PHYSICAL REVIEW VOLUME 139, NUMBER 6B 20 SEPTEMBER 1965

# Gauge-Invariant Quantum Electrodynamics. II

I. GOLDBERG
*Clarkson College of Technology, Potsdam, New York*
(Received 10 May 1965)

A manifestly gauge-invariant formulation is introduced together with the procedure for quantizing the theory. This procedure is path-independent and leads to an electron operator which creates charged particles. The theory is nonlocal, but a procedure is introduced for determining the Hamiltonian of the theory.

## I. INTRODUCTION

RECENTLY there has been much discussion in the literature of the role of potentials in quantum electrodynamics.[1-4] It is the purpose of this note to describe an approach to quantum electrodynamics which obviates the need for potentials. The theory described here is not a new theory, but merely a reformulation of the ordinary theory. However, the reformulation enables one to discuss the role of potentials, and their relation to locality and manifest covariance. Furthermore, this paper will endeavor to add to an earlier work on a gauge-invariant formulation of quantum electrodynamics, henceforth to be referred to as I.[5]

The basic approach of I was to start from the usual Lagrangian for electrodynamics, and carry out a quantization procedure which would not alter the basic gauge structure of the theory. In this approach the potentials entered explicitly although always in a guage-invariant way. The reformulation to be carried out here still requires a special type of quantization. However, this is intimately tied to the gauge invariance of the theory owing to the fact that any theory possessing any type of gauge invariance yields relations between coordinates and momenta not involving time derivatives. These relations are usually called constraints, since they place limits on the theory. When this situation occurs, the usual quantization procedure [Poisson bracket (P.B.)] breaks down, owing to the fact that the P.B. approach is based on a homomorphism between the infinitesimal unitary transformations and the infinitesimal canonical transformations. The particular set of transformations most useful is composed of those transformations generated by the coordinates and momenta. The P.B. approach fails in theories containing constraints because the coordinates and momenta generate transformations which violate the constraints. The correct approach to be followed in this case was first introduced by Bergmann,[6] and consists of choosing a subgroup of the canonical transformations which does not alter the constraints. The generators of the so-called group commutators are then not the Poisson brackets, but a new type of bracket which, under the homomorphism, is to be related to the ordinary commutators of the quantum theory.

## II. MANIFESTLY COVARIANT FORMULATION

In this section a reformulation of electrodynamics will be discussed, to be obtained from the usual formulation by a unitary transformation. Therefore, the new formulation will be physically identical with the old, although all reference to potentials will have vanished from the Lagrangian and Hamiltonian. In order to eliminate the potentials, one must render the electron operators gauge invariant since the usual potential term is necessary to cancel the results of a gauge transformation on the electron field. The first transformation is manifestly covariant but nonlocal and leads to seemingly nonlocal Lagrangian and Hamiltonian functions. The second is local, but not manifestly covariant, so that in the potential-free approach there is a relation between locality and manifest covariance.

We introduce the gauge-invariant electron operator $\phi$:

$$\phi=e^{iC}\psi\,,\qquad C=e\int d^4x'C_\mu(x,x')A_\mu(x')\,. \tag{1}$$

$\psi$ is the usual electron-field operator, and $C_\mu$ satisfies the equation,

$$\partial_\mu C_\mu(x,x')=-\delta^4(x-x')\,, \tag{2}$$

where

$$\partial_\mu=\partial/\partial x_\mu\,.$$

The gauge invariance of the operator $\phi$ is obvious since

$$\psi\to e^{ie\lambda}\psi\,,\quad A_\mu\to A_\mu+\partial_\mu\lambda\,,$$

$$\phi\to\exp\left\{ie\left[\lambda+\int d^4x'C_\mu(x,x')(A_\mu(x')+\partial_{\mu'}\lambda(x'))\right]\right\}\psi=\phi \tag{3}$$

after an integration by parts.

