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High-Frequency Conductivity of a Plasma in Quasi-Equilibrium. II. Effect of a Uniform Magnetic Field*

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A general expression for high-frequency conductivity is derived to include the effect of an external magnetic field. The limit of large ion mass is also discussed. For the special case that the unperturbed plasma is in thermodynamic equilibrium, the result obtained in the present paper reduces immediately to that previously discussed by Oberman and Shure.

I. INTRODUCTION

IN a previous paper[1] (hereafter called I), an expression was derived for the high-frequency conductivity of a plasma not necessarily in thermodynamic equilibrium. This result has been applied to the cases in which the electrons and ions have different temperatures. This study will be reported in forthcoming papers.

The purpose of the present paper is to extend the analysis of I to the case of a plasma in a uniform magnetic field, an extension of obvious practical interest. As in I, we shall assume: (1) the unperturbed plasma is stable, according to the Vlasov theory, and homogeneous; (2) the high-frequency electric field has a wavelength very long compared to the electron and ion Debye lengths, and therefore may be treated as spatially homogeneous; and (3) in the present study, the unperturbed distribution functions are isotropic in the plane normal to the magnetic field and vary slowly in time (in the sense discussed in I). Furthermore, we shall postulate that the frequency of the electric field is much higher than the cyclotron frequency (and the collision frequency, of course).

In Sec. II, we shall first discuss the governing equations and generalize an operator method which was discussed in Ref. 2 to include the effect of a uniform external magnetic field. Then in Sec. III, we study the solution of the pair-correlation function including the effect of the magnetic field. In Sec. IV, we derive a general expression for the high-frequency conductivity, and in V, discuss its possibile simplification in the limit of large ion mass. Finally, we present a summary and some concluding remarks in Sec. VI.

II. MATHEMATICAL FORMULATION AND METHOD OF SOLUTION

The Governing Equations

To facilitate our discussion, we introduce the following Fourier transforms:

$$G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t)=\int d^3r_1 e^{-i\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)}G_{sr}(\mathbf{r}_1-\mathbf{r}_2,\ \mathbf{v}_1,\mathbf{v}_2,t)\,, \qquad (1)$$

$$g_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t)=\int d^3r_1 e^{-i\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)}g_{sr}(\mathbf{r}_1-\mathbf{r}_2,\ \mathbf{v}_1,\mathbf{v}_2,t)\,. \qquad (2)$$

Here, following I, G_{sr} and g_{sr} are the unperturbed and perturbed pair-correlation functions, respectively. (Throughout, unless specifically noted, we shall follow the notations designated in I.) With these comments, we write the first two members of the Bogoliubov-Born-Green-Kirkwood-Yron hierarchy (after linearization) as follows:

$$\frac{\partial F_s}{\partial t}+\frac{e_s}{m_s c}(\mathbf{v}_1\times\mathbf{B}_0)\cdot\frac{\partial F_s}{\partial \mathbf{v}_1}=-\frac{\partial}{\partial \mathbf{v}_1}\frac{i}{(2\pi)^3}\cdot\int d^3k\frac{4\pi e_s\mathbf{k}}{m_s k^2}\sum_r n_r e_r\int d^3v_2 G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t)\,, \qquad (3)$$

$$\left[\frac{\partial}{\partial t}+i\mathbf{k}\cdot\mathbf{v}_1-i\mathbf{k}\cdot\mathbf{v}_2+\frac{e_s}{m_s c}(\mathbf{v}_1\times\mathbf{B}_0)\cdot\frac{\partial}{\partial\mathbf{v}_1}+\frac{e_r}{m_r c}(\mathbf{v}_2\times\mathbf{B}_0)\cdot\frac{\partial}{\partial\mathbf{v}_2}\right]G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t)$$
$$-\frac{4\pi e_s}{m_s k^2}i\mathbf{k}\cdot\frac{\partial F_s}{\partial\mathbf{v}_1}\sum_q n_q e_q\int d^3v_3 G_{rq}(-\mathbf{k},\mathbf{v}_2,\mathbf{v}_3)+\frac{4\pi e_r}{m_r k^2}i\mathbf{k}\cdot\frac{\partial F_r}{\partial\mathbf{v}_2}\sum_q n_q e_q\int d^3v_3 G_{sq}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_3)$$
$$=\frac{4\pi i e_s e_r\mathbf{k}}{k^2}\cdot\left[\frac{F_r(\mathbf{v}_2)}{m_s}\frac{\partial F_s(\mathbf{v}_1)}{\partial\mathbf{v}_1}-\frac{F_s(\mathbf{v}_1)}{m_r}\frac{\partial F_r(\mathbf{v}_2)}{\partial\mathbf{v}_2}\right], \qquad (4)$$

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[1] C.-S. Wu, Phys. Rev. 138, A51 (1965).

$$\frac{\partial f_s}{\partial t}+\frac{e_s}{m_s c}(\mathbf{v}_1\times\mathbf{B}_0)\cdot\frac{\partial f_s}{\partial \mathbf{v}_1}=-\frac{e_s}{m_s}\mathbf{E}\cdot\frac{\partial F_s}{\partial \mathbf{v}_1}-\frac{\partial}{\partial \mathbf{v}_1}\frac{i}{(2\pi)^3}\cdot\int d^3k\frac{4\pi e_s\mathbf{k}}{m_s k^2}\sum_r n_r e_r\int d^3v_2 g_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t)\,, \tag{5}$$

$$\left[\frac{\partial}{\partial t}+i\mathbf{k}\cdot\mathbf{v}_1-i\mathbf{k}\cdot\mathbf{v}_2+\frac{e_s}{m_s c}(\mathbf{v}_1\times\mathbf{B}_0)\cdot\frac{\partial}{\partial \mathbf{v}_1}+\frac{e_r}{m_r c}(\mathbf{v}_2\times\mathbf{B}_0)\cdot\frac{\partial}{\partial \mathbf{v}_2}\right]g_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t)$$
$$-\frac{4\pi e_s}{m_s k^2}i\mathbf{k}\cdot\frac{\partial F_s}{\partial \mathbf{v}_1}\sum_q n_q e_q\int d^3v_3 g_{rq}(-\mathbf{k},\mathbf{v}_1,\mathbf{v}_3,t)+\frac{4\pi e_r}{m_r k^2}i\mathbf{k}\cdot\frac{\partial F_r}{\partial \mathbf{v}_2}\sum_q n_q e_q\int d^3v_3 g_{sq}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_3,t)$$
$$=R_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t)\,, \tag{6}$$

where

$$R_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t)=\frac{4\pi e_s e_r i\mathbf{k}}{k^2}\cdot\left(\frac{1}{m_s}\frac{\partial}{\partial \mathbf{v}_1}-\frac{\partial}{m_r}\frac{\partial}{\partial \mathbf{v}_2}\right)[f_r(\mathbf{v}_2,t)F_s(\mathbf{v}_1)+f_s(\mathbf{v}_1,t)F_r(\mathbf{v}_2)]+\frac{e_s}{m_s}\frac{\partial f_s}{\partial \mathbf{v}_1}\cdot\frac{4\pi i\mathbf{k}}{k^2}\sum_q n_q e_q$$
$$\times\int d^3v_3 G_{rq}(-\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)-\frac{e_r}{m_r}\frac{\partial f_r}{\partial \mathbf{v}_2}\cdot\frac{4\pi i\mathbf{k}}{k^2}\sum_q n_q e_q\int d^3v_3 G_{sq}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)-\mathbf{E}\cdot\left(\frac{e_s}{m_s}\frac{\partial}{\partial \mathbf{v}_1}+\frac{e_r}{m_r}\frac{\partial}{\partial \mathbf{v}_2}\right)G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)\,. \tag{7}$$

It is understood that in (5) and (7), the electric field $\mathbf{E}$ is described by

$$\mathbf{E}=\mathbf{E}_0 e^{i\omega t}. \tag{8}$$

In order to proceed with our discussion, we must now generalize the previous method[2] of solution of the correlation function to include the magnetic field.

