Transformation of Brown's Linear Equations for an Infinitely Long Prism

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In the original system of Brown's linear equations the components of the magnetic vector normal to the applied magnetic field and the magnetostatic scalar potential are mixed. The present work shows how, for an infinitely long prism along the applied magnetic field, Brown's linear equations can be transformed into a new system of equations, where the components of the magnetic vector normal to the applied magnetic field and the magnetostatic scalar potential satisfy separate differential equations. The transformed system of equations can ease the calculation of the nucleation field for long ferromagnetic particles with different cross sections.

I. INTRODUCTION

UNTIL now a rigorous solution of Brown's[1] linear equations for an infinitely long prism has only been obtained for a circular cross section.[1–3] In the case of rectangular cross sections,[4,5] however, only approximations to the nucleation field were obtained. One of the reasons for not solving the problem for other cross sections is the fact that the components of the magnetic vector normal to the applied magnetic field, and the magnetostatic potential function are mixed in Brown's linear equations.[6]

The aim of this work is to transform the system of Brown's linear equations into a system of separate differential equations for the different functions. This object is achieved for an infinitely long prism along the applied magnetic field and with the easy direction of magnetization parallel to the length of the prism.

In Part IV the case of an infinite circular cylinder[7] is solved by using the transformed system of Brown's linear equations. This example is presented in order to show a more systematic way of calculating the nucleation field from the Brown's linear equations.

The transformed system can be useful also in analyzing the contribution of line defects to the parasitic paramagnetism in ferromagnetic materials,[8] and in using models[9,10] for lowering the nucleation field.

II. BROWN'S LINEAR EQUATIONS

Consider an infinitely long ferromagnetic cylinder magnetized to saturation along its axis taken as the z direction and the direction of the easy magnetization along the cylinder. Brown's linear equations[6] are

$$-(2A'/I_s)\nabla'^2\alpha'+\partial U'/\partial x+(2K/I_s+H)\alpha'=0\,, \quad (1a)$$

$$-(2A'/I_s)\nabla'^2\beta'+\partial U'/\partial y+(2K/I_s+H)\beta'=0\,, \quad (1b)$$

$$\nabla'^2 U'=4\pi I_s(\partial\alpha'/\partial x+\partial\beta'/\partial y)\,, \quad (1c)$$

$$\nabla'^2 U_{\text{out}}'=0;\quad U_{\text{out}}^{(\infty)}=0\,, \quad (2)$$

where $\nabla'^2=\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2$, α' and β' are the direction cosines of the spin in the x and y direction, respectively. U' is the scalar magnetostatic potential inside the ferromagnetic crystal. U_{out}' represents the magnetostatic potential outside the crystal. Here A' is the exchange energy constant, I_s the saturation magnetization of the material, K the magnetocrystalline anisotropy coefficient, and H the applied magnetic field along the $+z$ direction.

The boundary conditions are

$$\partial\alpha'/\partial n=\partial\beta'/\partial n=0\,, \quad (3a)$$

$$U'=U_{\text{out}}'\,, \quad (3b)$$

$$4\pi I_n=\partial U'/\partial n-\partial U_{\text{out}}'/\partial n\,, \quad (3c)$$

where n is a unit vector normal to the boundary and,

$$I_n=\mathbf{I}_s\cdot\mathbf{n};\quad \mathbf{I}_s=I_s(\alpha i+\beta j)\,.$$

It is assumed that the functions α', β', and U_{out}' are functions of the variable z as follows:

$$\alpha'(x,y,z)=\alpha(x,y)\sin(kz)\,, \quad (4a)$$

$$\beta'(x,y,z)=\beta(x,y)\sin(kz)\,, \quad (4b)$$

$$U'(x,y,z)=U(x,y)\sin(kz)\,, \quad (4c)$$

$$U_{\text{out}}'(x,y,z)=U_{\text{out}}(x,y)\sin(kz)\,, \quad (4d)$$

where k is a real parameter.

