Analysis of 3587 τ^+ Decays*

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An analysis of 3587 $\tau^+ \to \pi^- + \pi^+ + \pi^+$ decay events is presented. These events were found in an area scanning of an emulsion stack of 600-μ Ilford G5 emulsion pellicles exposed to a 300-MeV/c separated K^+ beam at the Bevatron of the Lawrence Radiation Laboratory. The pion energy spectra are compared with the predictions of linear-matrix-element theory, the pion-pole model, and the s-wave resonance model, and with the existing spectra of the secondaries from η, $\tau^{\pm}$, τ', and K_2^0.

I. INTRODUCTION

In the recent literature, considerable interest has been shown in the three-pion decay modes of the K and η mesons, and their connection with low-energy pion-pion interactions. Early data on the τ^+ decay mode of the K^+ meson ($K^+ \to \pi^- + \pi^+ + \pi^+$)[1,2] showed systematic deviations in the pion spectra from that determined by the phase space alone. The deviation is such that the unlike pion has a higher probability of being emitted with high energy. Subsequent measurements of τ^- decay[3,4] spectra confirmed the effect. Attempts to explain the pion asymmetries by final-state pion-pion interactions,[5–7] using the scattering-length approximation and neglecting p-wave and higher partial-wave effects, led to the requirements that the s-wave pion-pion scattering lengths in the $T=0$ and $T=2$ states be $a_0 \sim -\lambda_\pi$ and $a_2 \sim -0.3\lambda_\pi$.[6] However, experiments utilizing other means of measuring these parameters,[8–10] and theoretical considerations,[11,12] led to the requirements that $a_0 \sim \lambda_\pi$ with a_2 small and positive. Other attempts to explain the final-state pion asymmetries included the hypothesis of a K^* intermediate state,[13] and inclusion of p-wave pion-pion final-state interaction, both with[14] and without[15] "intrinsic" structure in the weak interactions.

The π^+ energy distribution in τ' ($K^+ \to \pi^+ + \pi^0 + \pi^0$) decays was seen to have a related deviation from phase space,[16–20] as predicted by Weinberg.[21] Similarly, the π^0 energy spectrum divided by phase space (the reduced π^0 energy spectrum) in $K_2^0 \to \pi^0 + \pi^+ + \pi^-$ was found to be a decreasing function of the π^0 energy,[22] in agreement with theoretical predictions.[23] Similarities to the kaon decay spectra were also noticed in the Dalitz plots of $\eta \to \pi^0 + \pi^+ + \pi^-$ decays.[24–26] The present theoretical models which could explain the similar final-state spectra in all these decays are (a) the pion-pole model,[15,27–29] and (b) an s-wave π-π resonance,[30,31] as summarized in Kacser's article.[32]

Analyses of about 2300 τ^- decays[3,4] and 900 τ^+ decays[1,2,33] have been reported. The present work roughly

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doubles this sample with the addition of 3587 τ^+ decay events.[34] The secondary spectra from these τ decays are compared with the existing decay spectra for τ, τ', K_2^0, and η. Comparison of the data with the predictions of the theories (a) and (b) above is made.

II. EXPERIMENTAL PROCEDURE

A. Exposure and Scanning

An 84-pellicle stack of 6 in.×8 in.×600-μ Ilford G5 emulsion was exposed to a 300 MeV/c separated K^+ beam at the Bevatron of the Lawrence Radiation Laboratory of the University of California.[35] The beam kaons came to rest near the center of each plate, in an area ~1.5 cm×4 cm. In order that the scanning time required to find a kaon decay event be fairly short, it was desired that the density of stopping kaons be relatively high. Therefore the stack was inserted in the beam after only one stage of separation, and a background of approximately 10 beam pions for each stopping K^+ was present. These pions were of minimum ionization, and traversed the entire stack. The density of kaon endings in the stopping region of the exposed stack was $\sim 2\times 10^4$ K^+'s/cm^3. The individual pellicles were aligned for scanning and track following by the method outlined in Ref. 36.

The stack was systematically area scanned for K^+-meson endings with multiple secondaries.[37] 3359 three-secondary events, each with two or three of the secondaries having greater than 1.5 times minimum ionization, were separated from the total scanning sample. Three events with a minimum-ionizing secondary are inconsistent with the τ-decay mode, and were excluded. One of these events was shown to be an example of the decay mode K_{e4} ($K^+ \to \pi^+ + \pi^- + e^+ + \nu$), and has been previously reported.[38] The other two events are also consistent with this decay mode, but are not sufficiently complete to allow clear identification. The remaining sample of 3356 "τ-like" events, together with 97 events from the Columbia stack C and 94 events from the Columbia stack D[39] were included in the detailed analysis described below. Six obvious decays in flight were excluded. In addition, 71 previously analyzed events from the Columbia stacks[40] A and B are included in the total sample.

B. Determination of Secondary Energies

The measurement of the pion energies for each event was carried out as follows:

(a). The plane angles between the secondaries (the angles between the projections of the pion tracks on the plane of the emulsion) for each event were measured, using a 6-in. protractor mounted on one of the microscope eyepiece tubes. The estimated errors for these measurements are ±2°, including distortion effects. The tangents of the dip angles between the plane of the emulsion and the secondaries were measured by using the fine z motion of the microscope and a calibrated eyepiece grid. Errors in the z measurements are estimated to be of the order of 1 μ. The uncertainty of the emulsion shrinkage factor is of the order of 10%. Secondary tracks were followed, recording any scatters of ~20° or greater, until the π^- was identified. π^+ secondaries are identified by their characteristic $\pi \to \mu$ decays; π^- secondaries are absorbed in the emulsion nuclei, giving rise to stars with zero or more ionizing prongs. Events for which the π^- could not be identified are discussed below.