The Generalized Q_{sr} Operator

Let us rewrite Eqs. (4) and (6) as follows

$$\left[\frac{1}{\partial t}+H_s(\mathbf{k},\mathbf{v}_1)+H_r(-\mathbf{k},\mathbf{v}_2)\right]G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t)=\frac{4\pi e_s e_r i\mathbf{k}}{k^2}\cdot\left[\frac{F_r}{m_s}\frac{\partial F_s}{\partial \mathbf{v}_1}-\frac{F_s}{m_r}\frac{\partial F_r}{\partial \mathbf{v}_2}\right]\equiv B_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t)\,, \tag{9}$$

$$\left[\frac{\partial}{\partial t}+H_s(\mathbf{k},\mathbf{v}_1)+H_r(-\mathbf{k},\mathbf{v}_2)\right]g_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t)=R_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t)\,, \tag{10}$$

where

$$H_l(\mathbf{k},\mathbf{v})=i\mathbf{k}\cdot\mathbf{v}+\frac{e_l}{m_l c}(\mathbf{v}\times\mathbf{B}_0)\cdot\frac{\partial}{\partial \mathbf{v}}-\frac{4\pi e_l i}{m_l k^2}\mathbf{k}\cdot\frac{\partial F_l}{\partial \mathbf{v}}\sum_l n_l e_l\int d^3v\,. \tag{11}$$

If we introduce a cylindrical coordinate system in velocity space as shown in Fig. 1, we can re-express H_t as

$$H_l(\mathbf{k},\mathbf{v})=i\mathbf{k}\cdot\mathbf{v}+\Omega_l\frac{\partial}{\partial\phi}-\frac{4\pi e_l i}{m_l k^2}\mathbf{k}\cdot\frac{\partial F_l}{\partial \mathbf{v}}\sum_l n_l e_l\int d^3v\,, \tag{12}$$

where $\Omega_l=e_lB_0/m_lc$ and ϕ is the azimuthal angle. Following closely the discussion presented in Ref. 2, we can write

$$\sum_r n_r e_r\int d^3v_2 G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t)=Q_{sr}(\mathbf{v}_1|\,\mathbf{v}_1',\mathbf{v}_2',\mathbf{k},t)G_{sr}(t=0)+\int_0^t d\tau Q_{sr}(\mathbf{v}_1|\,\mathbf{v}_1',\mathbf{v}_2',\mathbf{k},\tau)B_{sr}(\mathbf{k},\mathbf{v}_1',\mathbf{v}_2')\,, \tag{13}$$

$$\sum_r n_r e_r\int d^3v_2 g_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t)=Q_{sr}(\mathbf{v}_1|\,\mathbf{v}_1',\mathbf{v}_2',\mathbf{k},t)g_{sr}(t=0)+\int_0^t d\tau Q_{sr}(\mathbf{v}_1|\,\mathbf{v}_1',\mathbf{v}_2',\mathbf{k},\tau)R_{sr}(\mathbf{k},\mathbf{v}_1^l,\mathbf{v}_2^l,\,t-\tau)\,, \tag{14}$$

where the operator Q_{sr}, as discussed in Appendix A, has the form

$$Q_{sr}(\mathbf{v}_1|\,\mathbf{v}_1',\mathbf{v}_2',\mathbf{k},t)=\frac{1}{(2\pi i)^2}\int_{-\infty+i0_-}^{+\infty+i0_-}d\omega_1\int_{-\infty+i0_-}^{+\infty+i0_-}d\omega_2 e^{i(\omega_1+\omega_2)t}\frac{1}{\omega_1+\mathbf{k}\cdot\mathbf{v}_1+i\Omega_s\partial\phi_1}$$
$$\times\left[\int d^3v_1'\delta(\mathbf{v}_1-\mathbf{v}_1')-\frac{kD_s(\mathbf{v}_1,\mathbf{k})}{\epsilon(\omega_1,\mathbf{k})}\int d^3v_1'\frac{\sum_s n_s e_s}{\omega_1+\mathbf{k}\cdot\mathbf{v}_1+i\Omega_s\partial\phi_1}\right]\int d^3v_2'\frac{\sum_r n_r e_r}{\epsilon(\omega_2,\,-\mathbf{k})(\omega_2-\mathbf{k}\cdot\mathbf{v}_2+i\Omega_r\partial\phi_2)}\,. \tag{15}$$

[2] C.-S. Wu, J. Math. Phys. 5, 1701 (1964).

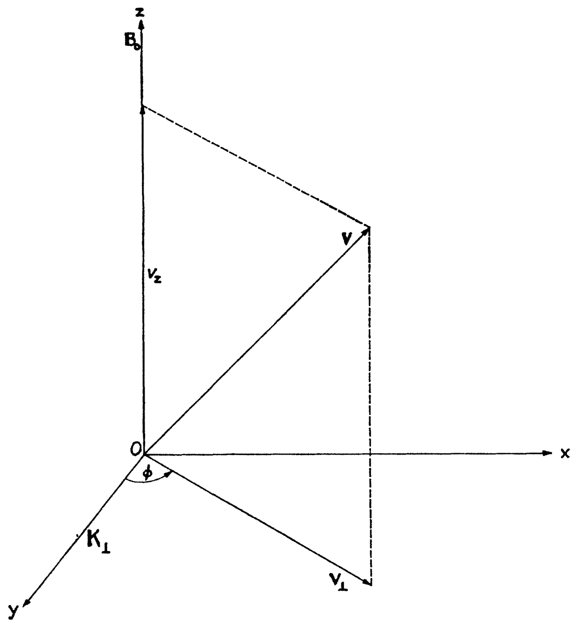

FIG. 1. Coordinates in velocity space.

In (15), $(\omega\pm\mathbf{k}\cdot\mathbf{v}+i\Omega_l\partial\phi)^{-1}$ again has the meaning of an angular operator such that

$$\frac{1}{\omega\pm\mathbf{k}\cdot\mathbf{v}+i\Omega_l\partial\phi}=\frac{1}{2\pi i\Omega_l}\int_0^{2\pi}d\phi\int_{\pm\infty}^{\phi}d\phi'\sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty}J_m\left(\frac{k_\perp v_\perp}{\Omega_l}\right)J_n\left(\frac{k_\perp v_\perp}{\Omega_l}\right)e^{-(i/\Omega_l)[(\omega\pm k_z v_z)(\phi'-\phi)\pm\Omega_l(n\phi'-m\phi)]},\tag{16}$$

where J is Bessel function of first kind and m and n are integers. Moreover, $\epsilon(\omega,\mathbf{k})$, the dielectric constant, in the present case can be written as

$$\epsilon(\omega_1\pm\mathbf{k})=1-\sum_s\frac{\omega_s^2}{k^2}\sum_{n=-\infty}^{+\infty}\int d^3v\frac{J_n^2(k_\perp v_\perp/\Omega_s)}{(k_zv_z+n\Omega_s\pm\omega)}\left(k_z\frac{\partial F_s}{\partial v_z}+\frac{n\Omega_s}{v_\perp}\frac{\partial F_s}{\partial v_\perp}\right).\tag{17}$$

As we shall see later, a quantity more interesting than $Q_{sr}(t)$ is its Laplace transform,

$$\tilde{Q}_{sr}(\mathbf{v}_1|\mathbf{v}_1',\mathbf{v}_2',\mathbf{k},\omega)=\int_0^\infty dte^{-i\omega t}Q_{sr}(\mathbf{v}_1|\mathbf{v}_1',\mathbf{v}_2',\mathbf{k},t).\tag{18}$$

Evidently,

$$\tilde{Q}_{sr}(\mathbf{v}_1|\mathbf{v}_1',\mathbf{v}_2',\mathbf{k},\omega)=\frac{1}{(2\pi i)^2}\int_{-\infty+i0-}^{+\infty+i0-}d\omega_1\int_{-\infty+i0-}^{+\infty+i0-}d\omega_2\frac{1}{i(\omega-\omega_1-\omega_2)(\omega_1+\mathbf{k}\cdot\mathbf{v}_1+i\Omega_s\partial\phi_1)}\Bigg[\int d^3v_1'\delta(\mathbf{v}_1-\mathbf{v}_1')$$
$$-\frac{kD_s(\mathbf{v}_1,\mathbf{k})}{\epsilon(\omega,\mathbf{k})}\int d^3v_1'\frac{\sum_s n_se_s}{\omega_1+\mathbf{k}\cdot\mathbf{v}_1+i\Omega_s\partial\phi_1}\Bigg]\int d^3v_2'\frac{\sum_r n_se_r}{\epsilon(\omega_2,-\mathbf{k})(\omega_2-\mathbf{k}\cdot\mathbf{v}_2'+i\Omega_r\partial\phi_2)}.\tag{19}$$

In (18), we require that

$$\mathrm{Im}(\omega-\omega_1-\omega_2)<0\tag{20}$$

and thus that the pole, $\omega_2=\omega-\omega_1$, be located below the path of integration in the complex ω_2 plane. Furthermore, since the function

$$[\epsilon(\omega_2,-\mathbf{k})(\omega_2-\mathbf{k}\cdot\mathbf{v}_2+i\Omega_r\partial\phi_2)]^{-1}$$

is analytic in the domain $0>\mathrm{Im}\omega_2>-\infty$, we may close the contour of the ω_2 integration in the lower half-plane. Thus,

$$\tilde{Q}_{sr}=\frac{1}{2\pi i}\int_{-\infty+i0-}^{+\infty+i0-}d\omega_1\frac{1}{i(\omega_1+\mathbf{k}\cdot\mathbf{v}_1+i\Omega_s\partial\phi_1)}\Bigg[\int d^3v_1'\delta(\mathbf{v}_1-\mathbf{v}_1')-\frac{kD_s(\mathbf{v}_1,\mathbf{k})}{\epsilon(\omega_1,\mathbf{k})}\int d^3v_1'\frac{\sum_s n_se_s}{\omega_1+\mathbf{k}\cdot\mathbf{v}_1'+i\Omega_s\partial\phi_1}\Bigg]$$
$$\times\int d^3v_2'\frac{\sum_r n_re_r}{\epsilon(\omega-\omega_1,-\mathbf{k})(\omega-\omega_1-\mathbf{k}\cdot\mathbf{v}_2'+i\Omega_r\partial\phi_2)}\tag{21}$$

with $\mathrm{Im}(\omega-\omega_1)<0$ and $\mathrm{Im}\omega_1<0$.

