.40) is confirmed.

For the purposes of the next section it is useful to give still another, more sophisticated, derivation of the behavior of $V_{2\gamma}^{(+)}$ for large r. Let $\phi(x)$ denote the quantized field associated with the neutral spinless particle B^0. Consider B^0 moving in the presence of an external electromagnetic field $A_\mu{}^{\text{ex}}(x)$. We wish to construct a phenomenological interaction Lagrangian $\mathcal{L}_I$, which can describe the scattering of B^0 in this field. $\mathcal{L}_I$ must then be bilinear in $\phi(x)$, and gauge-invariant. Since B^0 has zero charge, a coupling of the form

$$j^\mu(x)A_\mu{}^{\text{ex}}(x),$$

with $j^\mu(x)$, the current of B^0, given by

$$j^\mu(x)=\frac{1}{2iM}[\phi^\dagger(x)\partial^\mu\phi(x)-\phi(x)\partial^\mu\phi^\dagger(x)],$$

is inadmissible.

The simplest possibility is to take

$$\mathcal{L}_I(x)=\lambda\phi^\dagger(x)\phi(x)F_{\mu\nu}(x)F^{\mu\nu}(x), \tag{3.45}$$

where λ is a constant and

$$F_{\mu\nu}(x)=\partial_\mu A_\nu{}^{\text{ex}}(x)-\partial_\nu A_\mu{}^{\text{ex}}(x)$$

is the tensor of the external electromagnetic field strengths. We can now assume that $A_\mu{}^{\text{ex}}(x)$ is produced by a particle "2" of charge e, at rest at the origin of the coordinate system. Since

$$F_{\mu\nu}F^{\mu\nu}=-\mathbf{E}^2+\mathbf{H}^2,$$

where E and H are the electric and magnetic fields, we see that the Hamiltonian density $\mathcal{H}_I=-\mathcal{L}_I$ reduces to

$$\mathcal{H}_I(x)=\lambda\phi^\dagger(x)\phi(x)e^2/|\mathbf{x}|^4.$$

The potential energy V_I of B^0 in the field of the "2" is thus

$$V_I=\lambda e^2/|\mathbf{x}|^4.$$

Since $\mathcal{L}_I$ is quadratic in A_μ^{ex}, it is justifiable to interpret Eq. (3.45) as corresponding to an interaction by two-photon exchange, in the limit of a heavy particle "2" whose recoil may be neglected. Thus we have once again that

$$V_{2\gamma}^{(+)}\sim 1/r^4$$

for large r.

If the spinless particle is described by a real field $\pi(x)$ rather than by a complex field, we simply replace $\mathcal{L}_I(x)$ by

$$\lambda\pi^2(x)F_{\mu\nu}(x)F^{\mu\nu}(x)$$

and the result is the same. Thus the two-photon exchange force between a charged and neutral particle is independent of the identity of particle and antiparticle.

IV. THREE-PHOTON EXCHANGE FORCE BETWEEN A NEUTRAL AND A CHARGED PARTICLE

To study the asymptotic form of $V_{3\gamma}^{(+)}$, the potential between B^0 and a charged particle arising from three-photon exchange, we could in principle proceed in the same way as we did in Sec. IIIC for $V_{2\gamma}^{(+)}$. That is, we could write down the general form of $\Gamma_{\mu\nu\rho}(q,q',q'';P)$, the amplitude for emission of three photons by B^0, as a linear combination of the $4^3=64$ tensors of rank three which may be constructed from q, q', q'', and P, and the 12 additional tensors which may be formed with the help of $g_{\mu\nu}$. The restrictions on the coefficients arising from the conditions,

$$q^\mu\Gamma_{\mu\nu\rho}=q'^\nu\Gamma_{\mu\nu\rho}=q''^\rho\Gamma_{\mu\nu\rho}=0$$

as well as symmetry conditions related to the identity of the photons, could then be found. The resulting form of $\Gamma_{\mu\nu\rho}$ could next be coupled with the charged particle three-photon form factor $\Phi_{\mu\nu\rho}$, taken from third-order perturbation theory, and the integral

$$\int \Gamma^{\mu\nu\rho}\Phi_{\mu\nu\rho}{}^*\delta(q^2)\delta(q'^2)\delta(q''^2)\times\delta(Q-q-q'-q'')d^4q d^4q' d^4q''$$

could be studied for $t\gtrsim 0$.