If one transforms the Lagrangian $L$,

$$L=\int\left[-\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu}+\tfrac{1}{2}(\bar\psi\gamma^\mu\partial_\mu\psi-\partial_\mu\bar\psi\gamma^\mu\psi)-m\bar\psi\psi-ie\bar\psi\gamma^\mu\psi A_\mu\right]d^4x\,, \tag{4}$$

[1] Y. Aharonov and D. Bohm, Phys. Rev. **130**, 1625 (1963).
[2] S. Mandelstam, Ann. Phys. (N. Y.) **19**, 1 (1962).
[3] F. Rohrlich and F. Strocchi, Phys. Rev. **139**, B476 (1965).
[4] B. DeWitt, Phys. Rev. **125**, 2189 (1962).
[5] I. Goldberg, Phys. Rev. **112**, 1361 (1958).
[6] P. G. Bergmann and I. Goldberg, Phys. Rev. **98**, 531 (1955).

by means of the transformation (1), one finds

$$L=\int d^4x\Big[-\tfrac{1}{4}F_{\mu\xi}F_{\mu\xi}+\tfrac{1}{2}i(\bar{\phi}\gamma^\mu\partial_\mu\phi-\partial_\mu\bar{\phi}\gamma^\mu\phi)$$
$$-m\bar{\phi}\phi+ie\int d^4x'\bar{\phi}(x)\gamma^\mu C_\xi(x,x')F_{\mu\xi}(x')\phi(x)\Big]. \quad (5)$$

The Lagrangians (4) and (5), are, of course, identical, although expressed in terms of different variables. The gauge-invariant operators $\phi$ create and destroy charged electrons, that is to say, the electron with its Coulomb, Biot-Savart field. However, the interaction term is not a local operator, locality having been sacrificed for manifest gauge invariance. The field equations are given by

$$\partial_\xi F_{\mu\xi}=j_\mu-ie\int\bar{\phi}(x')\gamma^\mu\partial_\mu C_\xi(x,x')\phi(x')d^4x', \quad (6a)$$

$$i\gamma^\mu\partial_\mu\phi-m\phi-ie\int d^4x'\gamma^\mu C_\xi(x,x')F_{\mu\xi}(x')\phi(x)=0, \quad (6b)$$

$$i\partial_\mu\bar{\phi}\gamma^\mu+m\bar{\phi}+ie\bar{\phi}\int d^4x'\gamma^\mu C_\xi(x,x')F_{\mu\xi}(x')=0, \quad (6c)$$

where

$$j_\mu=+ie\bar{\phi}\gamma^\mu\phi.$$

The second term on the right-hand side of (6a) vanishes after an integration by parts by virtue of the current conservation guaranteed by (6b) and (6c). Equations (6b) and (6c) may be put into a more physical form by a proper choice of the function

$$C_\mu=\partial_\mu\Delta_{\rm ret}(x-x'). \quad (7)$$

It is clear that this function satisfies Eq. (2) and with an integration by parts Eq. (6) yields

$$i\gamma^\mu\partial_\mu\phi-m\phi=e^2\int j^\mu(x')\gamma^\mu\Delta_{\rm ret}(x-x')d^4x'\phi(x),$$
$$i\gamma^\mu\partial_\mu\bar{\phi}+m\bar{\phi}=-e^2\bar{\phi}\int j_\mu(x')\gamma^\mu\Delta_{\rm ret}(x-x')d^4x'. \quad (8)$$

That is, the electromagnetic interactions of a given electron are due to the presence of other electrons.

The momenta conjugate to $A_\mu$ are obtained in the usual way:

$$\pi_\mu=\partial L/\partial\dot{A}_\mu=F_{\mu0}-Q_\mu, \quad (9)$$

where

$$Q_\mu=ie\int\bar{\phi}(x')[\gamma^\mu C_0(x',x)-\gamma^4 C_\mu(x',x)]\phi(x')d^4x'.$$

The quantities $\pi_\mu$ obtained above are identical with the in or out fields of the Yang-Feldman formalism.[7] The restriction to in or out field is determined by whether $C_\mu$ is obtained from the advanced or retarded Green's function. The relation of $\pi_\mu$ to the asymptotic fields is clear from the commutation relations, which are obtained in precisely the same way as in I and are identical at least in so far as the $\pi_\mu$ are concerned. The equal-time commutation relations are

$$[\pi_\mu,\phi]=0,\quad [\phi^\dagger(x),\phi(x')]_+=(1/i)\delta(x-x'),$$
$$[\pi_\mu,\pi_\xi]=0,\quad [\phi(x),\phi(x')]_+=[\bar{\phi}(x),\bar{\phi}(x')]_+=0, \quad (10)$$
$$[\pi_i,F_{kl}]=(1/i)[\delta_{ik}\partial_l\delta(x-x')-\delta_{il}\partial_k\delta(x-x')].$$

