Consistency Conditions on the Strong Interactions Implied by a Partially Conserved Axial-Vector Current. II

STEPHEN L. ADLER*
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts
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Consequences of the partially conserved axial-vector current (PCAC) hypothesis are explored. A set of simple rules is derived which relate the matrix element for any strong interaction process with the matrix element for the corresponding process in which an additional zero-mass, zero-energy pion is emitted or absorbed. A generalization to include lowest order electromagnetic processes is given. A theorem is stated and proved which shows how divergence equations of the form $\partial_\lambda J_\lambda = D$ are modified when a minimal electromagnetic interaction is switched on.

INTRODUCTION

In an earlier paper[1] it was shown that the hypothesis of partially conserved $\Delta S=0$ axial-vector current (PCAC) leads to consistency conditions involving solely the strong interactions. One of these conditions, relating the pion-nucleon scattering amplitude $A^{\pi N(+)}$ and the pion-nucleon coupling constant g_r, was shown to agree with experiment to within 10%. In this note we give a simplified and generalized derivation of the consistency conditions implied by PCAC. We will derive a set of simple rules which relate the matrix element for any strong interaction or first-order electromagnetic process with the matrix element for the corresponding process in which an additional zero-mass, zero-energy pion is emitted or absorbed. The rules are closely connected with the "chirality conservation" formulas of Nambu, Lurié, and Shrauner.

Let us begin by recalling certain definitions from (I). We denote by $J_\lambda{}^A$ the strangeness-conserving weak axial current. By partially conserved axial-vector current we mean the hypothesis that

$$\partial_\lambda J_\lambda{}^A = \frac{-i\sqrt{2}M_N M_\pi{}^2 g_A{}^N(0)}{g_r K^{NN\pi}(0)}\phi_\pi + R. \tag{1}$$

Here M_N is the nucleon mass, M_π is the pion mass, $g_A{}^N(0)$ is the β-decay axial-vector coupling constant $[g_A{}^N(0)\approx 1.2\times 10^{-5}/M_N{}^2]$, g_r is the rationalized, renormalized pion-nucleon coupling constant ($g_r{}^2/4\pi \approx 14.6$), and ϕ_π is the renormalized field operator which creates the π^+. The quantity $K^{NN\pi}(0)$ is the pionic form factor of the nucleon evaluated at zero virtual pion mass; $K^{NN\pi}$ is normalized so that $K^{NN\pi}(-M\pi^2)=1$. In order to give content to the definition, we must specify properties of the residual operator R. We suppose that for states $\langle\beta(p_F)|$ and $|\alpha(p_I)\rangle$ for which $\langle\beta|\phi_\pi|\alpha\rangle\neq 0$, and for momentum transfer near the one pion pole at $-M_\pi{}^2$ [say, for $-M\pi^2<(p_F-p_I)^2<M_\pi{}^2$], the matrix element of R is much smaller than the matrix element of the pion operator term. In other words, we postulate that if $\langle\beta|\phi_\pi|\alpha\rangle\neq 0$ and if $|(p_F-p_I)^2|<M_\pi{}^2$, then

$$\frac{|\langle\beta|R|\alpha\rangle|}{[\sqrt{2}M_N M_\pi{}^2 g_A{}^N(0)/g_r K^{NN\pi}(0)]|\langle\beta|\phi_\pi|\alpha\rangle|}\ll 1. \tag{2}$$

In what follows we derive equalities which hold rigorously if the residual operator R is zero. If R is not zero, but satisfies the inequality of Eq. (2), the "equals" signs should be replaced by "approximately equals" signs.

It will be helpful to introduce a number of abbreviations and definitions. We denote by k the momentum transfer p_F-p_I. Let us introduce the isotopic vector quantities $J_\lambda{}^{Aa}$, $\phi_\pi{}^a$ ($a=1,2,3$), in terms of which

$$J_\lambda{}^A=\tfrac{1}{2}(J_\lambda{}^{A1}+iJ_\lambda{}^{A2}),\quad \phi_\pi=(1/\sqrt{2})(\phi_\pi{}^1+i\phi_\pi{}^2). \tag{3}$$

We denote the product $g_r\, K^{NN\pi}(0)$ by $g_r{}^{\pi N}(0)$. Then the generalization of Eq. (1) to all three isospin components $J_\lambda{}^{Aa}$ is (neglecting R)

$$\partial_\lambda J_\lambda{}^{Aa} = -i(2M_N M_\pi{}^2 g_A{}^N(0)/g_r{}^{\pi N}(0))\phi_\pi{}^a. \tag{4}$$

It will be convenient to introduce an isospin notation for the Σ and for the Ξ analogous to that for the nucleon N. We introduce isospinors and isospin column vectors as follows:

$$\Xi^0\to\begin{pmatrix}1\\0\end{pmatrix},\quad \Xi^-\to\begin{pmatrix}0\\1\end{pmatrix};$$

$$\Sigma^+\to\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\\0\end{pmatrix},\quad \Sigma^0\to\begin{pmatrix}0\\0\\1\end{pmatrix},\quad \Sigma^-\to\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\\0\end{pmatrix}. \tag{5}$$

By u_Ξ or u_Σ we will mean the ordinary Dirac spinor for the hyperon, multiplied by the appropriate isospinor or isospin column vector. Let τ^a denote the usual Pauli matrices, and let t^{Na}, $t^{\Xi a}$, and $t^{\Sigma a}$ be the matrices defined by

$$t^{Na}=t^{\Xi a}=\tau^a, \tag{6}$$

$$[t^{\Sigma a}]_{bc}=i\epsilon_{bca}. \tag{7}$$

Then we may write the baryon matrix elements of $J_\lambda{}^{Aa}$ and of $J_\pi{}^a=(-\Box+M_\pi{}^2)\phi_\pi{}^a$ as follows. (We omit the induced pseudoscalar terms in $J_\lambda{}^{Aa}$, since these are