From Eqs. (1)–(4), and using the notations $\nabla^2=\partial^2/\partial x^2+\partial^2/\partial y^2$, $A=I_s/2A'$, $B=(I_s/2A')(2K/I_s+H)$, and $C=4\pi I_s$, the following equations are obtained:

$$\nabla^2\alpha=A\,\partial U/\partial x+(B+k^2)\alpha\,, \quad (5a)$$

$$\nabla^2\beta=A\,\partial U/\partial y+(B+k^2)\beta\,, \quad (5b)$$

$$\nabla^2 U-k^2 U=C(\partial\alpha/\partial x+\partial\beta/\partial y)\,, \quad (5c)$$

$$\nabla^2 U_{\text{out}}=k^2 U_{\text{out}}\,, \quad (6)$$

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[6] W. F. Brown, Jr., Phys. Rev. **58**, 736 (1940).
[7] This problem is already solved by other methods, see Refs. 1–3.
[8] A. Aharoni, J. Appl. Phys. **35**, 913 (1964).
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with the boundary conditions:

$$\partial\alpha/\partial n=\partial\beta/\partial n=0\,, \tag{7a}$$

$$4\pi(\mathbf{I}_s\cdot\mathbf{n})=\partial U/\partial n-\partial U_{\text{out}}/\partial n\,, \tag{7b}$$

$$U=U_{\text{out}}. \tag{7c}$$

Differentiating (5a) with respect to x, (5b) with respect to y, and using (5c):

$$Ak^2U=\nabla^2(\partial\alpha/\partial x+\partial\beta/\partial y) -(AC+B+k^2)(\partial\alpha/\partial x+\partial\beta/\partial y). \tag{8}$$

Differentiating (5a) with respect to y, (5b) with respect to x

$$\nabla^2(\partial\alpha/\partial y-\partial\beta/\partial x)=(B+k^2)(\partial\alpha/\partial y-\partial\beta/\partial x). \tag{9}$$

Differentiating (8) with respect to y, (9) with respect to x and subtracting (9) from (8):

$$Ak^2\partial U/\partial y=\nabla^4\beta-(B+k^2)\nabla^2\beta -AC(\partial/\partial y)(\partial\alpha/\partial x+\partial\beta/\partial y). \tag{10}$$

Operating with ∇^2 on (10) and using (5b),

$$\nabla^6\beta-(2B+AC+3k^2)\nabla^4\beta+(3k^2+AC+B) \times(B+k^2)\nabla^2\beta-k^2(B+k^2)^2\beta=0. \tag{11a}$$

The differential Eq. (11a) is equivalent to the equation

$$(\nabla^2-\tau_1)(\nabla^2-\tau_2)(\nabla^2-\tau_3)\beta=0\,, \tag{11b}$$

where

$$\tau_1=B+k^2\,, \tag{11c}$$

$$\tau_2=\tfrac{1}{2}\{AC+B+2k^2 +[(AC+B+2k^2)^2-4k^2(B+k^2)]^{1/2}\}\,, \tag{11d}$$

$$\tau_3=\tfrac{1}{2}\{AC+B+2k^2 -[(AC+B+2k^2)^2-4k^2(B+k^2)]^{1/2}\}. \tag{11e}$$

The general solution of Eq. (11a) can be represented as

$$\beta=\beta_1+\beta_2+\beta_3\,, \tag{12}$$

where the functions $\beta_i(i=1,2,3)$ are general solutions of the differential equations,

$$\nabla^2\beta_1=\tau_1\beta_1\,, \tag{13a}$$

$$\nabla^2\beta_2=\tau_2\beta_2\,, \tag{13b}$$

$$\nabla^2\beta_3=\tau_3\beta_3. \tag{13c}$$

Similarly the general solution for the function α is

$$\alpha=\alpha_1+\alpha_2+\alpha_3\,, \tag{14}$$

where

$$\nabla^2\alpha_1=\tau_1\alpha_1\,, \tag{15a}$$

$$\nabla^2\alpha_2=\tau_2\alpha_2\,, \tag{15b}$$

$$\nabla^2\alpha_3=\tau_3\alpha_3. \tag{15c}$$

By operating with ∇^2 on (5c) and using (8)