Since the above straightforward procedure seems, at best, impossibly tedious, we consider an alternative approach, based on the construction of a phenomenological Lagrangian. This approach, when applied to $V_{2\gamma}^{(+)}$, correctly reproduced the $1/r^4$ behavior obtained from an exact quantum-mechanical calculation, so that we are justified in using it for $V_{3\gamma}^{(+)}$.

We thus look for $\mathcal{L}_I$ which are bilinear in $\phi(x)$, gauge-invariant, and now *trilinear* in $A_\mu^{\text{ex}}(x)$, in order to simulate the effects of three-photon exchange. Gauge invariance is assured by using only $F_{\mu\nu}(x)$. The only scalar which can be formed, without using derivatives, from $F_{\mu\nu}(x)$, trilinear in A_μ^{ex} is

$$s=F_{\mu\nu}(x)F^{\nu\rho}(x)F_\rho{}^\mu(x).$$

However, since

$$F_{\mu\nu}(x)=-F_{\nu\mu}(x)$$

it follows that

$$s\equiv 0.$$

Hence, it is necessary to use further derivatives, $\partial/\partial x^\mu$, in the construction of $\mathcal{L}_I$. If we wish $\mathcal{L}_I$ to be invariant under charge conjugation, we need only consider interactions with the current $j_\mu(x)$ of B^0. Thus, the simplest possibility for $\mathcal{L}_I$ is a sum of terms in which one of

$$j_\mu(x)\partial_\nu,\quad j_\nu(x)\partial_\mu$$

is contracted (on the right) with a second rank tensor $\tau^{\mu\nu}$ constructed as a trilinear function of $F^{\mu\nu}(x)$. The symbol ∂ need act only on one of the factors of $\tau^{\mu\nu}$. The distinct possibilities for $\tau^{\mu\nu}$ are[8]

$$F_{\alpha\beta}F^{\alpha\beta}F^{\mu\nu},\quad F_\alpha{}^\mu F^{\alpha\beta}F_\beta{}^\nu,$$

both of which are antisymmetric under $\mu\leftrightarrow\nu$. Since

$$\partial^\mu F_{\mu\nu}\propto j_\nu^{\text{ex}}(x),$$

the derivative ∂_ν or ∂_μ acting directly on $F^{\mu\nu}$ will give rise to a term in $\mathcal{L}_I$ proportional to $j_\nu^{\text{ex}}(x)$, i.e., a contact interaction. We thus find three distinct kinds of terms which may be of interest:

$$\mathcal{L}_I{}^{(1)}\sim j_\mu(x)F^{\mu\nu}\partial_\nu(F_{\alpha\beta}F^{\alpha\beta}),$$
$$\mathcal{L}_I{}^{(2)}\sim j_\mu(x)F_\alpha{}^\mu(\partial_\nu F^{\alpha\beta})F_\beta{}^\nu,$$
$$\mathcal{L}_I{}^{(3)}\sim j_\mu(x)(\partial_\nu F_\alpha{}^\mu)F^{\alpha\beta}F_\beta{}^\nu.$$

In the static limit for B^0, $j_\mu(x)=[j_0(x),\mathbf{j}(x)]\to[j_0(x),0]$. If the source of $A_\mu^{\text{ex}}(x)$ is again a charge e at the origin, producing an electrostatic field $\mathbf{E}$, we have

$$F^{ij}=0,\quad F^{0j}=E^j,$$

so that

$$\mathcal{L}_I{}^{(1)}\to j_0(\mathbf{E}\cdot\nabla)(\mathbf{E}^2),$$
$$\mathcal{L}_I{}^{(2)},\ \mathcal{L}_I{}^{(3)}\to j_0E^j(\mathbf{E}\cdot\nabla)E^j=\tfrac{1}{2}j_0(\mathbf{E}\cdot\nabla)(\mathbf{E}^2).$$

With $\mathbf{E}=e\hat{r}/r^2$ we thus find

$$\mathcal{L}_I\sim j_0(x)e^3/r^7.$$

Hence we conclude that

$$V_{3\gamma}^{(+)}\sim 1/r^7 \tag{4.1}$$

for large r.

The asymptotic form of $V_{3\gamma}^{(+)}$ thus coincides with that of $V_{2\gamma}^{(0)}$.