III. THE PAIR-CORRELATION FUNCTION

From Eqs. (5), (6), and (7), we see that in discussing the conductivity, it is desirable to determine G_{sr}. In this Section, we shall focus our attention on this subject, especially its asymptotic behavior as $t\to\infty$. First of all, we see from Eq. (4) that once $\sum_s n_s e_s \int d^3v_1 G_{sr}$ and $\sum_r n_r e_r \int d^3v_2 G_{sr}$ are determined, the solution of G_{sr} can be written down immediately. Thus, let us first study the quantities

$$\sum_s n_s e_s \int d^3v_1 G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t\to\infty)=\lim_{i\omega\to 0+}\tilde{Q}_{sr}(\mathbf{v}_2|\mathbf{v}_1',\mathbf{v}_2',\mathbf{k},\omega)\left\{\frac{4\pi e_s e_r i\mathbf{k}}{k^2}\cdot\left[\frac{F_r(\mathbf{v}_2')}{m_s}\frac{\partial F_s}{\partial \mathbf{v}_1'}-\frac{F_s(\mathbf{v}_1')}{m_r}\frac{\partial F_r}{\partial \mathbf{v}_2'}\right]\right\}, \quad (22)$$

$$\sum_r n_r e_r \int d^3v_2 G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t\to\infty)=\lim_{i\omega\to 0+}\tilde{Q}_{sr}(\mathbf{v}_1|\mathbf{v}_1',\mathbf{v}_2',\mathbf{k},\omega)\left\{\frac{4\pi e_s e_r i\mathbf{k}}{k^2}\cdot\left[\frac{F_r(\mathbf{v}_2')}{m_s}\frac{\partial F_s}{\partial \mathbf{v}_1'}-\frac{F_s(\mathbf{v}_1')}{m_r}\frac{\partial F_r}{\partial \mathbf{v}_2'}\right]\right\}. \quad (23)$$

To facilitate the discussion, we introduce a number of shorthand notations:

$$F_{sn}(\mathbf{v})=J_n^2(k_\perp v_\perp/\Omega_s)F_s(\mathbf{v}), \quad (24)$$

$$\left(\mathbf{k}\cdot\frac{\partial F_s}{\partial \mathbf{v}}\right)_n=J_n^2\left(\frac{k_\perp v_\perp}{\Omega_s}\right)\left(k_z\frac{\partial F_s}{\partial v_z}+\frac{n\Omega_s}{v_\perp}\frac{\partial F_s}{\partial v_\perp}\right), \quad (25)$$

$$(\mathbf{k}\cdot\mathbf{v})_{sn}=k_z v_z+n\Omega_s, \quad (26)$$

and

$$D_{sn}=-\frac{4\pi e_s}{m_s k^3}\left(\mathbf{k}\cdot\frac{\partial F_s}{\partial \mathbf{v}}\right)_n. \quad (27)$$

In addition, two useful relations should be mentioned

$$\frac{F_s(\mathbf{v})}{\omega\pm\mathbf{k}\cdot\mathbf{v}+i\Omega_s\partial\phi}=\sum_{n=-\infty}^{+\infty}\frac{F_{sn}}{\omega\pm(\mathbf{k}\cdot\mathbf{v})_n}, \quad (28)$$

$$\frac{1}{\omega\pm\mathbf{k}\cdot\mathbf{v}+i\Omega_s\partial\phi}\left(\mathbf{k}\cdot\frac{\partial F_s}{\partial \mathbf{v}}\right)=\pm\sum_{n=-\infty}^{+\infty}\frac{1}{\omega\pm(\mathbf{k}\cdot\mathbf{v})_n}\left(\mathbf{k}\cdot\frac{\partial F_s}{\partial \mathbf{v}}\right)_n, \quad (29)$$

where $F_s(\mathbf{v})=F_s(v_z,v_\perp)$. Making use of (17), (21), (23), (28), and (29), we obtain readily

$$\sum_r n_r e_r \int d^3v_2 G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t\to\infty)=\sum_{n=-\infty}^{+\infty}\left[\frac{e_s F_{sn}(\mathbf{v}_1)}{\epsilon^-[-(\mathbf{k}\cdot\mathbf{v}_1)_n,\mathbf{k}]}-e_s F_{sn}(\mathbf{v}_1)\right]$$
$$+k\sum_{n=-\infty}^{+\infty}D_{sn}(\mathbf{v}_1,\mathbf{k})\int_{-\infty}^{+\infty}d\omega\frac{1}{[\omega-(\mathbf{k}\cdot\mathbf{v}_1)_n+i\lambda]|\epsilon^-(-\omega,\mathbf{k})|^2}\sum_r n_r e_r^2\int d^3v_1\sum_{q=-\infty}^{+\infty}\delta[(\mathbf{k}\cdot\mathbf{v}_1')_q-\omega]F_{sq}(\mathbf{v}_1'), \quad (30)$$

where

$$\epsilon^\pm(\pm\omega,\mathbf{k})=1-\sum_s\frac{\omega_s^2}{k^2}\sum_{n=-\infty}^{+\infty}\int d^3v\frac{1}{(\mathbf{k}\cdot\mathbf{v})_n\pm(\omega\pm i\lambda)}\left(\mathbf{k}\cdot\frac{\partial F_s}{\partial \mathbf{v}}\right)_n.$$

At this point, we may comment that the kinetic equation for F_s can be derived immediately by inserting the above result into Eq. (3). The result is in agreement with that first derived by Rostoker[3] and later by Haggerty and deSobrino.[4]

Again, since from Eq. (4) we see that

$$G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t\to\infty)=G_{sr}^*(\mathbf{k},\mathbf{v}_2,\mathbf{v}_1,t\to\infty),$$

(where superscript * denotes complex conjugate), we can obtain $\sum_s n_s e_s \int d^3v_1 G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t\to\infty)$ simply by writing

[3] N. Rostoker, Phys. Fluids, **3**, 922 (1960).
[4] M. J. Haggerty and L. G. deSobrino, Can. J. Phys. **42**, 1969 (1964).

down the complex conjugate of (30) and replacing subscript s by r and $\mathbf{v}_1$ by $\mathbf{v}_2$. Thus,

$$\sum_s n_s e_s \int d^3v_1 G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2, t\to\infty) = \sum_{n=-\infty}^{+\infty}\left[\frac{e_r F_r(\mathbf{v}_2)}{\epsilon^+[-(\mathbf{v}_2\cdot\mathbf{k})_n, \mathbf{k}]} - e_r F_r(\mathbf{v}_2)\right] + k\sum_{n=-\infty}^{+\infty} D_{rn}(\mathbf{v}_2,\mathbf{k})$$
$$\times\int_{-\infty}^{+\infty} d\omega \frac{1}{[\omega-(\mathbf{k}\cdot\mathbf{v}_2)_n - i\lambda]\,|\epsilon^-(\omega,\mathbf{k})|^2}\sum_r n_r e_r^2 \int d^3v_2 \sum_{q=-\infty}^{+\infty}\delta[(\mathbf{k}\cdot\mathbf{v}_2)_q-\omega]F_{rq}(\mathbf{v}_2). \quad (31)$$

With (30) and (31), the pair-correlation function can be determined easily, since, from Eq. (4),

$$G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2, t\to\infty) = \lim_{i\omega\to 0+}\frac{1}{(2\pi)^2}\int_{-\infty+i0_-}^{+\infty+i0-} d\omega_1 \int_{-\infty+i0_-}^{+\infty+i0-} d\omega_2 \frac{1}{i(\omega-\omega_1-\omega_2)(i\omega_1+i\mathbf{k}\cdot\mathbf{v}_1-\Omega_s\partial\phi_1)}\,\frac{1}{(i\omega_2-i\mathbf{k}\cdot\mathbf{v}_2-\Omega_r\partial\phi_2)}$$
$$\times\left\{\frac{4\pi e_s e_r i\mathbf{k}}{k^2}\cdot\left[\frac{F_r}{m_s}\frac{\partial F_s}{\partial \mathbf{v}_1} - \frac{F_s}{m_r}\frac{\partial F_r}{\partial \mathbf{v}_2}\right] + \frac{4\pi e_s i\mathbf{k}}{m_s k^2}\cdot\frac{\partial F_s}{\partial \mathbf{v}_1}\sum_s n_s e_s \int d^3v_1 G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2, t\to\infty)\right.$$
$$\left. - \frac{4\pi e_r i\mathbf{k}}{m_r k^2}\cdot\frac{\partial F_r}{\partial \mathbf{v}_2}\sum_r n_r e_r \int d^3v_2 G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2, t\to\infty)\right\}, \quad (32)$$

where we remember $\mathrm{Im}(\omega-\omega_1-\omega_2)<0$. After inserting (30) and (31) into (32), and performing some straightforward manipulation, we came to the following result:

$$G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2, t\to\infty) = \sum_{l=-\infty}^{+\infty}\sum_{j=-\infty}^{+\infty}\frac{1}{[(\mathbf{k}\cdot\mathbf{v}_2)_j-(\mathbf{k}\cdot\mathbf{v}_1)_l+i\lambda]}\left\{kD_{sl}(\mathbf{v}_1,\mathbf{k})J_j^2\left(\frac{k_\perp v_{2\perp}}{\Omega_r}\right)\left[\sum_{n=-\infty}^{+\infty}\frac{e_r F_{rn}(\mathbf{v}_2)}{\epsilon^+[-(\mathbf{k}\cdot\mathbf{v}_2)_n, \mathbf{k}]}\right.\right.$$
$$\left. + \sum_{n=-\infty}^{+\infty} kD_{rn}(\mathbf{v}_2,\mathbf{k})\int d\tilde{\omega}\frac{F(\tilde{\omega})}{[\tilde{\omega}-(\mathbf{k}\cdot\mathbf{v}_2)_n-i\lambda]\,|\epsilon^-(-\tilde{\omega},\mathbf{k})|^2}\right] - kD_{rj}(\mathbf{v}_2,\mathbf{k})J_l^2\left(\frac{k_\perp v_{1\perp}}{\Omega_s}\right)$$
$$\left.\times\left[\sum_{n=-\infty}^{+\infty}\frac{e_s F_{sn}(\mathbf{v}_1)}{\epsilon^-[-(\mathbf{k}\cdot\mathbf{v}_1)_n, \mathbf{k}]} + \sum_{n=-\infty}^{+\infty} kD_{sn}(\mathbf{v}_1,\mathbf{k})\int d\tilde{\omega}\frac{F(\tilde{\omega})}{[\tilde{\omega}-(\mathbf{k}\cdot\mathbf{v}_1)_n+i\lambda]\,|\epsilon^-(-\tilde{\omega},\mathbf{k})|^2}\right]\right\}, \quad (33)$$

where

$$F(\tilde{\omega}) = \int d^3v_2 \sum_r n_r e_r^2 \sum_{q=-\infty}^{+\infty}\delta[(\mathbf{k}\cdot\mathbf{v}_2)_q-\tilde{\omega}]F_{rq}(\mathbf{v}_2).$$

At this point, we should remark that this expression of G_{sr} has already been averaged over the azimuthal angles in velocity space. Such a result, however, is all that we will need in the conductivity calculation as we shall see later. Although the general form of G_{sr} looks complicated, it yields much simpler expressions in two cases; namely (A) F_s and F_r are Maxwellian distributions with equal temperature, and (B) ion motion can be neglected $[m_i\to\infty]$. Let us discuss the two cases separately.

(A) Maxwellian Distributions

If F_s and F_r are Maxwellian distributions with equal temperature T, i.e.,

$$F_s(v) = (m_s/2\pi\kappa T)^{3/2}\exp(-m_s v^2/2\kappa T), \quad F_r(v) = (m_r/2\pi\kappa T)^{3/2}\exp(-m_r v^2/2\kappa T),$$

we can evaluate the $\tilde{\omega}$ integrals in (33) exactly, since

$$\sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\tilde{\omega}\frac{F(\tilde{\omega})}{[\tilde{\omega}-(\mathbf{k}\cdot\mathbf{v})_n\pm i\lambda]\,|\epsilon^-(-\tilde{\omega},\mathbf{k})|^2} = \sum_{n=-\infty}^{+\infty}\frac{1}{2\pi i}\int_{-\infty}^{+\infty} d\tilde{\omega}\frac{1}{[\tilde{\omega}-(\mathbf{k}\cdot\mathbf{v})_n\pm i\lambda]\tilde{\omega}}\left[\frac{1}{\epsilon^-(-\tilde{\omega},k)} - \frac{1}{\epsilon^+(-\tilde{\omega},\mathbf{k})}\right]\frac{k^2}{k_D^2}$$
$$= \sum_{n=-\infty}^{+\infty}\frac{1}{(\mathbf{k}\cdot\mathbf{v})_n\mp i\lambda}\left[\frac{1}{\epsilon(0)} - \frac{1}{\epsilon^\mp[-(\mathbf{k}\cdot\mathbf{v})_n, \mathbf{k}]}\right]\frac{k^2(\kappa T)}{4\pi}, \quad (34)$$

where $k_D^2=\sum_s(4\pi n_s e_s^2/\kappa T)$ and $\epsilon(0)=1+k_D^2/k^2$. Making use of the relations

$$kD_{sn}(\mathbf{v}\cdot\mathbf{k})=(4\pi e_s/k^2(\kappa T))(\mathbf{k}\cdot\mathbf{v})_n F_{sn}(\mathbf{v}) \quad\text{and}\quad \sum_{n=-\infty}^{+\infty} J_n^2=1,$$

we obtain immediately that

$$G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)=-(4\pi e_s e_r/\kappa T(k^2+k_D^2))F_s(v)F_r(v). \tag{35}$$

This result is more or less well known. One can derive the same result directly from Eq. (4) simply by assuming that

$$G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)=e_s e_r \Psi(k) F_s(v_1)F_r(v_2)$$

and then determining $\Psi(k)$. In this case, we see that the magnetic field produces no effect on G_{sr} since G_{sr} is isotropic in both $\mathbf{v}_1$ and $\mathbf{v}_2$ spaces.

(B) Limit of Infinite Ion Mass

Let us first suppose that the ion species possesses its own Maxwellian distribution, but is not necessarily equilibrium with the electron species, and then let us consider the limit of infinite ion mass. In this case, we can show that

$$G_{ie}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t\to\infty)=-\sum_{j=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty}\frac{kD_{ej}(\mathbf{v}_2)J_l^2(k_1v_{1\perp}/\Omega_i)}{[(\mathbf{k}\cdot\mathbf{v}_2)_j-(\mathbf{k}\cdot\mathbf{v}_1)_l+i\lambda]}\sum_{n=-\infty}^{+\infty}\frac{e_iF_{in}(\mathbf{v}_1)}{\epsilon^-[-(\mathbf{k}\cdot\mathbf{v}_1)_n,\mathbf{k}]}\simeq-\sum_{j=-\infty}^{+\infty}\frac{kD_{ej}(\mathbf{v}_2)e_iF_i(\mathbf{v}_1)}{[(\mathbf{k}\cdot\mathbf{v}_2)_j+i\lambda]\epsilon^-(0)} \tag{36}$$

and

$$G_{ei}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t\to\infty)=-\sum_{j=-\infty}^{+\infty}\frac{kD_{ej}(\mathbf{v}_1)e_iF_i(\mathbf{v}_2)}{[(\mathbf{k}\cdot\mathbf{v}_1)_j-i\lambda]\epsilon^+(0)}, \tag{37}$$

where $F_i(v)$ is a delta function, i.e.,

$$\lim_{m_i\to\infty}F_i(v)=\lim_{m_i\to\infty}(m_i/2\pi\kappa T_i)^{3/2}\exp(-m_iv^2/2\kappa T_i)=\delta(v_x)\delta(v_y)\delta(v_z).$$

IV. THE HIGH-FREQUENCY CONDUCTIVITY

Going back to Eqs. (5), (6), and (7), we obtain immediately that

$$\frac{\partial\mathbf{J}_s}{\partial t}-\Omega_s\mathbf{J}_s\times\hat{b}=\frac{\omega_s^2}{4\pi}\mathbf{E}_0e^{i\omega t}+\frac{i}{(2\pi)^3}\int d^3k\frac{\omega_s^2\mathbf{k}}{k^2}\int d^3v_1\sum_r n_re_r\int d^3v_2g_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t\to\infty), \tag{38}$$

where $\mathbf{J}_s=n_se_s\int d^3v\mathbf{v}f_s(\mathbf{v})$ is the current density of s species, and $\hat{b}$ is a unit vector parallel to $\mathbf{B}_0$. Furthermore, according to the discussion presented in Sec. II,

$$\sum_r n_re_r\int d^3v_2g_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t\to\infty)=\int_0^\infty d\tau Q_{sr}(\mathbf{v}_1|\mathbf{v}_1',\mathbf{v}_2',\mathbf{k},\tau)R_{sr}(\mathbf{k},\mathbf{v}_1',\mathbf{v}_2',t-\tau). \tag{39}$$

Here the operator Q_{sr} and the source term R_{sr} are defined by (15) and (7), respectively. Since the dominant contribution to the high-frequency conductivity is the reactive part, we may replace the perturbed distribution functions f_s and f_r in R_{sr} by the solution of Eq. (5) but without including the effect of correlation.[5] In this approximation we have found (see Appendix B) that

$$f_s(\mathbf{v},t)=-(e_s/im_s\omega)E_0e^{i\omega t}\hat{k}_{s0}\cdot(\partial F_s/\partial\mathbf{v}), \tag{40}$$

where $\hat{k}_{s0}$ is a vector such that

$$\hat{k}_{s0}=(\omega^2-\Omega_s^2)^{-1}[\omega^2\hat{e}-i\omega\Omega_s(\hat{e}\times\hat{b})-\Omega_s^2\hat{b}(\hat{e}\cdot\hat{b})] \tag{41}$$

and $\hat{e}=\mathbf{E}_0/E_0$.