$$\nabla^4U-(AC+B+2k^2)\nabla^2U+k^2(B+k^2)U=0. \tag{16}$$

The general solution of (16) is

$$U=U_2+U_3\,, \tag{17}$$

where U_2 and U_3 are solutions of the equations

$$\nabla^2U_2=\tau_2U_2\,, \tag{18a}$$

$$\nabla^2U_3=\tau_3U_3. \tag{18b}$$

The solution of the system of Eq. (5) contains 6 constants of integration, but the solutions represented by the functions (12), (14), and (17) contain 16 constants of integration between which there must be some relationship.

The functions (12) and (14) must satisfy the relations:

$$\partial\alpha_1/\partial x=-\partial\beta_1/\partial y\,, \tag{19a}$$

$$\partial\alpha_2/\partial y=\partial\beta_2/\partial x\,, \tag{19b}$$

$$\partial\alpha_3/\partial y=\partial\beta_3/\partial x. \tag{19c}$$

The function U can be expressed by using (8), (13b), (13c), (15b), and (15c) as

$$U=(2Ak^2)^{-1}\{(-AC-B+[(AC+B+2k^2)^2 -4k^2(B+k^2)]^{1/2})(\partial\alpha_2/\partial x+\partial\beta_2/\partial y) -(AC+B+[(AC+B+2k^2)^2-4k^2(B+k^2)]^{1/2}) \times(\partial\alpha_3/\partial x+\partial\beta_3/\partial y)\}. \tag{20}$$

From the relations (19) and (20) it is obvious that the solution of the transformed system, will contain at most 6 integration parameters different from zero.

III. CASE $k=0$

If the dependent functions of the system of the Eq. (1) are not functions of the coordinate z, then Brown's linear equations will be represented by (5) and (6) with $k=0$.

The solution of the system of equations will be represented again by (12), (14), and (17), but the different functions α_i, β_i, and U_i will satisfy different differential equations and there will be different relationships between them.

The differential equations are

$$\nabla^2\alpha_1=B\alpha_1\,, \tag{21a}$$

$$\nabla^2\alpha_2=(AC+B)\alpha_2\,, \tag{21b}$$

$$\nabla^2\alpha_3=0\,, \tag{21c}$$

$$\nabla^2\beta_1=B\beta_1\,, \tag{21d}$$

$$\nabla^2\beta_2=(AC+B)\beta_2\,, \tag{21e}$$

$$\nabla^2\beta_3=0\,, \tag{21f}$$

$$\nabla^2U_2=(AC+B)U_2\,, \tag{21g}$$

$$\nabla^2U_3=0. \tag{21h}$$

The relationships between the different functions are:

$$\partial\alpha_1/\partial x=-\partial\beta_1/\partial y\,, \quad (22a)$$

$$\partial\alpha_2/\partial y=\partial\beta_2/\partial x\,, \quad (22b)$$

$$\partial\alpha_3/\partial x=-\partial\beta_3/\partial y\,, \quad (22c)$$

$$\partial\alpha_3/\partial y=\partial\beta_3/\partial x\,, \quad (22d)$$

$$U_2=[C/(AC+B)](\partial\alpha_2/\partial x+\partial\beta_2/\partial y)\,, \quad (22e)$$

$$\partial U_3/\partial x=-(B/A)\alpha_3\,, \quad (22f)$$

$$\partial U_3/\partial y=-(B/A)\beta_3\,. \quad (22g)$$

IV. THE SOLUTION OF AN INFINITE CIRCULAR CYLINDER

Suppose an infinitely long circular cylinder of radius R, along the applied magnetic field H, which will be taken in the direction of the z axis. The direction cosines of the spin in the cylindrical coordinate system (r,φ,z) are α_r and α_φ.