V. APPLICATIONS

We have remarked in the Introduction that the force on a neutral particle coming from exchange of an even

[8] Another possibility, if we allow pseudoscalar interactions, is

$$T^{\mu\nu}=\epsilon_{\alpha\beta\rho\sigma}F^{\alpha\beta}F^{\rho\mu}F^{\sigma\nu}.$$

This vanishes for an external electrostatic field and hence is of shorter range than the ones we consider.

number of photons is the same for particle and antiparticle, while that coming from exchange of an odd number is opposite for particle and antiparticle. As a result of this, the exchange of an odd number of photons gives a force which can convert a K_1^0 into a K_2^0. Furthermore, the K_1^0-K_2^0 mass difference is so small, and the properties of these mesons are so different, that the regeneration of K_1^0 mesons in a beam of K_2^0 would probably be the most sensitive test of the occurrence of such a force. We have seen that the existence of a long-range force is equivalent to a certain behavior of the scattering amplitude at very small momentum transfer. Therefore, it would seem that it would be best to look at such regenerations at very small angles in the hope of distinguishing the effect of the long-range force from that of short-range interactions with the nuclei of atoms.

A number of experiments which examine the regeneration of K_1's from K_2's have been carried out. Two of the most recent ones are those of Leipuner *et al.*,[2] and of Eisler *et al.*[2] In these experiments, the regeneration is examined for angles satisfying $\cos\theta > 0.999$, which for mesons of momentum 1 BeV/c corresponds to momentum transfers on the order of 10 MeV/c. Clearly, if the effect of multiphoton exchange is to be important in these experiments, it is necessary that the multiphoton contribution to the scattering amplitude at such small momentum transfers be comparable to the contribution of the strong interaction. Therefore, we will use as our reference scattering amplitude the one obtained by exchanging a ρ meson between the K^0 and the proton

$$F_\rho(q^2) = g_\rho^2/(q^2+m_\rho^2) \tag{5.1}$$

with $g_\rho^2 \approx 1$. At zero momentum transfer this gives

$$F_\rho(0) \simeq \frac{1}{m_\rho^2} \sim \frac{1}{16}(\text{Fermi})^2. \tag{5.2}$$

Let us compare this with the contribution of a hypothetical long-range interaction, i.e., a potential with $V(r) \sim r^{-n}$ for large r. From the considerations of Sec. II, we see that corresponding to this potential, there is a discontinuity function $A(q)$ which goes as q^{+n-3}. We want to know what contribution $A(q)$ makes to the scattering amplitude $F(q)$ at small q. It is easy to see that when $n>3$, the contribution coming from the long-range interaction, i.e., the values of $A(q)$ for small q, are small compared to the short-range contribution, i.e., the values of $A(q)$ for non-small q. To see this, we write $F(q)$ as a sum of terms, one coming from small q, say less than $q_0 \sim 10$ MeV/c, and a remainder

$$F(q) = \int_0^{q_0} \frac{A(q')q'dq'}{q^2+q'^2} + \int_{q_0}^{\infty} \frac{A(q')q'dq'}{q^2+q'^2} \tag{5.3}$$

$$\equiv F_1 + F_2.$$

We note that the second term will in general behave like a constant, independent of q_0 for small q. On the other hand, in the first term, we substitute the small-q approximation to $A(q)$, i.e., $A(q) \sim q^{n-3}$. Then we get for F_1 as $q \to 0$

$$F_1 \approx q_0^{n-3}.$$

Clearly, when $n>3$, the small-momentum-transfer region makes a contribution negligible compared to the high-momentum-transfer region.

Since the three-photon exchange potential between a K^0 and a proton goes as r^{-7}, we see that its long-range part actually gives a contribution small compared to its short-range part. However, the latter is obviously much smaller than the nuclear-scattering amplitude (5.1), as it contains three powers of the fine-structure constant. We conclude that the three-photon exchange cannot possibly account for the results of the experiment of Leipuner *et al.*, or of the experiment of Christianson *et al.*