Substituting (40) into (7), we find after some rearrangement that

$$R_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t)=-E_0e^{i\omega t}R_{sr}^0(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2) \tag{42}$$

and

$$R_{sr}^0=\frac{e_s\hat{k}_{s0}}{m_s\omega}\cdot\frac{\partial}{\partial\mathbf{v}_1}\left[\omega G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)+\frac{e_s}{m_s}\frac{\partial F_s}{\partial\mathbf{v}_1}4\pi\frac{\mathbf{k}}{k^2}\sum_q n_qe_q\int d^3v_3G_{rq}(-\mathbf{k},\mathbf{v}_2,\mathbf{v}_3)+\frac{4\pi e_se_r}{k^2}\mathbf{k}\cdot\left(\frac{1}{m_s}\frac{\partial}{\partial\mathbf{v}_1}-\frac{1}{m_r}\frac{\partial}{\partial\mathbf{v}_2}\right)F_s(\mathbf{v}_1)F_r(\mathbf{v}_2)\right]$$

$$+\frac{e_r\hat{k}_{r0}}{m_r\omega}\cdot\frac{\partial}{\partial\mathbf{v}_2}\left[\omega G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)-\frac{e_r}{m_r}\frac{\partial F_r}{\partial\mathbf{v}_2}4\pi\frac{\mathbf{k}}{k^2}\sum_q n_qe_q\int d^3v_3G_{sq}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_3)+\frac{4\pi e_se_r}{k^2}\mathbf{k}\cdot\left(\frac{1}{m_s}\frac{\partial}{\partial\mathbf{v}_1}-\frac{1}{m_r}\frac{\partial}{\partial\mathbf{v}_2}\right)F_s(\mathbf{v}_1)F_r(\mathbf{v}_2)\right]. \tag{43}$$

[5] This approximation is valid only when ω is not in the vicinity of the cyclotron frequency and also much higher than the collision frequency. These conditions have been assumed to be true in the present discussion.

Notice that $G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t\to\infty)$ satisfies the following equation:

$$(-i\delta+\mathbf{k}\cdot\mathbf{v}_1-\mathbf{k}\cdot\mathbf{v}_2)G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)+\frac{e_s}{im_s}(\mathbf{v}_1\times\mathbf{B}_0)\cdot\frac{\partial G_{sr}}{\partial\mathbf{v}_1}+\frac{e_r}{im_r}(\mathbf{v}_2\times\mathbf{B}_0)\cdot\frac{\partial G_{sr}}{\partial\mathbf{v}_2}$$
$$-\frac{\mathbf{k}}{k^2}\cdot\frac{e_s}{m_s}\frac{\partial F_s}{\partial\mathbf{v}_1}4\pi\sum_q n_qe_q\int d^3v_3G_{rq}(-\mathbf{k},\mathbf{v}_2,\mathbf{v}_3)+\frac{\mathbf{k}}{k^2}\cdot\frac{e_r}{m_r}\frac{\partial F_r}{\partial\mathbf{v}_2}4\pi\sum_q n_qe_q\int d^3v_3G_{sq}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_3)$$
$$=\frac{4\pi e_se_r}{k^2}\mathbf{k}\cdot\left(\frac{1}{m_s}\frac{\partial}{\partial\mathbf{v}_1}-\frac{1}{m_r}\frac{\partial}{\partial\mathbf{v}_2}\right)F_s(\mathbf{v}_1)F_r(\mathbf{v}_2).\quad(44)$$

Consequently, we can write the quantity $\int d^3v_1\sum_r n_re_r\int d^3v_2g_{sr}$ in (38) in the following manner:

$$\int d^3v_1\sum_r n_re_r\int d^3v_2g_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t\to\infty)=-E_0e^{i\omega t}\int d^3v_1\tilde{Q}_{sr}(\omega)\left\{\frac{e_s}{m_s}\frac{\hat{k}_{s0}\cdot\mathbf{k}}{\omega}G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)-\frac{e_r}{m_r}\frac{\hat{k}_{r0}\cdot\mathbf{k}}{\omega}G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)\right.$$
$$\left.+\frac{e_s}{m_s\omega}\Omega_s(\hat{k}_{s0}\times\hat{b})\cdot\frac{\partial G_{sr}}{\partial\mathbf{v}_1}+\frac{e_r}{m_r\omega}\Omega_r(\hat{k}_{r0}\times\hat{b})\cdot\frac{\partial G_{sr}}{\partial\mathbf{v}_2}\right\}.\quad(45)$$

To obtain this result we have made use of some operational relations which provide considerable simplification, similar to those discussed in I. Inserting (45) into Eq. (38), we see that the last two terms in (45) do not actually contribute since they will vanish after the integration in $\mathbf{k}$ space. Thus,

$$\frac{\partial\mathbf{J}_s}{\partial t}-\Omega_s\mathbf{J}_s\times\hat{b}=\frac{\omega_s^2}{4\pi}\mathbf{E}_0e^{i\omega t}-\frac{iE_0e^{i\omega t}}{(2\pi)^3\omega}\int d^3k\frac{\omega_s^2\mathbf{k}}{k^2}$$
$$\times\int d^3v_1\tilde{Q}_{sr}(\mathbf{v}_1|\mathbf{v}_1',\mathbf{v}_2',\mathbf{k},\omega)\left\{\frac{e_s}{m_s}(\hat{k}_{s0}\cdot\mathbf{k})-\frac{e_r}{m_r}(\hat{k}_{r0}\cdot\mathbf{k})\right\}G_{sr}(\mathbf{k},\mathbf{v}_1',\mathbf{v}_2',t\to\infty).\quad(46)$$

Assuming a solution of the form

$$\mathbf{J}_s=\mathbf{J}_{s0}(\omega)e^{i\omega t},$$

we see that Eq. (46) becomes a vector equation for $\mathbf{J}_s$ and can be solved readily. The solution takes the following form:

$$\mathbf{J}_s=\frac{\omega_s^2\hat{k}_{s0}E_0e^{i\omega t}}{4\pi(\omega^2-\Omega_s^2)i\omega}-\frac{E_0e^{i\omega t}\omega_s^2}{(2\pi)^3\omega^2(\omega^2-\Omega_s^2)}\int d^3k\frac{1}{k^2}[\omega^2\mathbf{k}-i\omega\Omega_s(\mathbf{k}\times\hat{b})-\Omega_s^2\hat{b}(\mathbf{k}\cdot\hat{b})]$$
$$\times\int d^3v_1\tilde{Q}_{sr}(\mathbf{v}_1|\mathbf{v}_1',\mathbf{v}_2',\mathbf{k},\omega)\left\{\frac{e_s}{m_s}(\hat{k}_{s0}\cdot\mathbf{k})-\frac{e_r}{m_r}(\hat{k}_{r0}\cdot\mathbf{k})\right\}G_{sr}(\mathbf{k},\mathbf{v}_1',\mathbf{v}_2',t\to\infty).\quad(47)$$

If we now introduce a cylindrical coordinate system in $\mathbf{k}$ space in which k_z is parallel to the magnetic field and if we decompose the field $\mathbf{E}_0$ into components parallel and perpendicular to the magnetic field, we can rewrite Eq. (47) as follows:

$$J_{sz}=\frac{\omega_s^2E_{0z}e^{i\omega t}}{4\pi i\omega}-\frac{\omega_s^2E_{0z}e^{i\omega t}}{2(2\pi)^3\omega^2}\int d^3kk_z^2\frac{1}{k^2}\int d^3v_1\tilde{Q}_{sr}(\mathbf{v}_1|\mathbf{v}_1',\mathbf{v}_2',\mathbf{k},\omega)\left(\frac{e_s}{m_s}-\frac{e_r}{m_r}\right)G_{sr}(\mathbf{k},\mathbf{v}_1',\mathbf{v}_2'),\quad(48)$$

$$\mathbf{J}_{s\perp}=\frac{\omega_s^2e^{i\omega t}}{4\pi i\omega(\omega^2-\Omega_s^2)}[\omega_0^2\mathbf{E}_{0\perp}-i\omega\Omega_s(\mathbf{E}_{0\perp}\times\hat{b})]-\frac{\omega_s^2e^{i\omega t}}{8\pi^2\omega^2}\frac{1}{(2\pi)}\int d^3k\frac{k_\perp^2}{k^2}\left\{\frac{\omega^2\mathbf{E}_{0\perp}-i\omega\Omega_s(\mathbf{E}_{0\perp}\times\hat{b})}{\omega^2-\Omega_s^2}\int d^3v_1\tilde{Q}_{sr}\right.$$
$$\left.\times\left[\frac{e_s\omega^2}{m_s(\omega^2-\Omega_s^2)}-\frac{e_r\omega^2}{m_r(\omega^2-\Omega_r^2)}\right]G_{sr}-\frac{\omega^2(\mathbf{E}_{0\perp}\times\hat{b})+i\omega\Omega_s\mathbf{E}_{0\perp}}{\omega^2-\Omega_s^2}\int d^3v_1\tilde{Q}_{sr}\left[\frac{e_si\omega\Omega_s}{m_s(\omega^2-\Omega_s^2)}-\frac{e_ri\omega\Omega_r}{m_r(\omega^2-\Omega_r^2)}\right]G_{sr}\right\}.\quad(49)$$