* Junior Fellow, Society of Fellows.
[1] Stephen L. Adler, Phys. Rev. 137, B1022 (1965). We will refer to this paper as (I).

treated separately in the derivation below. See Refs. 4 and 6.)

$$\langle B(p_F)|J_\lambda^{Aa}|B(p_I)\rangle = \left(\frac{M_B}{p_{F0}}\frac{M_B}{p_{I0}}\right)^{1/2} \bar{u}_B(p_F) g_A{}^B \gamma_\lambda\gamma_5 t^{Ba} u_B(p_I)\,,$$
$$\langle B(p_F)|J_\pi{}^a|B(p_I)\rangle = \left(\frac{M_B}{p_{F0}}\frac{M_B}{p_{I0}}\right)^{1/2} \bar{u}_B(p_F) i g_r{}^{\pi B}\gamma_5 t^{Ba} u_B(p_I)\,. \tag{8}$$

Here B denotes N, Σ, or Ξ.

Using these definitions of the coupling constants, and Eq. (4), it is an easy matter to see that

$$\frac{M_N g_A{}^N(0)}{g_r{}^{\pi N}(0)} = \frac{M_\Sigma g_A{}^\Sigma(0)}{g_r{}^{\pi\Sigma}(0)} = \frac{M_\Xi g_A{}^\Xi(0)}{g_r{}^{\pi\Xi}(0)}\,. \tag{9}$$

Equation (9) will permit us to eliminate the axial-vector coupling constants $g_A{}^N$, $g_A{}^\Sigma$, and $g_A{}^\Xi$ from the consistency conditions obtained in the next section.

I. DERIVATION OF CONSISTENCY CONDITIONS

We take the matrix element of both sides of Eq. (4) between states $\langle\beta(p_F)^{\rm out}|$ and $|\alpha(p_I)^{\rm in}\rangle$, where β and α are any systems of strongly interacting particles. This gives

$$k_\lambda\langle\beta(p_F)^{\rm out}|J_\lambda{}^{Aa}|\alpha(p_I)^{\rm in}\rangle$$
$$= (2M_N M_\pi{}^2 g_A{}^N(0)/g_r{}^{\pi N}(0))\langle\beta(p_F)^{\rm out}|\phi_\pi{}^a|\alpha(p_I)^{\rm in}\rangle\,,$$
$$= \frac{2M_N g_A{}^N(0)}{g_r{}^{\pi N}(0)}\frac{M_\pi{}^2}{M_\pi{}^2+k^2}\langle\beta(p_F)^{\rm out}|J_\pi{}^a|\alpha(p_I)^{\rm in}\rangle\,. \tag{10}$$

Let us examine what happens in the limit as $k\to 0$ ($p_F\to p_I$). The right-hand side of Eq. (10) in most cases approaches a finite limit, since

$$\lim_{p_F\to p_I}\langle\beta(p_F)^{\rm out}|J_\pi{}^a|\alpha(p_I)^{\rm in}\rangle \tag{11}$$

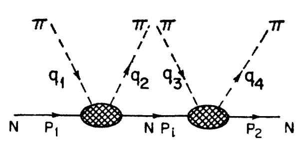

FIG. 1. The sort of situation which is excluded by the requirement that we avoid singularities of $\langle\beta^{\rm out}|\alpha^{\rm in}\rangle$. When $p_i{}^2 = (p_1+q_1-q_2)^2 = -M_N{}^2$, the diagram illustrated is infinite because the nucleon propagator joining the two bubbles is infinite. Such infinities can arise in general from pole diagrams contributing to $\langle\beta^{\rm out}|\alpha^{\rm in}\rangle$. (Pole diagrams are those which can be divided into two disconnected parts by cutting a single internal line.) We restrict ourselves in the text to values of the external four-momenta for which all pole diagrams contributing to $\langle\beta^{\rm out}|\alpha^{\rm in}\rangle$ are nonsingular.

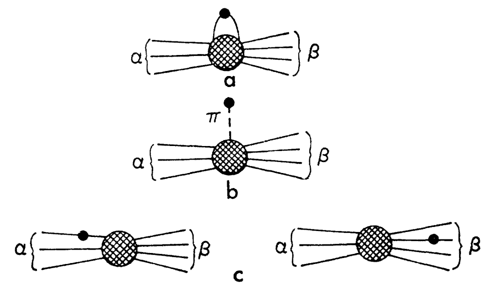

FIG. 2. Ways of attaching the proper vertex of $J_\lambda{}^A$, represented by a heavy dot. The proper vertex can be (a) attached to an internal line, (b) attached to a terminating external pion line, (c) attached to a nonterminating external line.

is just the matrix element for

$$\alpha\to\beta+(\text{zero-mass, zero-energy pion})\,,$$

and is in general nonzero.[2] Thus, the matrix element $\langle\beta(p_F)^{\rm out}|J_\lambda{}^{Aa}|\alpha(p_I)^{\rm in}\rangle$ must contain pole terms which go as $1/k$, in order that the scalar product of k with this matrix element have a finite limit. Clearly, if we can develop a simple set of rules for calculating these pole terms, we can calculate $\langle\beta(p_F)^{\rm out}|J_\pi{}^a|\alpha(p_I)^{\rm in}\rangle$ to zeroth order in k.