Let us ask what type of long-range, odd-C interaction could be observable in regeneration experiments. If we substitute $A(q') \sim (q')^{n-3}$ into F_1, and ask for the leading term in powers of q, for $q \ll q_0$, then we see that this term behaves as $q^{n-3}\ln q$ for n-odd, or as q^{n-3} for n even. The total scattering amplitude will then behave as

$$F \sim C + \lambda q^{n-3}[\ln q], \tag{5.4}$$

where the bracket indicates the presence or absence of a $\ln q$ depending on whether n is odd or even. The effect of short-range interactions is represented by the constant C, and λ is another constant. Clearly, if $n>3$, the long-range interaction is dominated by the short-range interactions, even when $q \to 0$. Such interactions only show up in the discontinuities of the *derivatives* of F with respect to q^2. It appears hopeless to obtain such a detailed measurement of the scattering amplitude by any techniques known to us. If, however, $n=3$, then

$$F \sim C + \lambda \ln q \tag{5.5}$$

and there is a small region about $q=0$ where the long-range interaction actually dominates.

The most likely effect of the long-range interaction to be observed is on the forward-scattering amplitude or on the index of refraction. We see from (5.5) that if the amplitude is taken literally at $q=0$, then the forward-scattering amplitude is infinite. There are at least two mechanisms by which the amplitude could be modified. The first of these is screening, familiar from the Coulomb interaction. In order for this to occur, the long-range interaction would have to act between K mesons and electrons as well as between K mesons and protons. In this case, the contribution of the long-range interaction is modified at distances beyond the Bohr radius, and (5.5) is modified by

$$F \sim c + \lambda \ln(q^2 + a_0^{-2}), \tag{5.6}$$

corresponding to a potential,

$$V(r)=\frac{\lambda e^{-r/a_0}}{m_\rho^2 r^3}, \quad (r>1/m_\rho). \tag{5.7}$$

With $\lambda\sim10^{-2}$, this would give a 10% change in the forward-scattering amplitude from that computed in (5.2). Such an interaction, which is much larger and of longer range than any we can anticipate for neutral K mesons, would probably be detectable in the current type of experiments. Smaller interactions, such as we have considered, are very unlikely to be detected.

If there is no screening, as would be the case if the long-range interaction does not occur with electrons, then the approximation of treating this interaction in Born approximation is not sufficient. It would then be necessary to solve the Schrödinger equation for the interaction of two particles, with this potential. This problem, while interesting, is beyond the scope of the present paper.

VI. CONCLUSIONS

We have discovered some general features of the long-range electromagnetic forces which are expected to act on neutral particles. We have shown that the long-range potential can be extracted directly from the knowledge of the discontinuity function in the momentum transfer of the scattering amplitude. This is done by use of Eq. (2.8). In particular, we find that if the discontinuity function $A(t)$, coming from a set of graphs, behaves as $q^n(t^{n/2})$ for small q, then the corresponding long-range potential will go as r^{-n-3}.

The most interesting case this may be applied to is the two-photon exchange potential for two neutral spinless systems. For this case, an analysis of the two-photon vertex of a neutral particle is required. This was done, through the use of Ward's identity, in Sec. III. We find that this vertex is characterized by seven form factors which depend on four invariant scalar quantities. On the photon mass shell, which is relevant to the calculation of the discontinuity functions, there are, of course, only two invariants, which are s and t. The requirement that none of the form factors have singularities as functions of the invariants then determines the behavior of the form factors for small values of the momentum transfer variables. This behavior is given in Eq. (3.19).

We then calculate the two-photon exchange graphs for two such neutral systems and find that the leading term in an expansion of $A(q^2)$ in powers of q^2 for small q^2 behaves as q^4. The rule (2.8) then implies that the longest range potential goes as r^{-7}. This result is in agreement with that discovered by Casimir and Polder in 1949. We believe that our demonstration is somewhat more transparent because the potential comes directly from a single covariant expression for the scattering amplitude rather than as a cancellation between distinct noncovariant terms, as in the calculation of Casimir and Polder. Also, our formula is valid for any two neutral spinless particles.[9]

One difference between the case of two atoms and the case of two neutral elementary particles is that the atoms are bound by electromagnetic interactions, while the particles are not. Hence, the atoms have excited states separated by energies of order $\alpha^2 m_e$ from the ground state, whereas the excited states of the particles, if any, are separated by energies of the order of the pion mass. As a consequence of this, the potential in the atomic case contains terms with decreasing exponentials of the form $\exp(-\alpha r/a_0)$, where a_0 is of the order of the Bohr radius, while in the particle case the corresponding terms are $\exp(-r/\lambda_\pi)$, where λ_π is the pion Compton wavelength. For the atom, the exponential terms may dominate the true long-range force for many atomic radii. It appears that this is the origin of the fact that in the work of Casimir and Polder, the r^{-7} potential is not dominant until $r\sim(a_0/\alpha)$.