In order to get the transformed Brown's linear equation for $\alpha_r(r,\varphi,z)$, $\alpha_\varphi(r,\varphi,z)$ and the magnetostatic scalar potential $U(r,\varphi,z)$, two transformations are required, one for the components of the magnetic vector:

$$\alpha=\alpha_r\cos\varphi-\alpha_\varphi\sin\varphi\,, \quad (23a)$$

$$\beta=\alpha_r\sin\varphi+\alpha_\varphi\cos\varphi\,, \quad (23b)$$

and a second for the coordinates:

$$x=r\cos\varphi\,, \quad (24a)$$

$$y=r\sin\varphi\,. \quad (24b)$$

From (23) and (24) is obtained

$$\nabla^2\alpha_i=\cos\varphi\nabla_1^2\alpha_{r_i}-\sin\varphi\nabla_1^2\alpha_{\varphi_i}-r^{-2}\alpha_{r_i}\cos\varphi +r^{-2}\alpha_{\varphi_i}\sin\varphi-2r^{-2}(\sin\varphi\partial\alpha_{r_i}/\partial\varphi +\cos\varphi\partial\alpha_{\varphi_i}/\partial\varphi)\,, \quad (25a)$$

$$\nabla^2\beta_i=\sin\varphi\nabla_1^2\alpha_{r_i}+\cos\varphi\nabla_1^2\alpha_{\varphi_i}-r^{-2}\alpha_{r_i}\sin\varphi -r^{-2}\alpha_{\varphi_i}\cos\varphi+2r^{-2}(\cos\varphi\partial\alpha_{r_i}/\partial\varphi -\sin\varphi\partial\alpha_{\varphi_i}/\partial\varphi)\,, \quad (25b)$$

where $\nabla^2=\partial^2/\partial x^2+\partial^2/\partial y^2$, and

$$\nabla_1^2=\partial^2/\partial r^2+r^{-1}\partial/\partial r+r^{-2}\partial^2/\partial\varphi^2.$$

The relations equivalent to (19) are

$$\partial\alpha_{\varphi_1}/\partial\varphi=-(r\partial\alpha_{r_1}/\partial r+\alpha_{r_1}) \quad (26a)$$

for (19a), and

$$\partial\alpha_{r_{2,3}}/\partial\varphi=r\partial\alpha_{\varphi_{2,3}}/\partial r+\alpha_{\varphi_{2,3}} \quad (26b)$$

for (19b) and (19c), respectively.

Using (25), (26) the expressions (13) and (15) are

$$\nabla_1^2\alpha_{r_1}+r^{-2}\alpha_{r_1}+2r^{-1}\partial\alpha_{r_1}/\partial r=\tau_1\alpha_{r_1}\,,$$

or

$$\partial^2\alpha_{r_1}/\partial r^2+3r^{-1}\partial\alpha_{r_1}/\partial r+r^{-2}\alpha_{r_1} +r^{-2}\partial^2\alpha_{r_1}/\partial\varphi^2=\tau_1\alpha_{r_1}\,, \quad (27a)$$

$$\partial\alpha_{\varphi_1}/\partial\varphi=-(r\partial\alpha_{r_1}/\partial r+\alpha_{r_1})\,, \quad (27b)$$

$$\partial^2\alpha_{\varphi_{2,3}}/\partial r^2+3r^{-1}\partial\alpha_{\varphi_{2,3}}/\partial r+r^{-2}\alpha_{\varphi_{2,3}} +r^{-2}\partial^2\alpha_{\varphi_{2,3}}/\partial r^2=\tau_{2,3}\alpha_{\varphi_{2,3}}\,, \quad (27c)$$

$$\partial\alpha_{r_{2,3}}/\partial\varphi=r\partial\alpha_{\varphi_{2,3}}/\partial r+\alpha_{\varphi_{2,3}}\,, \quad (27d)$$

and from (20)

$$U=(2Ak^2R)^{-1}\{(2k^2R^2-\tau_3) \times[\partial\alpha_{r_2}/\partial r+r^{-1}\alpha_{r_2}+r^{-1}\partial\alpha_{\varphi_2}/\partial\varphi]-(\tau_2-2k^2R^2) \times[\partial\alpha_{r_3}/\partial r+r^{-1}\alpha_{r_3}+r^{-1}\partial\alpha_{\varphi_3}/\partial\varphi]\}\,. \quad (27e)$$