Calculation of the pole terms in $\langle\beta(p_F)^{\rm out}|J_\lambda{}^{Aa}|\alpha(p_I)^{\rm in}\rangle$ turns out to be quite easy. Let us restrict ourselves to values of the momenta of the particles in α and in β for which the matrix element $\langle\beta^{\rm out}|\alpha^{\rm in}\rangle$ has no singularities. (The sort of situation we wish to exclude is illustrated in Fig. 1.) The renormalized matrix element for $\langle\beta(p_F)^{\rm out}|J_\lambda{}^{Aa}|\alpha(p_I)^{\rm in}\rangle$ is obtained as follows[3]: First we write down a complete set of irreducible or "skeleton" diagrams for the matrix element. Then we make a series of insertions in the skeleton diagrams. We replace each bare propagator by the renormalized propagator, each bare strong-interaction vertex by the renormalized proper strong-interaction vertex, and each bare vertex where $J_\lambda{}^A$ acts by the renormalized proper vertex of $J_\lambda{}^A$.[4] We can divide the diagrams so obtained into three categories, according to where the proper vertex of $J_\lambda{}^A$ is attached: (a) The proper vertex of $J_\lambda{}^A$ is attached to an *internal* line [Fig. 2(a)]; (b) the proper vertex of $J_\lambda{}^A$ is attached to an *external pion* line which terminates [Fig. 2(b)]; (c) the proper vertex of $J_\lambda{}^A$ is attached to an *external line* which does not terminate [Fig. 2(c)].

[2] Note that the value of the limit depends in general on the *direction* in which k approaches zero.

[3] Let us review some definitions. The *skeleton* of a diagram is obtained by replacing all vertex parts by bare vertices and by omitting all self-energy parts from the propagators, so that only bare propagators appear. An *irreducible* or "*skeleton*" *diagram* is a diagram which is identical with its own skeleton. A *proper* vertex diagram is one which cannot be divided into two disconnected diagrams by cutting a single internal line.

[4] Note that the dominant part of the induced pseudoscalar coupling arises from the diagrams which give the one-pion pole term in dispersion theory. These diagrams are *improper* when considered as baryon-$J_\lambda{}^A$ vertices, and thus are not included in the proper baryon vertices of $J_\lambda{}^A$.

Corresponding to this division, we can write

$$\langle\beta(p_F)^{\rm out}|k_\lambda J_\lambda^{Aa}|\alpha(p_I)^{\rm in}\rangle = \langle\beta(p_F)^{\rm out}|k_\lambda J_\lambda^{Aa}|\alpha(p_I)^{\rm in}\rangle^{\rm INT} + \langle\beta(p_F)^{\rm out}|k_\lambda J_\lambda^{Aa}|\alpha(p_I)^{\rm in}\rangle^{\rm PION} + \langle\beta(p_F)^{\rm out}|k_\lambda J_\lambda^{Aa}|\alpha(p_I)^{\rm in}\rangle^{\rm EXT}. \quad (12)$$

We now analyze in turn the contribution of each of the terms in Eq. (12):

(a) First let us consider the case where the proper vertex of J_λ^A is attached to an internal line. Each diagram contributing to $\langle\beta(p_F)^{\rm out}|J_\lambda^A|\alpha(p_I)^{\rm in}\rangle^{\rm INT}$ corresponds to a diagram for $\langle\beta^{\rm out}|\alpha^{\rm in}\rangle$, but has an additional internal propagator. The requirement that $\langle\beta^{\rm out}|\alpha^{\rm in}\rangle$ be nonsingular means that all internal momenta are either integrated over or are off the mass shell. Thus the additional propagator cannot give rise to an infinity as $k\rightarrow 0$, and we conclude that $\langle\beta(p_F)^{\rm out}|k_\lambda J_\lambda^A|\alpha(p_I)^{\rm in}\rangle^{\rm INT}$ is of order k.[5]

(b) The sum of all diagrams where the proper vertex of J_λ^A is attached to a terminating external pion line is proportional to

$$\langle\beta(p_F)^{\rm out}|J_\pi^c|\alpha(p_I)^{\rm in}\rangle[1/(k^2+M_\pi^2)]\langle\pi^c|J_\lambda^{Aa}|0\rangle. \quad (13)$$

Using Eq. (4) to evaluate $\langle\pi^c|J_\lambda^{Aa}|0\rangle$ gives the result

$$\langle\beta(p_F)^{\rm out}|k_\lambda J_\lambda^{Aa}|\alpha(p_I)^{\rm in}\rangle^{\rm PION} = \frac{-k^2}{k^2+M_\pi^2}\frac{2M_N g_A^N(0)}{g_r^{\pi N}(0)}\langle\beta(p_F)^{\rm out}|J_\pi^a|\alpha(p_I)^{\rm in}\rangle. \quad (14)$$

This is of order k^2 and may be neglected.[6]

(c) We next consider diagrams where the proper vertex of J_λ^A is attached to a nonterminating external line. (We restrict ourselves to external lines of particles in the pseudoscalar meson or baryon octets.) These diagrams may be divided into two types, according to whether J_λ^A changes or does not change the *mass* of the external particle.[7] The only case where the mass is changed is that where J_λ^A changes an external Σ to a Λ or an external Λ to a Σ. Both of these cases make a contribution to $\langle\beta(p_F)^{\rm out}|k_\lambda J_\lambda^{Aa}|\alpha(p_I)^{\rm in}\rangle^{\rm EXT}$ which is of order k, since the propagator which follows the proper vertex of J_λ^A behaves as $(M_\Sigma^2-M_\Lambda^2)^{-1}$ as $k\rightarrow 0$, and thus is nonsingular. Finally, we will show that the diagrams where J_λ^A is attached to a nonterminating external line, and does not change the mass, are of order k^{-1}. Insertion of J_λ^A into a pseudoscalar meson line is forbidden by parity; insertion of J_λ^A into a Λ line is forbidden by isospin. Thus, we need only consider insertions of J_λ^A into external N, Σ, and Ξ lines. The contribution of the insertion of J_λ^A into the line of a final baryon B of four-momentum p_B is

$$\left(\frac{M_B}{p_{B0}}\right)^{1/2}\bar{u}_B(p_B)g_A^B\gamma_\lambda\gamma_5 t^{Ba}\frac{1}{\not p_B-\not k-iM_B}\mathfrak{M}. \quad (15)$$