Finally, we have computed the three-photon exchange potential between a K^0 meson and a proton. This potential is odd under charge conjugation and hence can regenerate K_1 mesons in a beam of K_2 mesons. We find that the potential falls off as r^{-7} here also, and is much too small to explain the anomalous regeneration reported by Leipuner *et al.*

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APPENDIX

A discussion of the vertex functions for photon emission by a neutral spinless particle B does not seem to be available in the literature. We therefore give a brief treatment of this topic here, from the point of view of field theory.[10]

Let $\phi(x)$ and $A_\mu(y)$ denote the (unrenormalized) Heisenberg fields associated with B and the photon, respectively. The electromagnetic current $j_\mu(y)$ is defined by

$$\Box_y A_\mu(y)=j_\mu(y) \tag{A1}$$

and assumed to be conserved:

$$\partial_y{}^\mu j_\mu(y)=0. \tag{A2}$$

[9] The sign of the potential is easily obtained in the simple case when only the form factors A, C of Eq. (3.10a) are present. In this case, essentially that treated in Ref. 4, there is an attractive force between identical spinless particles. For the general case, we have no rigorous argument about the sign of the potential, although arguments can be given which suggest that the two-photon exchange potential between similar particles is always attractive.

[10] Our treatment parallels that of K. Nishijima, Phys. Rev. **119**, 485 (1960), for charged particles.

We shall also need the equal-time commutation relations

$$[\dot{A}_\mu(y),A_\nu(y')]_{y_0=y_0'}=ig_{\mu\nu}\delta(\mathbf{y}-\mathbf{y}'). \qquad \text{(A3)}$$

As a field-theoretic definition of the neutrality of B we take the relation

$$[j_0(y),\phi(x)]_{y_0=x_0}=0. \qquad \text{(A4)}$$

This is motivated by the fact that if a Schrödinger picture is introduced at, say, $t=0$, the expression for $j_\mu^s(\mathbf{y})=j_\mu(\mathbf{y},0)$ will depend explicitly only on the Schrödinger fields associated with charged particles and so will commute with $\phi^s(\mathbf{x})=\phi(\mathbf{x},0)$.[11] In particular (A4) holds for A_μ itself, i.e.,

$$[j_0(y),A_\nu(y')]_{y_0=y_0'}=0. \qquad \text{(A5)}$$

From (A3) it follows readily that

$$\Box_y T[A_\mu(y)A_\nu(y')]=T[j_\mu(y)A_\nu(y')]+ig_{\mu\nu}\delta(y-y')$$

and from (A5) it follows that

$$\partial_y{}^\mu T[j_\mu(y)A_\nu(y')]=T[\partial_y{}^\mu j_\mu(y)A_\nu(y')].$$

Because of (A2), we then have

$$\partial_y{}^\mu\Box_y T[A_\mu(y)A_\nu(y')]=\frac{i\partial}{\partial_y{}^\nu}\delta(y-y'). \qquad \text{(A6)}$$

We now define

$$V_\mu(x',x;y)=T[\phi^\dagger(x')\phi(x)A_\mu(y)]$$

and

$$V_{\mu\nu}(x',x;y,y')=T[\phi^\dagger(x')\phi(x)A_\mu(y)A_\nu(y')].$$

Using the fact that $\phi(x)$ and $A_\mu(y)$ are kinematically independent fields so that

$$[\phi(x),A_\mu(y)]_{x_0=y_0}=[\phi(x),\dot{A}_\mu(y)]_{x_0=y_0}=0$$

we get

$$\Box_y V_\mu(x',x;y)=T[\phi^\dagger(x')\phi(x)j_\mu(y)], \qquad \text{(A7a)}$$

$$\Box_y V_{\mu\nu}(x',x;y,y')=T[\phi^\dagger(x')\phi(x)j_\mu(y)A_\nu(y')] +ig_{\mu\nu}\delta(y-y')T[\phi^\dagger(x')\phi(x)]. \qquad \text{(A7b)}$$