If we now denote

$$J_{sz}=\sigma_{sz}E_{0z}e^{i\omega t}, \tag{50}$$

$$\mathbf{J}_{s\perp}=\bar{\bar{\sigma}}_{s\perp}\mathbf{E}_{0\perp}e^{i\omega t}, \tag{51}$$

and $\sigma=\sigma^0+\sigma^1$ where σ^0 is the reactive part and σ^1 the resistive part, we conclude that

$$\sigma_{sz}{}^0=\omega_s{}^2/4\pi i\omega, \tag{52}$$

$$\bar{\bar{\sigma}}_{s\perp}{}^0=\frac{\omega_s{}^2}{4\pi i\omega}\begin{pmatrix}\omega^2/(\omega^2-\Omega_s{}^2) & i\omega\Omega_s/(\omega^2-\Omega_s{}^2)\\ -i\omega\Omega_s/(\omega^2-\Omega_s{}^2) & \omega^2/(\omega^2-\Omega_s{}^2)\end{pmatrix}, \tag{53}$$

$$\sigma_{sz}{}^1=-\frac{\omega_s{}^2}{8\pi^2\omega^2}\frac{1}{(2\pi)}\int d^3k k_z{}^2\frac{1}{k^2}\int d^3v_1\tilde{Q}_{sr}(\omega)\left(\frac{e_s}{m_s}-\frac{e_r}{m_r}\right)G_{sr}, \tag{54}$$

$$\begin{aligned}\bar{\bar{\sigma}}_{s\perp}{}^1=-\frac{\omega_s{}^2}{8\pi^2\omega^2}\frac{1}{(2\pi)}\int d^3k\frac{k_\perp{}^2}{k^2}\Bigg\{&\begin{pmatrix}\omega^2/(\omega^2-\Omega_s{}^2) & i\omega\Omega_s/(\omega^2-\Omega_s{}^2)\\ -i\omega\Omega_s/(\omega^2-\Omega_s{}^2) & \omega^2/(\omega^2-\Omega_s{}^2)\end{pmatrix}\int d^3v_1\tilde{Q}_{sr}\left[\frac{e_s}{m_s}\frac{\omega^2}{(\omega^2-\Omega_s{}^2)}-\frac{e_r}{m_r}\frac{\omega^2}{(\omega^2-\Omega_r{}^2)}\right]G_{sr}\\ &-\begin{pmatrix}i\omega\Omega_s/(\omega^2-\Omega_s{}^2) & -\omega^2/(\omega^2-\Omega_s{}^2)\\ \omega^2/(\omega^2-\Omega_s{}^2) & i\omega\Omega_s/(\omega^2-\Omega_s{}^2)\end{pmatrix}\int d^3v_1\tilde{Q}_{sr}\left[\frac{e_s}{m_s}\frac{i\omega\Omega_s}{(\omega^2-\Omega_s{}^2)}-\frac{e_r}{m_r}\frac{i\omega\Omega_r}{(\omega^2-\Omega_r{}^2)}\right]G_{sr}\Bigg\}.\end{aligned} \tag{55}$$

It is obvious from (54) and (55), that we only need the information of $\tilde{Q}_{sr}$ and G_{sr} averaged over the azimuthal angles in velocity space.

V. LIMIT OF INFINITE ION MASS

It is conceivable that if the frequency is very high, say $\omega\geq\omega_e$, and if the electron temperature is equal to or higher than the ion temperature, the ions will not respond to the fast oscillations and thus can be treated as immovable. In other words, in such a case, dominant results can be obtained by considering the limit as $m_i^{-1}\to 0$. In this limit, considerable simplification can be gained. We can show (Appendix C) that

$$\int d^3v_1\tilde{Q}_{sr}\left(\frac{e_s}{m_s}\theta_s-\frac{e_r}{m_r}\theta_r\right)G_{sr}=-\frac{4\pi i n_i e_i{}^2e_e{}^2\theta_e}{m_e{}^2\epsilon^-(-w,\mathbf{k})k^2}\int_{-\infty}^{+\infty}du\frac{\tilde{\Phi}_e(u)}{u-w+i\lambda}, \tag{56}$$

where θ_s is a constant (which depends upon e_s/m_s), $w=\omega/k$, and

$$\tilde{\Phi}_e(u)=\sum_{n=-\infty}^{+\infty}\int d^3v\delta[(\hat{k}\cdot\mathbf{v})_n-u]J_n{}^2\left(\frac{k_\perp v_\perp}{\Omega_e}\right)\sum_{m=-\infty}^{+\infty}([(\mathbf{k}\cdot\mathbf{v})_m+i\lambda]\epsilon^-(0))^{-1}(\mathbf{k}\cdot(\partial F_e/\partial\mathbf{v}))_m. \tag{57}$$

With the help of (57), we are ready to discuss the conductivity tensor. Clearly, since the ion motions are neglected in this limit, the effective current density is due to the electrons. Hence $\mathbf{J}=\mathbf{J}_e$, and according to Equations (53) to (56), we conclude that

$$\sigma^1=-\frac{i\omega_e{}^2}{4\pi\omega}\begin{bmatrix}\omega^2/(\omega^2-\Omega_e{}^2) & i\omega\Omega_e/(\omega^2-\Omega_e{}^2) & 0\\ -i\omega\Omega_e/(\omega^2-\Omega_e{}^2) & \omega^2/(\omega^2-\Omega_e{}^2) & 0\\ 0 & 0 & 1\end{bmatrix}, \tag{58}$$

$$\begin{aligned}\sigma^1=\frac{i\omega_e{}^2}{8\pi^2\omega^2}\frac{1}{(2\pi)}\int d^3k&\begin{bmatrix}[(\omega^4+\omega^2\Omega_e{}^2)/(\omega^2-\Omega_e{}^2)^2]k_\perp{}^2 & 2i\omega^3\Omega_e k_\perp{}^2/(\omega^2-\Omega_e{}^2)^2 & 0\\ -2i\omega^3\Omega_e k_\perp{}^2/(\omega^2-\Omega_e{}^2)^2 & [(\omega^4+\omega^2\Omega_e{}^2)/(\omega^2-\Omega_e{}^2)^2]k_\perp{}^2 & 0\\ 0 & 0 & k_z{}^2\end{bmatrix}\\ &\times\frac{4\pi n_i e_i{}^2e_e{}^2}{k^5m_e{}^2\epsilon^-(-w)}\int_{-\infty}^{+\infty}du\frac{\tilde{\Phi}_e(u)}{u-w+i\lambda}.\end{aligned} \tag{59}$$

For the special case of electrons possessing a Maxwellian distribution and in equilibrium with the ions, (59) reproduces the result previously obtained by Oberman and Shure.[6]

[6] C. Oberman and F. Shure, Phys. Fluids 6, 834 (1963).

VI. SUMMARY AND REMARKS

The foregoing discussion represents an extension of the previous paper (I) to include the effect of a uniform external magnetic field. A general expression for the high-frequency conductivity is obtained. The results is valid under the following conditions: (1) the frequency is assumed to be not in the vicinity of the cyclotron frequency, (2) the unperturbed distribution function may be anisotropic $F_s(\mathbf{v})=F_s(v_z,v_\perp)$ but must be stable, and (3) the time of relaxation of F_s toward complete thermodynamic equilibrium is long compared with the period of the oscillating field.

The limit of infinite ion mass is also discussed. In this limit the general expression is considerably simplified. For the case of an equilibrium plasma, the result obtained by Oberman and Shure[6] can be reproduced instantly. However, it is not easy to compare the present result with that obtained by Dupree[7] who recently discussed the absorption coefficient for radiation, since his approach and formalism are different, and since his result is expressed in terms of the quantity $\langle\delta\rho_i\delta N_e\rangle$ for which, besides the equilibrium case, no explicit expression for the general case was given. However, if the unperturbed plasma is in thermodynamic equilibrium, both of our results agree with that derived by Oberman and Shure.

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APPENDIX A

Identification of the Operators $\sum_s n_se_s\int d^3vP_s(\mathbf{v}|\mathbf{v}',\mathbf{k},t)$ and $P_s(\mathbf{v}|\mathbf{v}',\mathbf{k},t)$

Let us consider the fundamental equation

$$[\partial/\partial t+H_s(\mathbf{k},\mathbf{v},\phi)]\psi_s(\mathbf{v},\mathbf{k},t)=0\,, \tag{A1}$$

where

$$H_s(\mathbf{k},\mathbf{v},\phi)=i\mathbf{k}\cdot\mathbf{v}-\Omega_s\frac{\partial}{\partial\phi}-\frac{4\pi ie_s}{m_sk^2}\mathbf{k}\cdot\frac{\partial F_s}{\partial\mathbf{v}}\int d^3v\sum_s n_se_s\,. \tag{A2}$$

Evidently, Eq. (A1) is the usual linearized Vlasov equation with an external magnetic field $\mathbf{B}_0$. We can write formally

$$\psi_s(\mathbf{v},\mathbf{k},t)=P_s(\mathbf{v}|\,\mathbf{v}',\mathbf{k},t)\psi_s(\mathbf{v}',\mathbf{k},0)\,. \tag{A3}$$

Here $\psi_s(\mathbf{v},\mathbf{k},0)$ is an initial condition and $P_s(\mathbf{v}|\,\mathbf{v}',\mathbf{k},t)$ is an operator to be identified.