Here $\mathfrak{M}$ is the matrix element for the process $\alpha\rightarrow\beta$, with the final baryon B *virtual*. Since $p_B^2=-M_B^2$, the propagator can be written as

$$\frac{1}{\not p_B-\not k-iM_B}=\frac{\not p_B-\not k+iM_B}{-2p_B\cdot k+k^2}, \quad (16)$$

showing that there is indeed a singularity as $k\rightarrow 0$. To lowest order in k, we can neglect k in calculating $\mathfrak{M}$ and can retain only the term of order k^{-1} in Eq. (16). Thus, the insertion becomes

$$\left(\frac{M_B}{p_{B0}}\right)^{1/2}\bar{u}_B(p_B)g_A^B\gamma_\lambda\gamma_5 t^{Ba}\frac{\not p_B+iM_B}{-2p_B\cdot k}\mathfrak{M}(k=0). \quad (17)$$

Calculating $\mathfrak{M}$ with $k=0$ means that we keep the final baryon B on the mass shell. Furthermore, $\not p_B+iM_B$ is just the positive frequency projection operator for B, with the property

$$(\not p_B+iM_B)\not p_B=(\not p_B+iM_B)iM_B. \quad (18)$$

Let us denote by $\mathfrak{M}^c$ the matrix element obtained by bringing all $\not p_B$ in $\mathfrak{M}(k=0)$ to the left and replacing them by iM_B. Then the insertion becomes, finally,

$$\left(\frac{M_B}{p_{B0}}\right)^{1/2}\bar{u}_B(p_B)g_A^B\gamma_\lambda\gamma_5 t^{Ba}\frac{\not p_B+iM_B}{-2p_B\cdot k}\mathfrak{M}^c. \quad (19)$$

The crucial point is that

$$\langle\beta^{\rm out}|\alpha^{\rm in}\rangle=\delta_{\beta\alpha}+(2\pi)^4 i\delta(p_F-p_I)\mathfrak{M}(\alpha\rightarrow\beta), \quad (20a)$$

$$-i\mathfrak{M}(\alpha\rightarrow\beta)=\left(\frac{M_B}{p_{B0}}\right)^{1/2}\bar{u}_B(p_B)\mathfrak{M}^c \quad (20b)$$

[5] We assume, of course, that none of the proper vertices of J_λ^A have a singularity as $k\rightarrow 0$.

[6] These diagrams form the dominant part of the induced pseudoscalar coupling. A statement much stronger than that they are of order k^2 can be made. Referring to Eq. (10), we note that the right-hand side may be written

$$(2M_N g_A^N(0)/g_r^{\pi N}(0))[1-k^2/(k^2+M_\pi^2)]\langle\beta(p_F)^{\rm out}|J_\pi^a|\alpha(p_I)^{\rm in}\rangle.$$

The part of this proportional to $k^2/(k^2+M_\pi^2)$ *exactly cancels* the contribution, given by Eq. (14), of the diagrams where J_λ^A is attached to a terminating external pion line. Now $k^2/(k^2+M_\pi^2)$ has the property

$$\lim_{k\rightarrow 0}\lim_{M_\pi^2\rightarrow 0}k^2/(k^2+M_\pi^2)=1,$$
$$\lim_{M_\pi^2\rightarrow 0}\lim_{k\rightarrow 0}k^2/(k^2+M_\pi^2)=0,$$

whereas the terms in Eq. (12) labeled INT and EXT are independent of the order of the limiting operations:

$$\lim_{k\rightarrow 0}\lim_{M_\pi^2\rightarrow 0}\langle\beta(p_F)^{\rm out}|k_\lambda J_\lambda^{Aa}|\alpha(p_I)^{\rm in}\rangle^{\rm INT,EXT} = \lim_{M_\pi^2\rightarrow 0}\lim_{k\rightarrow 0}\langle\beta(p_F)^{\rm out}|k_\lambda J_\lambda^{Aa}|\alpha(p_I)^{\rm in}\rangle^{\rm INT,EXT}.$$

Hence the exact cancellation of terms proportional to $k^2/(k^2+M_\pi^2)$ means that the limit, as $M_\pi^2\rightarrow 0$, of the consistency conditions of Eq. (24) is identical with the consistency conditions which would be obtained in a theory in which the pion mass was set equal to zero at the outset. Note that by virtue of Eq. (4), in such a theory the axial-vector current would be exactly conserved.

[7] We are neglecting the electromagnetic interactions, so all particles in the same isospin multiplet are of equal mass.

is just the matrix element which describes the strong process $\alpha \to \beta$, with all particles on the mass shell. Thus, $\mathfrak{M}^c$ can be measured experimentally. Similar arguments show that the insertion of $J_\lambda{}^A$ into an initial baryon line gives

$$\left(\frac{M_B}{p_{B0}}\right)^{1/2} \mathfrak{M}'^c \frac{p_B+iM_B}{2p_B\cdot k} g_A{}^B \gamma_\lambda\gamma_5 t^{Ba} u_B(p_B)\,, \quad (21)$$

with

$$-i\mathfrak{M}(\alpha\to\beta)=(M_B/p_{B0})^{1/2}\mathfrak{M}'^c u_B(p_B)\,. \quad (22)$$