Furthermore, from (A4) and (A2) we find

$$\partial_y{}^\mu T[\phi^\dagger(x')\phi(x)j_\mu(y)]=0 \qquad \text{(A8a)}$$

and from (A5), (A4), and (A2), that

$$\partial_y{}^\mu T[\phi^\dagger(x')\phi(x)j_\mu(y)A_\nu(y)]=0. \qquad \text{(A8b)}$$

We now define three-point and four-point functions associated with emission of one or two photons by

$$W_\mu(x',x;y)=\langle V_\mu(x',x;y)\rangle \qquad \text{(A9a)}$$

and

$$W_{\mu\nu}(x',x;y,y')=\langle V_{\mu\nu}(x',x;y,y')\rangle -\Delta_F'(x'-x)D_{F_{\mu\nu}}'(y-y,y'). \qquad \text{(A9b)}$$

[11] For this it is sufficient to assume minimal electromagnetic coupling.

Here, Δ_F' and $D_{F_{\mu\nu}}'$ are the (unrenormalized) propagators defined by

$$\Delta_F'(x'-x)=\langle T[\phi^\dagger(x')\phi(x)]\rangle \qquad \text{(A10a)}$$

and

$$D_{F_{\mu\nu}}'(y-y')=\langle T[A_\mu(y)A_\nu(y')]\rangle. \qquad \text{(A10b)}$$

A special case of Eq. (A6) is therefore

$$\partial_y{}^\mu\Box_y D_{F_{\mu\nu}}'(y-y')=\frac{i\partial}{\partial_y{}^\nu}\delta(y-y'). \qquad \text{(A11)}$$

Using Eqs. (A7)–(A11) we see that

$$\partial_y{}^\mu\Box_y W_\mu(x',x;y)=0 \qquad \text{(A12a)}$$

and

$$\partial_y{}^\mu\Box_y W_{\mu\nu}(x',x;y,y')=0. \qquad \text{(A12b)}$$

On transformation to momentum space, via

$$W_\mu(x',x;y)=\int e^{i(p'\cdot x'-p\cdot x+q\cdot y)}\tilde{W}_\mu(p',p;q)d^4p'd^4pd^4q,$$

$$W_{\mu\nu}(x',x;y,y')=\int e^{i(p'\cdot x'-p\cdot x+q\cdot y+q'\cdot y')} \times\tilde{W}_{\mu\nu}(p',p;q,q')d^4p'd^4pd^4qd^4q',$$

Eqs. (A12a) and (A12b) reduce to

$$q^\mu q^2\tilde{W}_\mu(p',p;q)=0, \qquad \text{(A13a)}$$

and

$$q^\mu q^2\tilde{W}_{\mu\nu}(p',p;q,q')=0. \qquad \text{(A13b)}$$

On writing

$$D_{F_{\mu\nu}}'(y)=\frac{-i}{(2\pi)^4}\int D_{F_{\mu\nu}}'(k^2)e^{-ik\cdot y}d^4k$$

Eq. (A11) reduces to

$$k^\mu k^2 D_{F_{\mu\nu}}'(k^2)=k_\nu. \qquad \text{(A14)}$$

The vertex functions Γ_μ and $\Gamma_{\mu\nu}$ are now defined, up to a factor, by

$$\tilde{W}_\mu(p',p;q)=(\text{const})\Delta_F'(p'^2)\Delta_F'(p^2)\Gamma^{\mu'}D_{F_{\mu'\mu}}'(q^2) \times\delta(p'+q-p) \qquad \text{(A15a)}$$

and

$$\tilde{W}_{\mu\nu}(p',p;q,q')=(\text{const})\Delta_F'(p'^2)\Delta_F'(p^2)\Gamma^{\mu'\nu'}D_{F_{\mu'\mu}}'(q^2) \times D_{F_{\nu'\nu}}'(q'^2)\delta(p'+q'+q-p). \qquad \text{(A15b)}$$

From Eqs. (A13a), (A14), and (A15a) one infers that[12]

$$q_{\mu'}\Gamma^{\mu'}=0. \qquad \text{(A16)}$$

From Eqs. (A13b), (A14), and (A15b) we get

$$H^{\nu'}D_{F_{\nu'\nu}}'(q'^2)=0, \qquad \text{(A17)}$$

where

$$H^{\nu'}=q_{\mu'}\Gamma^{\mu'\nu'}.$$

[12] We assume that Γ^μ and $\Gamma^{\mu\nu}$ are free of δ-function singularities.