The commutation relations of $F_{\mu0}$ are not readily obtainable in this formulation. $F_{\mu0}$ depends upon both the $\pi_\mu$ and the electron-field variables at all times. This is, of course, a consequence of the fact that commutation relations are not obtainable for interacting fields. If one expresses $F_{\mu0}$ in terms of $\pi_\mu$ and $Q_\mu$ and tries to determine the commutator $[\phi,F_{\mu0}]$, one obtains $[\phi,F_{\mu0}]=[\phi,Q_\mu]$. This commutator cannot be evaluated explicitly owing to the occurrence of electron operators for all time in $Q_\mu$. However, one can expand the electron operators in terms of the field-free functions. Thus to first approximation

$$[\phi,Q_\mu(x')]=\frac{1}{i}\int d^4x''$$
$$\times\{S(x-x')[\gamma^\mu\partial_4\Delta(x''-x')-\gamma^4\partial_\mu\Delta(x-x')]\phi(x'')\},$$

where the term in square brackets represents the electromagnetic field associated with the electron and

$$S(x)=(i\gamma^\mu\partial_\mu+m)\Delta(x).$$

## III. LOCAL FORMULATION

It is more simple to observe the properties of the gauge-invariant formalism in the local formalism, which however is not manifestly covariant. In this case the operators $\phi$ are defined by[8] $\phi=e^{iC'}\psi$, where

$$C'=\int d^3x'C_s(x,x')A_s(x'),$$
$$\partial_s C_s'(x-x')=-\delta^3(x-x'). \quad (11)$$

The method to be employed here is identical with the procedure used above. The Lagrangian has the form,

$$L=\int d^4x\Big(-\tfrac{1}{4}F_{\mu\xi}F_{\mu\xi}+\tfrac{1}{2}i[\bar{\phi}\gamma^\mu\partial_\mu\phi-\partial_\mu\phi\gamma^\mu\phi]$$
$$-m\bar{\phi}\phi+ie\int d^3x'\bar{\phi}(x)\gamma^\mu\phi(x)C_s(x,x')F_{\mu s}(x')\Big). \quad (12)$$

The field equations are

$$\gamma^\mu\partial_\mu\phi-m\phi+ie\int d^3x'\gamma^\mu C_s(x,x')F_{\mu s}(x')\phi(x)=0. \quad (13)$$

[7] C. N. Yang and D. Feldman, Phys. Rev. **79**, 972 (1950).

[8] P. A. M. Dirac, Proc. Roy. Soc. (London) **235**, 138 (1950).

The momenta are defined by

$$\pi^4=F_{44}=0\,,\qquad \pi^s=F_{s4}+ie\int d^3x'\bar{\phi}(x')\gamma^4C_s(x',x)\phi(x)\,. \tag{14}$$

It is clear that in this case the electric field is given by $\pi_s$ plus the Coulomb field of the electrons, since $\bar{\phi}\gamma^4\phi=\phi^\dagger\phi$, the charge density, and $C_s(x,x')=(x^s-x'^s)/|x^s-x'^s|^3$. The commutation relations are identical with those above except that it is now possible to determine the commutation relations between the electric-field operators and the operator for the electric field $F_{s4}$.

$$[F_{s4},\phi]=[\pi_s,\phi]+ie\left[\int d^3x'\bar{\phi}(x')\gamma^4C_s(x',x)\phi(x'),\phi(x'')\right]=ie\int\gamma^4C_s(x',x)\phi(x')\delta(x''-x')d^3x'\,. \tag{15}$$

This is just the electric field due to the particle. The Hamiltonian can be readily obtained in this formulation and can shed considerable light on the nonlocal formulation.

$$H=\int d^3x\,\pi_\mu A_\mu+\pi\dot{\phi}+\pi^\dagger\dot{\phi}^\dagger-L\,. \tag{16}$$