To sum up, we have analyzed the behavior of each of the terms in Eq. (12). Let us collect the results and write

$$(2M_N g_A{}^N(0)/g_r{}^{\pi N}(0))\langle\beta(p_F)^{\text{out}}|J_\pi{}^a|\alpha(p_I)^{\text{in}}\rangle+O(k^2) =O(k)+O(k^2) + \sum_{\substack{\text{external}\\ \text{lines}}} [\text{insertions in} -i\mathfrak{M}(\alpha\to\beta)]+O(k)\,. \quad (23)$$

The three terms on the right-hand side of Eq. (23) refer, respectively, to the internal line, the terminating external pion line, and the nonterminating external line insertions of $J_\lambda{}^A$. Multiplying through by $g_r{}^{\pi N}(0)/[2M_N g_A{}^N(0)]$ and using Eq. (9) to eliminate the ratios $g_A{}^\Sigma(0)/g_A{}^N(0)$ and $g_A{}^\Xi(0)/g_A{}^N(0)$ in terms of strong-interaction coupling constants leads to the following set of rules:

$$\langle\beta(p_F)^{\text{out}}|J_\pi{}^a|\alpha(p_I)^{\text{in}}\rangle =O(k)+ \sum_{\substack{\text{external}\\ \text{lines}}} [\text{insertions in} -i\mathfrak{M}(\alpha\to\beta)]\,. \quad (24)$$

Insertions

For external π, K, η, Λ, the insertion is zero. For external N, Σ, Ξ, denoted by B, the insertions are

final B:

$$\bar{u}_B(p_B)\to \bar{u}_B(p_B)\left[\frac{g_r{}^{\pi B}(0)}{2M_B} k\gamma_5 t^{Ba}\right]\frac{p_B+iM_B}{-2p_B\cdot k} \quad (25\text{a})$$

initial B:

$$u_B(p_B)\to \frac{p_B+iM_B}{2p_B\cdot k}\left[\frac{g_r{}^{\pi B}(0)}{2M_B} k\gamma_5 t^{Ba}\right] u_B(p_B)\,. \quad (25\text{b})$$

These rules are the generalization to arbitrary processes of the consistency conditions derived in (I). It is an interesting fact that these rules are just what would be obtained if the effective pion-baryon coupling for pions with four-momentum near zero were pseudovector rather than pseudoscalar. This intimate connection between PCAC and gradient coupling theories was first noted by Feynman.[8]

As an illustration of the above rules, let us consider a special case. Let α be a single nucleon of four-momentum p_1 and any number of pions; similarly, let β be a single nucleon of four-momentum p_2 and any number of pions. Then we may write

$$\mathfrak{M}(\alpha\to\beta)=(M_N{}^2/p_{10}p_{20})^{1/2}\bar{u}_N(p_2)\mathfrak{M}u_N(p_1)\,. \quad (26)$$

According to the rules derived above,

$$\langle\beta(p_F)^{\text{out}}|J_\pi{}^a|\alpha(p_I)^{\text{in}}\rangle =O(k)-\left(\frac{M_N{}^2}{p_{10}p_{20}}\right)^{1/2} i\bar{u}_N(p_2)\left\{\left[\frac{g_r{}^{\pi N}(0)}{2M_N}k\gamma_5\tau^a\right] \times\left[\frac{p_2+iM_N}{-2p_2\cdot k}\right]\mathfrak{M}+\mathfrak{M}\left[\frac{p_1+iM_N}{2p_1\cdot k}\right] \times\left[\frac{g_r{}^{\pi N}(0)}{2M_N}k\gamma_5\tau^a\right]\right\}u(p_1)\,. \quad (27)$$

It is easily seen that Eq. (27) is equivalent to the "chirality conservation" formula obtained by Nambu and Lurié[9] in a theory in which the pion mass is zero and in which the axial-vector current is exactly conserved.[6] Nambu and Shrauner[10] and Shrauner[11] applied Eq. (27) to the case when $\alpha,\beta=\pi+N$ and found possible consistency with experiment. A simpler case was studied in (I), where we took $\alpha=N$, $\beta=\pi^b+N$. In this case $\mathfrak{M}$ is just the pion-nucleon vertex $ig_r\tau^b\gamma_5$ and $\langle(\pi^b N)^{\text{out}}|J_\pi{}^a|N^{\text{in}}\rangle$ is the pion-nucleon scattering amplitude. Introducing the usual pion-nucleon scattering-energy and momentum-transfer variables ν and ν_B,

$$p_1\cdot k=-M_N(\nu-\nu_B)\,,\quad p_2\cdot k=-M_N(\nu+\nu_B)\,, \quad (28)$$

we get from Eq. (27)

$$\langle(\pi^b N)^{\text{out}}|J_\pi{}^a|N^{\text{in}}\rangle=\left(\frac{M_N{}^2}{p_{10}p_{20}}\right)^{1/2} K^{NN\pi}(0)\bar{u}_N(p_2) \times\left\{\frac{g_r{}^2}{M_N}\delta_{ab}-ik\frac{g_r{}^2}{2M_N}\left[\frac{\tau^b\tau^a}{\nu_B-\nu}-\frac{\tau^a\tau^b}{\nu_B+\nu}\right]\right\}u_N(p_1)\,. \quad (29)$$

The term $(g_r{}^2/M_N)\delta_{ab}$ leads to the consistency condition

$$\frac{A^{\pi N(+)}(\nu=0,\,\nu_B=0,\,k^2=0)}{K^{NN\pi}(0)}=\frac{g_r{}^2}{M_N}, \quad (30)$$

which was discussed in detail in (I).