Supposing solutions of the form:

$$\alpha_{r_1}(r,\varphi)=A_{r_1}(r)\sin(m\varphi-\varphi_{0m}) \quad (28a)$$

$$\alpha_{\varphi_{2,3}}(r,\varphi)=A_{\varphi_{2,3}}(r)\cos(n_{2,3}\varphi-\varphi_{0n_{2,3}})\,, \quad (28b)$$

where the parameters φ_{0m}, $\varphi_{0n_{2,3}}$ are real and m, n_2, and n_3 are integers, then in order to fit the boundary conditions at $r=R$, it is necessary that $\varphi_{0m}=\varphi_{0n_{2,3}}$ and $m=n_{2,3}$.

The Eq. (27) will be

$$\partial^2A_{r_1}/\partial t^2+3t^{-1}\partial A_{r_1}/\partial t +[-\tau_1'+(1-m^2)t^{-2}]A_{r_1}=0\,, \quad (29a)$$

$$mA_{\varphi_1}=t\partial A_{r_1}/\partial t+A_{r_1}\,, \quad (29b)$$

$$\partial^2A_{\varphi_{2,3}}/\partial t^2+3t^{-1}\partial A_{\varphi_{2,3}}/\partial t +[-\tau_{2,3}'+(1-m^2)t^{-2}]A_{\varphi_{2,3}}=0\,, \quad (29c)$$

$$mA_{r_{2,3}}=t\partial A_{\varphi_{2,3}}/\partial t+A_{\varphi_{2,3}}\,, \quad (29d)$$

where $\tau_i'=R^2\tau_i$, $t=r/R$.

The solutions of (29) are

$$A_{r_1}(t)=A_1t^{-1}J_m([-\tau_1']^{1/2}r)\,, \quad (30a)$$

$$mA_{\varphi_1}(t)=A_1(-\tau_1')^{1/2}\{m(-\tau_1')^{-1/2}t^{-1} \times J_m([-\tau_1']^{1/2}t)-J_{m-1}([-\tau_1']^{1/2}t)\}\,, \quad (30b)$$

$$A_{\varphi_{2,3}}(t)=A_{2,3}t^{-1}J_m([-\tau_{2,3}']^{1/2}t)\,, \quad (30c)$$

$$mA_{r_{2,3}}(t)=A_{2,3}(-\tau_{2,3}')^{1/2}\{m(-\tau_{2,3}')^{-1/2}t^{-1} \times J_m([-\tau_{2,3}']^{1/2}t)-J_{m-1}([-\tau_{2,3}']^{1/2}t)\}\,. \quad (30d)$$

The magnetostatic scalar potential will be

$$U=(2Ak^2R)^{-1}\{(\tau_3'-2k^2R^2)A_2[\tau_2'J_m([-\tau_2']^{1/2}t) +m(m-1)t^{-2}J_m[(-\tau_2']^{1/2}t)]+(\tau_2'-2k^2R^2)A_3 \times[\tau_3'J_m([-\tau_3']^{1/2}t)+m(m-1)t^{-2}J_m([-\tau_3']^{1/2}t)]\}\,.$$

By adding and subtracting A_{r_i} and $A_{\varphi_i}(i=1,2,3)$ it is obvious that we obtain the same set of solutions as Aharani and Shtrikman.[3]

The transformed system of Brown's equations can be successfully applied to the calculation of the nucleation field of an infinitely long ferromagnetic prism with rectangular or ellipitcal cross sections. This will be the subject of a future paper.

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