Now let us return to Eq. (A1) and introduce a Laplace transform

$$\tilde{\psi}_s(\mathbf{v},\mathbf{k},\omega)=\int_0^\infty dte^{-i\omega t}\psi_s(\mathbf{v},\mathbf{k},t) \tag{A4}$$

with $\mathrm{Im}\omega<0$. The equation then reduces to

$$\frac{\partial\tilde{\psi}_s}{\partial\phi}=\frac{1}{\Omega_s}i(\omega+\mathbf{k}\cdot\mathbf{v})\tilde{\psi}_s=\frac{1}{\Omega_s}\left[-\psi_s(0)-\frac{4\pi ie_s}{m_sk^2}\mathbf{k}\cdot\frac{\partial F_s}{\partial\mathbf{v}}\sum_s n_se_s\int d^3v\tilde{\psi}_s\right]. \tag{A5}$$

Thus, we obtain

$$\tilde{\psi}_s=-\frac{1}{\Omega_s}\int_{\pm\infty}^{\phi}d\phi' I_s(\omega,\phi'-\phi,\sin\phi'-\sin\phi)\left\{\psi_s(0)-\frac{4\pi ie_s}{m_sk^2}\mathbf{k}\cdot\frac{\partial F_s}{\partial\mathbf{v}}\sum_s n_se_s\int d^3v\tilde{\psi}_s\right\}, \tag{A6}$$

where the correct sign for the lower integration limit depends upon the sign of Ω_s (it should be so chosen that the integral converges). Furthermore in (A6), I_s is an integration factor, i.e.,

$$I_s(\omega,\phi,\sin\phi)=\exp\{-(i/\Omega_s)[(\omega+k_zv_z)\phi+k_\perp v_\perp\sin\phi]\}\quad\text{with}\quad\mathrm{Im}\omega<0\,. \tag{A7}$$

In the following we assume $\psi_s(0)=\psi_s(v_z,v_\perp,\mathbf{k},0)$ and shall be interested only in the result of $\tilde{\psi}$ averaged over ϕ.[8] Thus

$$\tilde{\psi}_s(\mathbf{v},\mathbf{k},\omega)=-\frac{1}{2\pi\Omega_s}\int_0^{2\pi}d\phi\int_{\pm\infty}^{\phi}d\phi' I_s(\omega,\,\phi'-\phi,\,\sin\phi'-\sin\phi)\left\{\psi_s(0)-\frac{4\pi ie_s}{m_sk^2}\mathbf{k}\cdot\frac{\partial F_s}{\partial\mathbf{v}}\sum_s n_se_s\int d^3v\tilde{\psi}_s\right\} \tag{A8}$$

[7] T. Dupree, Phys. Fluids, 7, 923 (1964).

[8] This step of averaging over the azimuthal angle facilitates the discussion in the present paper. However, one ought to be reminded that such an operation may not be useful in the discussion of other problems.

or

$$\sum_s n_s e_s \int d^3v \tilde{\psi}_s(\mathbf{v},\mathbf{k},\omega) = -\frac{1}{\epsilon(\omega,\mathbf{k})} \frac{\sum_s n_s e_s}{2\pi\Omega_s} \int d^3v \int_0^{2\pi} d\phi \int_{\pm\infty}^{\phi} d\phi' I_s(\omega, \phi'-\phi, \sin\phi'-\sin\phi)\psi_s(0), \tag{A9}$$

where

$$\epsilon(\omega,\mathbf{k}) = 1 - \sum_s \frac{\omega_s^2}{k^2} \frac{i}{2\pi\Omega_s} \int_0^{2\pi} d\phi \int_{\pm\infty}^{\phi} d\phi' I_s(\omega, \phi'-\phi, \sin\phi'-\sin\phi)\left[k_z \frac{\partial F_s}{\partial v_z} + k_\perp \cos\phi' \frac{\partial F_s}{\partial v_\perp}\right]. \tag{A10}$$

Making use of the Bessel identities

$$e^{\pm(i/\Omega_s)k_\perp v_\perp \sin\phi} = \sum_{n=-\infty}^{+\infty} J_n(k_\perp v_\perp/\Omega_s) e^{\pm in\phi}, \tag{A11}$$

$$\cos\phi\, e^{\pm(i/\Omega_s)k_\perp v_\perp \sin\phi} = \sum_{n=-\infty}^{+\infty} (n\Omega_s/k_\perp v_\perp) J_n(k_\perp v_\perp/\Omega_s) e^{\pm in\phi}, \tag{A12}$$

we obtain

$$\epsilon(\omega,\mathbf{k}) = 1 - \sum_s \frac{\omega_s^2}{k^2} \sum_{n=-\infty}^{+\infty} \int d^3v \frac{J_n^2(k_\perp v_\perp/\Omega_s)}{(\omega + n\Omega_s + k_z v_z)} \left(k_z \frac{\partial F_s}{\partial v_z} + \frac{n\Omega_s}{v_\perp} \frac{\partial F_s}{\partial v_\perp}\right). \tag{A13}$$

For simplicity, a short-hand operator notation is preferable, i.e.,

$$(i(\omega + \mathbf{k}\cdot\mathbf{v}) - \Omega_s \partial\phi)^{-1} = -\frac{1}{2\pi\Omega_s} \int_0^{2\pi} d\phi \int_{\pm\infty}^{\phi} d\phi' \sum_n \sum_m J_n\left(\frac{k_\perp v_\perp}{\Omega_s}\right) J_m\left(\frac{k_\perp v_\perp}{\Omega_s}\right) e^{-(i/\Omega_s)[(\omega + k_z v_z)(\phi'-\phi) + \Omega_s(n\phi' - m\phi)]}. \tag{A14}$$

In terms of this notation, we see immediately that

$$P_s(\mathbf{v}|\mathbf{v}',\mathbf{k},t) = \frac{1}{2\pi} \int_{-\infty+i0_-}^{+\infty+i0_-} d\omega e^{i\omega t} \frac{1}{i(\omega + \mathbf{k}\cdot\mathbf{v}) - \Omega_s\partial\phi} \left[\int d^3v_1' \delta(\mathbf{v}_1' - \mathbf{v}_1) - \frac{ikD_s(\mathbf{v}_1,\mathbf{k})}{\epsilon(\omega,\mathbf{k})} \int d^3v_1' \frac{\sum_s n_s e_s}{i(\omega + \mathbf{k}\cdot\mathbf{v}_1') - \Omega_s\partial\phi}\right], \tag{A15}$$

and

$$\sum_s n_s e_s \int d^3v P_s(\mathbf{v}|\mathbf{v}',\mathbf{k},t) = \frac{1}{2\pi} \int_{-\infty+i0_-}^{+\infty+i0_-} d\omega e^{i\omega t} \frac{1}{\epsilon(\omega,\mathbf{k})} \int d^3v_1' \frac{\sum_s n_s e_s}{i(\omega + \mathbf{k}\cdot\mathbf{v}) - \Omega_s\partial\phi}. \tag{A16}$$

With (A15) and (A16) the operator $Q_{sr}(\mathbf{v}_1|\mathbf{v}_1',\mathbf{v}_2',\mathbf{k},t)$, defined as

$$Q_{sr}(\mathbf{v}_1|\mathbf{v}_1',\mathbf{v}_2',\mathbf{k},t) = P_s(\mathbf{v}_1|\mathbf{v}_1',\mathbf{k},t) \sum_r n_r e_r \int d^3v_2 P_r(\mathbf{v}_2|\mathbf{v}_2',\mathbf{k},t),$$

is thus in principle determined.

APPENDIX B

Reactive Approximation of $f_s(\mathbf{v},t)$

In this Appendix, we study the solution to the following equation:

$$\frac{\partial f_s}{\partial t} + \frac{e_s}{m_s c}(\mathbf{v}\times\mathbf{B}_0)\cdot\frac{\partial f_s}{\partial \mathbf{v}} = -\frac{e_s}{m_s}\mathbf{E}_0 e^{i\omega t}\cdot\frac{\partial F_s}{\partial \mathbf{v}} \tag{B1}$$

in which F_s is assumed to be known and to be weakly time-dependent. In order to solve Eq. (B1) we apply the usual technique of characteristics. The following contains some details concerning the method of solution. First of all, we may solve Eq. (B1) formally such that

$$f_s(\mathbf{v},t) = \exp[-\Omega_s(\mathbf{v}\times\hat{b})\cdot\nabla_{\mathbf{v}}t] f_s(\mathbf{v},0) - \frac{e_s}{m_s}\mathbf{E}_0 e^{i\omega t}\cdot \int_0^t d\tau e^{-i\omega t} \exp[-\Omega_s(\mathbf{v}\times\hat{b})\cdot\nabla_{\mathbf{v}}t] \frac{\partial F_s}{\partial \mathbf{v}}, \tag{B2}$$

where $\hat{b} = \mathbf{B}_0/B_0$ and $\exp[-\Omega_s(\mathbf{v}\times\hat{b})\cdot\nabla_{\mathbf{v}}t]$ represents an operator which can be easily identified. To do this, we consider the equation

$$\frac{\partial f_s}{\partial t}+\frac{e_s}{m_s c}(\mathbf{v}\times\mathbf{B}_0)\cdot\frac{\partial f_s}{\partial \mathbf{v}}=0. \tag{B3}$$

By integrating along the particle trajectory, we can show that any solution to Eq. (B3) of the form

$$f_s(\mathbf{v},t)=f_s[(\mathbf{v}\cdot\hat{b})\hat{b}-(\mathbf{v}\times\hat{b})\times\hat{b}\cos\Omega_s t-(\mathbf{v}\times\hat{b})\sin\Omega_s t,\,0] \tag{B4}$$

satisfies the equation and the initial condition $f_s(\mathbf{v},0)$. Therefore, we conclude that the following prescription holds.