II. MODIFICATION IN THE PRESENCE OF THE ELECTROMAGNETIC INTERACTIONS

It is interesting to see how the rules derived above are modified when the electromagnetic interactions are taken into account. Since isotopic spin is not a good

[8] R. P. Feynman, *Proceedings of the Aix-en-Provence International Conference on Elementary Particles* (Centre d'Etudes Nucléaires de Saclay, Seine et Oise, 1961), Vol. II, p. 210. I am very grateful to Dr. M. Veltman for calling my attention to this reference and for emphasizing the connection between PCAC and gradient coupling of the pion.

[9] Y. Nambu and D. Lurié, Phys. Rev. **125**, 1429 (1962); Y. Nambu and E. Shrauner, *ibid.* **128**, 862 (1962).

[10] Y. Nambu and E. Shrauner, Ref. 9.

[11] E. Shrauner, Phys. Rev. **131**, 1847 (1963).

quantum number in the presence of electromagnetism, we will work only with fields and currents with definite charge transformation properties. Thus, we replace the three equations contained in Eq. (4) by the equations

$$\partial_\lambda J_\lambda{}^{A(\pm)} = C\phi_\pi{}^{(\pm)}, \quad \partial_\lambda J_\lambda{}^{A(0)} = \sqrt{2}C\phi_\pi{}^{(0)}, \tag{31}$$

where

$$J_\lambda{}^{A(\pm)} = \tfrac{1}{2}(J_\lambda{}^{A1} \mp iJ_\lambda{}^{A2}), \qquad J_\lambda{}^{A(0)} = J_\lambda{}^{A3};$$
$$\phi_\pi{}^{(\pm)} = (1/\sqrt{2})(\phi_\pi{}^1 \mp i\phi_\pi{}^2), \qquad \phi_\pi{}^{(0)} = \phi_\pi{}^3; \tag{32}$$
$$C = \frac{-i\sqrt{2}M_N M_\pi{}^2 g_A{}^N(0)}{g_r K^{NN\pi}(0)}.$$

[The superscript $(\pm)$ refers to the charge destroyed.] It is shown in the Appendix that to first order in the electric charge e $(e>0)$, the modification of Eqs. (31) in the presence of the electromagnetic interactions is

$$(\partial_\lambda \mp ieA_\lambda)J_\lambda{}^{A(\pm)} = C\phi_\pi{}^{(\pm)}, \quad \partial_\lambda J_\lambda{}^{A(0)} = \sqrt{2}C\phi_\pi{}^{(0)}. \tag{33}$$

As is customary, A_λ denotes the electromagnetic field. Since all electromagnetic corrections to masses and coupling constants are of second order in e, questions such as whether to use the charged or neutral pion mass in computing C do not arise.

Equations (33) permit us to state a simple set of rules for computing (up to terms linear in the four-momentum of the added pion) the matrix elements $\langle\beta^{\text{out}}|J_\pi{}^{(\pm 0)}|(\alpha\gamma)^{\text{in}}\rangle$, where α and β are any systems of strongly interacting particles and where the initial photon γ may be real or virtual. The terms $\partial_\lambda J_\lambda{}^{A(\pm 0)}$ in Eqs. (33) give rise to insertions into the external baryon lines of $-i\mathfrak{M}(\alpha\gamma\to\beta)$ identical with those of Eq. (25), apart from trivial changes in the isospin factors arising from the use of fields and currents of definite charge. In addition, we must add to $\langle\beta^{\text{out}}|J_\pi{}^{(\pm)}|(\alpha\gamma)^{\text{in}}\rangle$ the term

$$\frac{\pm e g_r{}^{\pi N}(0)}{\sqrt{2}M_N g_A{}^N(0)}\langle\beta^{\text{out}}|A_\lambda J_\lambda{}^{A(\pm)}|(\alpha\gamma)^{\text{in}}\rangle \tag{34}$$

arising from the term $A_\lambda J_\lambda{}^{A(\pm)}$ in Eq. (33). Using the standard reduction formulas, we find to lowest order in e that

$$\langle\beta^{\text{out}}|A_\lambda(y)J_\lambda{}^{A(\pm)}(y)|(\alpha\gamma)^{\text{in}}\rangle = \frac{\exp(ik'\cdot y)}{(2k_0')^{1/2}}\langle\beta^{\text{out}}|\epsilon_\lambda J_\lambda{}^{A(\pm)}(y)|\alpha^{\text{in}}\rangle, \tag{35}$$

where k' is the four-momentum and ϵ_λ the polarization four-vector of the photon γ. Equations (33), (34), and (35) allow us to calculate the matrix element for the emission of a zero-energy, zero-mass pion in photo- and electroproduction reactions. They are equivalent to the formalism derived for this purpose by Nambu and Shrauner,[9] who also discuss a detailed application to the reaction $e+N\to e+N+\pi$.

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APPENDIX

We give here a fairly general treatment of the way in which divergence equations of the form

$$\partial_\lambda J_\lambda = D \tag{A1}$$

are modified in the presence of electromagnetic interactions. We state the result in the form of a theorem.[12]

Theorem. Let ψ_j be the unrenormalized fields of particles of charge e_j. Let us consider a strong-interaction theory with the Lagrangian $\mathfrak{L}[\{\psi\},\{\partial_\sigma\psi\}]$, where $\{\psi\}$ denotes the set of the ψ_j. Let J_λ be a current with definite charge transformation properties (charge e_J) derived by making an infinitesimal gauge transformation on the fields ψ_j in the following manner[13]:

$$\psi_j \to \psi_j' = \psi_j + \Lambda F_j[\{\psi\}], \quad \mathfrak{L}\to\mathfrak{L}' = \mathfrak{L}[\{\psi'\},\{\partial_\sigma\psi'\}], \quad J_\lambda = [\delta\mathfrak{L}'/\delta(\partial_\lambda\Lambda)]_{\Lambda=0}. \tag{A2}$$