The first term may be rewritten $\pi_\mu[F_{\mu4}+\partial_\mu A_4]$. The second vanishes as $\pi_4=0$ and $\partial_s\pi_s=0$. Then

$$H=\int d^3x[\pi_sF_{s4}-\tfrac{1}{2}F_{s4}F_{s4}+\tfrac{1}{4}F_{st}F_{st}+ie\int d^3x'\bar{\phi}(x)\gamma^4\bar{\phi}(x)C_s(x,x')F_{s4}(x')$$
$$-ie\int d^3x'\bar{\phi}(x)\gamma^s\phi(x)C_t(x,x')F_{st}(x')+\tfrac{1}{2}(\bar{\phi}\gamma^s\partial_s\phi-\partial_s\phi\gamma^s\phi)+m\bar{\phi}\phi]\,, \tag{17}$$

where terms have been explicitly resolved into space and time components. This expression can be simplified by noting that the fourth term may be written $-ie\int d^3x'\bar{\phi}(x')\gamma^4\phi(x')C_s(x',x)F_{s4}(x)$, and when combined with the first line may be written

$$H=\int d^3x[\tfrac{1}{2}F_{s4}F_{s4}+\tfrac{1}{4}F_{st}F_{st}+\tfrac{1}{2}(\bar{\phi}\gamma^s\partial_s\phi+\partial_s\bar{\phi}\gamma^s\phi)+m\bar{\phi}\phi-ie\int\bar{\phi}(x)\gamma^s\phi C_t(x,x')F_{st}(x')d^3x']\,. \tag{18}$$

This is precisely the Hamiltonian that would be obtained if the transformation of Eq. (11) were applied to the usual Hamiltonian, e.g., the Hamiltonian of I. As a simple test of the consistency of the commutation relations we check the equations of motion obtained from the Hamiltonian:

$$\dot{\pi}^s=\frac{1}{i}[\pi^s,H]=\frac{1}{i}\int d^3x'[\pi^s(x),\tfrac{1}{4}F_{rt}F_{rt}]-ie\frac{1}{i}\int d^3x'd^3x''\bar{\phi}(x')\gamma^r\phi(x')C_t(x',x'')[\pi^s(x),F_{rt}(x'')]; \tag{19}$$

all other commutators vanish.

Since $\frac{1}{4}F_{rt}^2=\frac{1}{2}\mathbf{H}^2$ and $[\pi^s(x),H_t(x')]=\epsilon_{stl}\partial_l\delta(x-x')$, the first integral reduces to $\nabla\times\mathbf{H}(x)$. The second term is

$$+ie\int d^3x'd^3x''\bar{\phi}(x')\gamma^r\phi(x')C_t(x',x'')\epsilon_{rtl}[\pi^s,H_l]$$
$$=ie\int\int d^3x'd^3x''\bar{\phi}(x')\gamma^r\phi(x')C_t(x',x'')[\delta_{rn}\delta_{ts}-\delta_{rs}\delta_{tn}]\partial_n\delta(x-x'')$$
$$-ie\int\int d^3x'd^3x''\partial_r[\bar{\phi}(x')\gamma^r\phi(x')]C_s(x',x'')+\bar{\phi}(x')\gamma^s\phi(x')\partial_r''C_r(x',x'')\delta(x-x'')$$
$$=ie\int d^3x'\frac{\partial}{\partial t}(\bar{\phi}(x')\gamma^4(x')C_s(x',x))+ie\phi(x)\gamma^s\phi(x)\,, \tag{20}$$

since

$$\pi^s=E^s+ie\int d^3x'\bar{\phi}(x')\gamma^4\phi(x')C_s(x',x)\,.$$

This equation is identical with the usual equation

$$\partial \mathbf{E}/\partial t = -\nabla\times\mathbf{H}+\mathbf{J}. \tag{21}$$

T he other equations of motion can be obtained in the same way.

$$\dot{\mathbf{H}} = \frac{1}{i}[\mathbf{H},H] = \frac{1}{i}[\mathbf{H},\tfrac{1}{2}\pi^s\pi^s] = \nabla\times\mathbf{E},$$

$$\dot{\phi} = \frac{1}{i}[\phi,H] = \gamma^4(\gamma^s\partial_s\phi - m\phi) + ie\tfrac{1}{2}\int d^3x' F_{s4}\gamma^4 C_s(x,x')\phi + \phi F_{s4}(x')\gamma^4 C_s(x,x') + ie\int C_s(x,x')\gamma^t F_{st}(x') d^3x'\phi(x). \tag{22}$$