$$\exp[-\Omega_s(\mathbf{v}\times\hat{b})\cdot\nabla_{\mathbf{v}}t]f_s(\mathbf{v},0)=f_s[(\mathbf{v}\cdot\hat{b})\hat{b}-(\mathbf{v}\times\hat{b})\times\hat{b}\cos\Omega_s t-(\mathbf{v}\times\hat{b})\sin\Omega_s t,\,0]. \tag{B5}$$

Returning to Eq. (B2), we can thus write

$$f_s(\mathbf{v},t)=f_s[\mathbf{V}(t),0]-\frac{e_s}{m_s}\mathbf{E}_0e^{i\omega t}\cdot\int_0^t d\tau e^{-i\omega t}\left(\frac{\partial F_s}{\partial \mathbf{v}}\right)_{\mathbf{v}\to\mathbf{V}(t)}, \tag{B6}$$

where

$$\mathbf{V}(t)=[(\mathbf{v}\cdot\hat{b})\hat{b}-(\mathbf{v}\times\hat{b})\times\hat{b}\cos\Omega_s t-(\mathbf{v}\times\hat{b})\sin\Omega_s t].$$

If we are only interested in the solution as $t\to\infty$ and if we suppose $f[\mathbf{V}(\infty),0]\to 0$, then we obtain

$$\begin{aligned}f_s(\mathbf{v},t)&=-\frac{e_s}{m_s}e^{i\omega t}\int_0^\infty d\tau e^{-i\omega\tau}[(\mathbf{E}_0\cdot\hat{b})\hat{b}-(\mathbf{E}_0\times\hat{b})\times\hat{b}\cos\Omega_s\tau+(\mathbf{E}\times\hat{b})\sin\Omega_s\tau]\cdot(\partial F_s/\partial\mathbf{v})\\&=-\frac{e_s}{m_s}E_0e^{i\omega t}(i\omega(\omega^2-\Omega_s^2))^{-1}[\omega^2\hat{e}-i\omega\Omega_s(\hat{e}\times\hat{b})-\Omega_s^2(\hat{e}\cdot\hat{b})\hat{b}]\cdot(\partial F_s/\partial\mathbf{v}),\end{aligned} \tag{B7}$$

where $\hat{e}=\mathbf{E}_0/E_0$. In obtaining (B7) we have made use of the property

$$F_s(\mathbf{v})=F_s(v_z,v_\perp).$$

APPENDIX C

The Integral Operation $\int d^3v_1\tilde{Q}_{sr}(\omega)((e_s/m_s)\theta_s-(e_r/m_r)\theta_r)G_{sr}(\mathbf{k},\mathbf{v}_1',\mathbf{v}_2',\,t\to\infty)$ in the Limit of Infinite Ion Mass

Making use of Eqs. (21), (36), and (37), we obtain in the limit $m_i^{-1}\to 0$ that

$$\begin{aligned}&\int d^3v_1\tilde{Q}_{sr}(\omega)\left(\frac{e_s}{m_s}\theta_s-\frac{e_r}{m_r}\theta_r\right)G_{sr}(\mathbf{k},\mathbf{v}_1',\mathbf{v}_2')\\&=\frac{1}{i}\int d^3v_1\sum_{n=-\infty}^{+\infty}J_n^2\left(\frac{k_\perp v_{1\perp}}{\Omega_e}\right)\frac{(e_in_ee_e^3\theta_e/m_e^2k^2)\Phi_e^-(\mathbf{v})}{\epsilon^-[-\omega-(\mathbf{k}\cdot\mathbf{v})_n,\mathbf{k}][\omega+(\mathbf{k}\cdot\mathbf{v})_n-i\lambda]}\\&=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}d\omega_1\frac{1}{i\epsilon^+(\omega_1-\omega,\mathbf{k})}\left\{\int d^3v_1'\sum_{n=-\infty}^{+\infty}\frac{J_n^2(k_\perp v_{1\perp}'/\Omega_e)}{[\omega_1+(\mathbf{k}\cdot\mathbf{v}_1')_n-i\lambda]}\frac{(n_ie_i^2e_e^2\theta_e/m_e^2k^2)\Phi_e^-(\mathbf{v}_1')}{(\omega-\omega_1-i\lambda)}\right.\\&\qquad\left.-\int d^3v_2'\sum_{n=-\infty}^{+\infty}\frac{J_n^2(k_\perp v_{1\perp}'/\Omega_e)}{[\omega-\omega_1-(\mathbf{k}\cdot\mathbf{v}_2')_n-i\lambda]}\frac{(n_ie_i^2e_e^2\theta_e/m_e^2k^2)\Phi_e^+(\mathbf{v}_2')}{(\omega_1-i\lambda)}\right\},\end{aligned} \tag{C1}$$

where θ_s is a constant that depends upon e_s/m_s and

$$\Phi_e^\pm(\mathbf{v})=-\sum_{j=-\infty}^{+\infty}\frac{(\mathbf{k}\cdot(\partial F_e/\partial\mathbf{v}))_j}{[(\mathbf{k}\cdot\mathbf{v})_j\pm i\lambda]\epsilon^\mp(0)}.$$

Since $\epsilon^+(\omega_1-\omega,\mathbf{k})$ is analytic in the upper half of the complex ω_1 plane, we can take advantage of this fact and evaluate the ω_1 integral in (C1) by contour integration. Consequently,

$$\int d^3v_1\tilde{Q}_{sr}(\mathbf{v}_1|\mathbf{v}_1',\mathbf{v}_2',\mathbf{k},\omega)\left(\frac{e_s}{m_s}\theta_s-\frac{e_r}{m_r}\theta_r\right)G_{sr}(\mathbf{k},\mathbf{v}_1',\mathbf{v}_2')=-\frac{iA}{\epsilon^-(-\omega,\mathbf{k})}\int d^3v_2'\sum_{n=-\infty}^{+\infty}\frac{J_n^2(k_\perp v_{2\perp}'/\Omega_e)\Phi_e^+(\mathbf{v}_2')}{[(\mathbf{k}\cdot\mathbf{v}_2')_n-\omega+i\lambda]}, \tag{C2}$$

where $A=n_ie_i^2e_e^2\theta_e/m_e^2k^2$.

In the foregoing discussion, we have used the definition

$$\epsilon^{\pm}(-\omega,\mathbf{k})=1-\sum_s \frac{\omega_s^2}{k^2}\sum_{n=-\infty}^{\infty}\int d^3v \frac{1}{(\mathbf{k}\cdot\mathbf{v})_n\pm(\omega\pm i\lambda)}\left(\mathbf{k}\cdot\frac{\partial F_s}{\partial \mathbf{v}}\right)_n,$$

where the superscripts ($\pm$) designate the sign in front of $i\lambda$ (as $\lambda\to 0$).

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Suppression at High Temperature of Effects Due to Statistics in the Second Virial Coefficient of a Real Gas*

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It is shown that the repulsive core present in realistic two-body potentials and in hard spheres leads to the rapid suppression of the effects of statistics in the second virial coefficient, except at very low temperatures. For hard spheres, an upper bound is obtained which goes down exponentially with temperature when the latter becomes large.

THE effects of quantum mechanics on the second virial coefficient may be formally separated into diffraction effects which obtain for a Boltzmann gas and exchange contributions associated with the Bose-Einstein or Fermi-Dirac character of the gas.[1] This separation arises very naturally in the formalism developed by Lee and Yang[2] and allows us to consider the virial as being the sum of a direct term

$$B_{\text{direct}}=-(N/2)\int d\mathbf{r}[2^{3/2}\lambda_T{}^3\langle\mathbf{r}|e^{-\beta H_{\text{rel}}}|\mathbf{r}\rangle-1],$$

which in the limit $h\to 0$ gives us the classical answer, and of an exchange term

$$B_{\text{exch}}=\mp(N/2)[1/(2S+1)]\int d\mathbf{r}2^{3/2}\lambda_T{}^3\langle\mathbf{r}|e^{-\beta H_{\text{rel}}}|-\mathbf{r}\rangle.$$

H_{rel} is the relative Hamiltonian, β^{-1} is Boltzmann's constant times the temperature, λ_T is the thermal wavelength defined as $h(2\pi mkt)^{-1/2}$, N is Avogadro's constant, S is the spin of the individual component, and the sign is negative for Bose-Einstein statistics and positive for Fermi-Dirac cases.

In the case of a perfect gas we have

$$B_{\text{exch}}=\mp N(\lambda_T{}^3/2^{5/2})[1/(2S+1)].$$

At high temperatures this value is customarily[1] used to represent the quantum-mechanical effects due to statistics of a gas such as helium, while a Wigner-Kirkwood expansion is used to evaluate the direct term.

The purpose of this note is to point out that, in fact, for a real gas the presence of a strong repulsive core entails a drastic suppression of the exchange effect at high temperature.[3] We first show this to be the case for

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[1] See J. O. Hirschfelder, C. F. Curtis, and R. B. Bird, *Molecular Theory of Gases and Liquids* (John Wiley & Sons, Inc., New York, 1954) with special reference to the article by J. deBoer and R. Byron Bird on the quantum theory and the equation of state.
[2] T. D. Lee and C. N. Yang, Phys. Rev. **113**, 1165 (1959).
[3] Lloyd D. Fosdick has, independently, reached similar conclusions (private communication).