Then,

(1) In the absence of electromagnetic interactions the current J_λ satisfies

$$\partial_\lambda J_\lambda = D, \tag{A3}$$

with J_λ and D both functions of the ψ_j and the $\partial_\sigma\psi_j$ only:

$$J_\lambda = J_\lambda[\{\psi\},\{\partial_\sigma\psi\}], \quad D = D[\{\psi\},\{\partial_\sigma\psi\}]. \tag{A4}$$

(2) Inclusion of the electromagnetic interactions, with minimal electromagnetic coupling, changes Eqs. (A3) and (A4) to

$$(\partial_\lambda - ie_J A_\lambda)J_\lambda[\{\psi\},\{\pi_\sigma\}] = D[\{\psi\},\{\pi_\sigma\}], \tag{A5}$$

where $\pi_{j\sigma}$ denotes the quantity $(\partial_\sigma - ie_j A_\sigma)\psi_j$.

Proof. We proceed as if the fields were classical quantities, ignoring questions of commutation and anticommutation. Let us first consider the case when there are no electromagnetic interactions. The Lagrange equation of motion for the field ψ_j is

$$\frac{\delta\mathfrak{L}}{\delta\psi_j} = \partial_\sigma\frac{\delta\mathfrak{L}}{\delta(\partial_\sigma\psi_j)}. \tag{A6}$$

Under the gauge transformation

$$\psi_j \to \psi_j' = \psi_j + \Lambda F_j[\{\psi\}], \tag{A7}$$

[12] I am grateful to Professor S. Coleman for assistance in proving the theorem.

[13] M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960).

the derivatives $\partial_\sigma\psi_j$ and the Lagrangian $\mathcal{L}$ change according to

$$\begin{aligned}\partial_\sigma\psi_j \to \partial_\sigma\psi_j' &= \partial_\sigma\psi_j + (\partial_\sigma\Lambda)F_j + \Lambda(\partial_\sigma F_j)\,,\\ \mathcal{L} \to \mathcal{L}' &= \mathcal{L}[\{\psi'\},\{\partial_\sigma\psi'\}]\\ &= \mathcal{L}[\{\psi+\Lambda F\},\{\partial_\sigma\psi + (\partial_\sigma\Lambda)F + \Lambda(\partial_\sigma F)\}]\,.\end{aligned} \tag{A8}$$

From Eq. (A8) we find for the first variations,

$$\begin{aligned}\frac{\delta\mathcal{L}'}{\delta\Lambda} &= \sum_j\left[\frac{\delta\mathcal{L}'}{\delta\psi_j'}F_j + \frac{\delta\mathcal{L}'}{\delta(\partial_\sigma\psi_j')}\partial_\sigma F_j\right],\\ \frac{\delta\mathcal{L}'}{\delta(\partial_\lambda\Lambda)} &= \sum_j \frac{\delta\mathcal{L}'}{\delta(\partial_\lambda\psi_j')}F_j\,.\end{aligned} \tag{A9}$$

Eq. (A8) also implies that

$$\left[\frac{\delta\mathcal{L}'}{\delta\psi_j'}\right]_{\Lambda=0} = \frac{\delta\mathcal{L}}{\delta\psi_j}\,,\quad \left[\frac{\delta\mathcal{L}'}{\delta(\partial_\lambda\psi_j')}\right]_{\Lambda=0} = \frac{\delta\mathcal{L}}{\delta(\partial_\lambda\psi_j)}\,. \tag{A10}$$

Together, Eqs. (A6), (A9), and (A10) imply that

$$\partial_\lambda\left[\frac{\delta\mathcal{L}'}{\delta(\partial_\lambda\Lambda)}\right]_{\Lambda=0} = \left[\frac{\delta\mathcal{L}'}{\delta\Lambda}\right]_{\Lambda=0}\,. \tag{A11}$$

We define

$$\begin{aligned}J_\lambda &\equiv [\delta\mathcal{L}'/\delta(\partial_\lambda\Lambda)]_{\Lambda=0}\,,\\ D &\equiv [\delta\mathcal{L}'/\delta\Lambda]_{\Lambda=0}\,;\end{aligned} \tag{A12}$$

these are clearly functions only of the $\{\psi\}$ and the $\{\partial_\sigma\psi\}$.

Let us now turn on the electromagnetic interactions. According to the hypothesis of minimal electromagnetic coupling, the Lagrangian is modified according to

$$\mathcal{L} \to \mathcal{L}^{\mathrm{EM}} = \mathcal{L}[\{\psi\},\{\pi_\sigma\}] + \mathcal{L}^{\mathrm{EM}0}\,, \tag{A13}$$

where $\mathcal{L}^{\mathrm{EM}0}$ is the kinetic Lagrangian of the electromagnetic field A_σ and where $\pi_{j\sigma}$ is $(\partial_\sigma - ie_jA_\sigma)\psi_j$. The new Lagrange equation for the field ψ_j is

$$\partial_\sigma(\delta\mathcal{L}^{\mathrm{EM}}/\delta(\partial_\sigma\psi_j)) = \delta\mathcal{L}^{\mathrm{EM}}/\delta\psi_j\,. \tag{A14}$$

Let us henceforth treat ψ_j and $\pi_{j\sigma}$, rather than ψ_j and $\partial_\sigma\psi_j$, as the independent variables in taking the variation of $\mathcal{L}^{\mathrm{EM}}$. Then the Lagrange equation becomes

$$\partial_\sigma\frac{\delta\mathcal{L}^{\mathrm{EM}}}{\delta\pi_{j\sigma}} = \frac{\delta\mathcal{L}^{\mathrm{EM}}}{\delta\psi_j} - ie_jA_\sigma\frac{\delta\mathcal{L}^{\mathrm{EM}}}{\delta\pi_{j\sigma}}\,. \tag{A15}$$