If we write $D_{F\nu'\nu}'(q'^2)$ in the form

$$D_{F'\nu'\nu}'=D_1g_{\nu'\nu}-D_2q_{\nu'}'q_\nu',$$

where D_1 and D_2 are functions of q'^2 only, then Eq. (A17) implies

$$D_1H_\nu-D_2(H\cdot q')q_\nu'=0. \tag{A18}$$

On multiplying Eq. (A17) by $(q')^2q'^\nu$ and using Eq. (A14) we see that $H\cdot q'=0$, so that Eq. (A18) implies

$$H_\nu=0.$$

Thus, we have

$$q_\mu\Gamma^{\mu\nu}=0 \tag{A19a}$$

and by symmetry,

$$q_\nu'\Gamma^{\mu\nu}=0. \tag{A19b}$$

Equations (A16) and (A19) constitute the analogs for a neutral field of the generalized Ward identities for charged fields.

Invariance under particle-antiparticle conjugation is equivalent to the existence of a unitary operator U_c which leaves the vacuum invariant and is such that

$$U_cA_\mu(y)U_c^{-1}=-A_\mu(y),$$

and

$$U_c\phi(x)U_c^{-1}=\phi^\dagger(x).$$

It follows that

$$W_\mu(x',x;y)=-W_\mu(x,x';y),$$

but

$$W_{\mu\nu}(x',x;y,y')=W_{\mu\nu}(x,x';y,y').$$

We then have

$$\tilde{W}_\mu(p',p;q)=-\tilde{W}_\mu(-p,-p';q),$$

and

$$\tilde{W}_{\mu\nu}(p',p;q,q')=\tilde{W}_{\mu\nu}(-p,p';q,q'),$$

so that, using Eqs. (A15a) and (A15b), and recalling that $P=p+p'$,

$$\Gamma_\mu(q;P)=-\Gamma_\mu(q;-P), \tag{A20a}$$

$$\Gamma_{\mu\nu}(q,q';P)=\Gamma_{\mu\nu}(q,q';-P). \tag{A20b}$$

Finally, since $W_{\mu\nu}(x',x;y,y')=W_{\nu\mu}(x',x;y',y)$ so that $\tilde{W}_{\mu\nu}(p',p;q,q')=\tilde{W}_{\nu\mu}(p',p;q',q)$, we have also

$$\Gamma_{\mu\nu}(q,q';P)=\Gamma_{\nu\mu}(q',q;P). \tag{A20c}$$

Equations (A20a)–(A20c) are the symmetry properties of Γ_μ and $\Gamma_{\mu\nu}$ used in Sec. III of this paper.

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Lie Groups, Lie Algebras, and the Troubles of Relativistic $SU(6)$

HARRY J. LIPKIN
The Weizmann Institute of Science, Rehovoth, Israel
(Received 5 May 1965)

Some of the difficulties of relativistic $SU(6)$ are examined. Those arising from the use of continuous groups can be avoided by the use of algebras of finite sets of operators which are sufficient to give the desired properties of elementary particles. The nonconservation of probability associated with the relativistic separation of space and spin is pointed out. Quantum electrodynamics applied to atomic structure is shown to exhibit the type of peculiar symmetry which leaves the interaction invariant but is broken by free Dirac propagators. The implications of this analogy for $SU(6)$ are discussed. The mixing of physical and nonphysical states (positive- and negative-energy quark states) leads to noninvariance of the vacuum under the symmetry group, and to a degenerate vacuum in the exact symmetry limit. The existence of open inelastic channels for low-mass boson production is relevant to unitarity calculations and is implied in all energy regions where the symmetry is not badly broken.

INTRODUCTION

THE successes of the $SU(6)$-symmetry scheme for elementary particles[1] and its relativistic generalizations[2] have been accompanied by an assortment of difficulties in principle and also by some predictions in disagreement with experiment.[3] A better picture of the relation between the successes and the difficulties can be obtained by examining the general assumptions underlying the proposed theories to determine which are really necessary to obtain the desired results. Analysis of the sources of some of the troubles may help in finding ways to get around them.

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