The term involving $F_{\mu 4}$ brings up the problem of factor sequence since $F_{\mu 4}$ and $\phi$ do not commute. If the interaction term which has the form $\int d^3x' J_\mu(x) C_s(x,x') F_{\mu s}(x')$ is symmetrized to take the form

$$\int d^3x' \tfrac{1}{2}[F_{r\mu}(x')J_\mu(x) + J_\mu(x)F_{r\mu}(x')]C_r(x,x'),$$

this difficulty will be removed and equations of motion derived from the commutation relations will be identical with the Euler-Lagrange equation. This approach, as has been pointed out, is not manifestly covariant, but it enables us to approach the covariant formalism with the knowledge that a manifestly gauge-invariant formalism does not suffer from internal inconsistencies.

The Hamiltonian for the covariant formalism cannot be obtained uniquely by forming the function

$$H = \int d^3x (\partial L/\partial \dot{y}_a)\dot{y}_a - L, \tag{23}$$

where the $y_a$ represent the field variables, owing to the nonlocality of the interaction term. However, if one transforms the usual Hamiltonian for quantum electrodynamics,

$$H = \int d^3x\, \tfrac{1}{2}(\mathbf{E}^2+\mathbf{H}^2) + \tfrac{1}{2}(\partial_s\bar{\psi}\gamma^s\psi - \bar{\psi}\partial_s\psi) + m\bar{\psi}\psi + ie\bar{\psi}\gamma^s\psi A_s, \tag{24}$$

one arrives at the correct Hamiltonian formulation. This makes use of the fact that the only difference between the theory we are describing and the nonmanifestly gauge-invariant approach is a unitary transformation. The two forms of the Lagrangian differ by a unitary transformation and therefore the two forms of the Hamiltonian should differ by the same transformation. Then

$$H = \int d^3x\, \tfrac{1}{2}(\mathbf{E}^2+\mathbf{H}^2) + \tfrac{1}{2}(\partial_s\phi\gamma^s\phi - \phi\gamma^s\partial_s\phi) + m\bar{\phi}\phi + ie\int d^4x' \phi(x)\gamma^s\phi(x) C_\mu(x') F_{\mu s}(x'). \tag{25}$$

In the preceding sections we have indicated a somewhat different approach to the gauge-invariant formulation of quantum electrodynamics. One of the major reasons for undertaking such an approach was to find a perturbation theory which is gauge invariant and to determine its relationship with the usual approach. It should be pointed out that the method of Feynman is gauge-invariant although the gauge invariance must be explicitly checked to ensure that no diagrams have been omitted. We can readily see the relationship between our formulation and some of the other approaches to quantum electrodynamics by examining the interaction term

$$L_{\text{int}} = ie\int\int d^4x d^4x' \bar{\phi}(x)\gamma^\mu\phi(x) C_\xi(x,x') F_{\mu\xi}(x'), \tag{26}$$

$$\begin{aligned} L_{\text{int}} &= -ie\int\int d^4x d^4x' \bar{\phi}(x)\gamma^\mu\phi(x)\partial_\xi\Delta(x-x')F_{\mu\xi}(x') \\ &= -ie^2\int\int d^4x d^4x' \bar{\phi}(x)\gamma^\mu\phi(x) \\ &\qquad \times\Delta(x-x')\bar{\phi}(x')\gamma^\mu\phi(x'), \end{aligned} \tag{27}$$

$$C_\xi = \partial_\xi\Delta(x-x').$$

In this picture all electromagnetic interactions occur as a result of electron-electron interactions and if $\Delta(x-y)$ is chosen to be the photon propagator the integral looks like the term one would write down for electron-electron scattering in the usual formalism. In this paper we have attempted to shed further light on the role of gauge invariance in quantum electrodynamics. The above formulation is manifestly gauge-invariant and does not require either subsidiary conditions or indefinite metric to guarantee consistency.


## ACKNOWLEDGMENT

I should like to thank Professor S. Mandelstam for stimulating discussions on the subject.