Now let us make the gauge transformation $\psi_j \to \psi_j' = \psi_j + \Lambda F_j$. The quantity $\pi_{j\sigma}$ and the Lagrangian $\mathcal{L}^{\mathrm{EM}}$ change according to

$$\begin{aligned}\pi_{j\sigma} \to \pi_{j\sigma}' &= \pi_{j\sigma} - ie_jA_\sigma\Lambda F_j + (\partial_\sigma\Lambda)F_j + \Lambda(\partial_\sigma F_j)\,,\\ \mathcal{L}^{\mathrm{EM}} \to \mathcal{L}^{\mathrm{EM}\prime} &= \mathcal{L}[\{\psi'\},\{\pi_\sigma'\}] + \mathcal{L}^{\mathrm{EM}0}\\ &= \mathcal{L}[\{\psi+\Lambda F\},\{\pi_\sigma - ieA_\sigma\Lambda F + (\partial_\sigma\Lambda)F + \Lambda(\partial_\sigma F)\}] + \mathcal{L}^{\mathrm{EM}0}\,.\end{aligned} \tag{A16}$$

The first variations are

$$\begin{aligned}\frac{\delta\mathcal{L}^{\mathrm{EM}\prime}}{\delta\Lambda} &= \sum_j\left[\frac{\delta\mathcal{L}^{\mathrm{EM}\prime}}{\delta\psi_j'}F_j + \frac{\delta\mathcal{L}^{\mathrm{EM}\prime}}{\delta\pi_{j\sigma}'}(\partial_\sigma F_j - ie_jA_\sigma F_j)\right],\\ \frac{\delta\mathcal{L}^{\mathrm{EM}\prime}}{\delta(\partial_\lambda\Lambda)} &= \sum_j\left[\frac{\delta\mathcal{L}^{\mathrm{EM}\prime}}{\delta\pi_{j\lambda}'}F_j\right].\end{aligned} \tag{A17}$$

Using the Lagrange equation, Eq. (A15), we see that

$$\partial_\lambda\left[\frac{\delta\mathcal{L}^{\mathrm{EM}\prime}}{\delta(\partial_\lambda\Lambda)}\right]_{\Lambda=0} = \left[\frac{\delta\mathcal{L}^{\mathrm{EM}\prime}}{\delta\Lambda}\right]_{\Lambda=0}\,. \tag{A18}$$

Let us make use of the fact that the current J_λ has definite charge transformation properties. Since $\delta\mathcal{L}/\delta(\partial_\lambda\psi_j)$ transforms as a field with charge $-e_j$, Eqs. (A9) and (A12) tell us that F_j must transform as a field with charge e_j+e_J. Thus,

$$F_j[\psi_1\exp(ie_1t),\psi_2\exp(ie_2t),\cdots] = \exp[i(e_j+e_J)t]F_j[\psi_1,\psi_2,\cdots]\,. \tag{A19}$$

Taking the first derivative with respect to t gives the identity

$$\textstyle\sum_l(\delta F_j/\delta\psi_l)e_l\psi_l = (e_j+e_J)F_j\,. \tag{A20}$$

Consequently, using $\partial_\sigma F_j = \sum_l(\delta F_j/\delta\psi_l)\partial_\sigma\psi_l$, we obtain

$$\begin{aligned}\partial_\sigma F_j - i(e_j+e_J)A_\sigma F_j &= \textstyle\sum_l(\delta F_j/\delta\psi_l)(\partial_\sigma - ie_lA_\sigma)\psi_l\\ &= \textstyle\sum_l(\delta F_j/\delta\psi_l)\pi_{l\sigma}\,.\end{aligned} \tag{A21}$$

In other words, $\partial_\sigma F_j - i(e_j+e_J)A_\sigma F_j$ is the same function of $\{\psi\}$, $\{\pi_\sigma\}$ as $\partial_\sigma F_j$ is of $\{\psi\}$, $\{\partial_\sigma\psi\}$. Hence, by comparison of Eq. (A17) with Eq. (A9) it is clear that

$$\begin{aligned}&[\delta\mathcal{L}^{\mathrm{EM}\prime}/\delta(\partial_\lambda\Lambda)]_{\Lambda=0} = J_\lambda[\{\psi\},\{\pi_\sigma\}]\,,\\ &\textstyle\sum_j\{[\delta\mathcal{L}^{\mathrm{EM}\prime}/\delta\psi_j']_{\Lambda=0}F_j + [\delta\mathcal{L}^{\mathrm{EM}\prime}/\delta\pi_{j\sigma}']_{\Lambda=0}\\ &\quad\times[\partial_\sigma F_j - i(e_j+e_J)A_\sigma F_j]\} = D[\{\psi\},\{\pi_\sigma\}]\,.\end{aligned} \tag{A22}$$

Thus, Eq. (A18) can be rewritten as

$$(\partial_\lambda - ie_JA_\lambda)J_\lambda[\{\psi\},\{\pi_\sigma\}] = D[\{\psi\},\{\pi_\sigma\}]\,. \tag{A23}$$

This completes the proof.

Equation (A23) involves unrenormalized quantities throughout and is *exact*. In the case of PCAC, as considered in the text, $D = C^u\phi_\pi$, where the superscript on C^u denotes that it is unrenormalized. It is trivial to pass from Eq. (A23) to Eq. (33) of the text, which involves only renormalized quantities, if we work to lowest order in the electromagnetic coupling e: All electromagnetic renormalization effects are of second order in e and may be neglected. All strong interaction renormalization effects are contained in the ratio C/C^u, where C is the renormalized constant appearing in Eq. (32) of the text.