(b). The point-to-point ranges, including scatter points, were used to find the energies for those pions which had been followed to an ending. An average emulsion thickness, measured before exposure, was used for each stack. The range-energy charts of Barkas and Young are used throughout this work.[41] The energies calculated directly from a range are expected to be good to ~3%. Using these energies and the measured angles, each event was analyzed on the IBM 1620 of the Stevens Computer Center to find the missing pion energies. Since the events are overdetermined, the Q value was calculated as a check, as was the quantity

$$\phi = \frac{\mathbf{P}_1 \cdot \mathbf{P}_2 \times \mathbf{P}_3}{|\mathbf{P}_1||\mathbf{P}_2||\mathbf{P}_3|},$$

where the $\mathbf{P}_i$'s are the secondary three-momenta. ϕ is an indication of the coplanarity of the event, and is an invariant under labeling of the tracks. A limit of 5 MeV was placed on the deviation of the calculated Q value from the accepted Q value, 75.11±0.14,[42] and the coplanarity measure was required to differ from zero by no more than 0.075.

(c). Any event falling outside the above limits was returned to the scanners for remeasurement. If the two measurements agreed within statistics an additional track was followed, and the event was returned to part (b) of the schedule. If all tracks had been followed, the event was examined for possible reinterpretations, such as secondary scatters near the K ending, inelastic

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[34] See T. Huetter, E. L. Koller, S. Taylor, P. Stamer, and J. Grauman, Bull. Am. Phys. Soc. **9**, 23 (1964) for a preliminary report on the first 1049 events.

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TABLE I. Summary of events. All three-secondary events from the total scanning sample having two or three secondaries with greater than 1.5 times minimum ionization are tabulated. Events inconsistent with τ decay but consistent with the K_{e4} ($K^+ \to \pi^+ + \pi^- + e^+ + \nu$) or the radiative τ ($K^+ \to \pi^+ + \pi^+ + \pi^- + \gamma$) decay modes, along with τ decays in flight, are excluded from the distributions. "Incomplete" events, for which the π^- could not be identified, are included in the distributions only if the difference between the unidentified pion energies was less than 9.6 MeV. Of the "excluded" events only the "incomplete" events could contribute to a bias in the distributions.

		Events excluded from distributions				Events used in distributions		
Stack	3-secondary events	K_{e4}	Radiative τ	τ decay in flight	"Incomplete"	τ	"Incomplete"	Total
Stevens *E*	3359	3	7	4	16	3317	12	3329
Columbia *C*[a]	97	...	...	...	1	93	3	96
Columbia *D*[a]	94	...	...	2	...	92	...	92
Columbia *A* and *B*[b]	71	...	...	...	1	68	2	70
Totals	3621	3	7	6	18	3570	17	3587

[a] See Ref. 38. [b] See Ref. 39.

scattering in the secondary, decay of the secondary in flight, or alternate decay mode. Seven of the events in the lower tail of the distribution were shown to be inconsistent with the τ-decay mode, and were identified as examples of the radiative τ-decay mode, $K^+ \to \pi^- + \pi^+ + \pi^+ + \gamma$.[43]

At the close of the analysis 51 events remained outside the 5-MeV Q-value limit. This would correspond to a standard deviation for the Q-value distribution for all events of $\sim$2.2 MeV, if the distribution were Gaussian. The mean Q value for 65 events for which it was necessary to follow all three tracks was 74.7 MeV; the standard deviation of their Q-value distribution was 1.7 MeV.

There were 35 "incomplete" events for which the π^- could not be identified, due to interactions in flight of the secondaries or secondary tracks leaving the stack. Each of these events had one π^+ energy measured by range, and were subjected to the same Q value and "coplanarity" tests described above. The energies were renormalized by preserving the ratio of the energies of the unidentified tracks as calculated from the space angles of the event, and requiring the sum of the three energies to be the accepted Q value. For 17 of these events, the difference between the two unidentified pion energies was less than 9.6 MeV. These 17 events are included in the distributions, using the mean of the two "missing" energies for the unidentified π^+ and π^-. Thus no energy is shifted by more than 4.8 MeV (the data is divided for analysis into 4.8-MeV groups; see below). The remaining 18 events are excluded from the distributions.

The pion energies used in the distributions for events having two followed tracks are the two measured energies and the difference between the known Q value and the sum of the measured energies. Pion energies for events for which one or three tracks were followed were renormalized to the known Q value by multiplying the raw pion energies by the ratio of the known Q value to the calculated Q value for the event.

C. The Data

Of 3605 stopplng τ decays, 3587 events are included in the final pion distributions. A summary of the different classes of events in the total sample is presented in Table I. Histograms of the π^- energy distribution, and the distribution $T_1 - T_2$, where T_1 is the kinetic energy of the more energetic π^+ and T_2 is the energy of the remaining π^+, are shown in Figs. 1 and 2.

Since only 18 events of a total sample of 3605

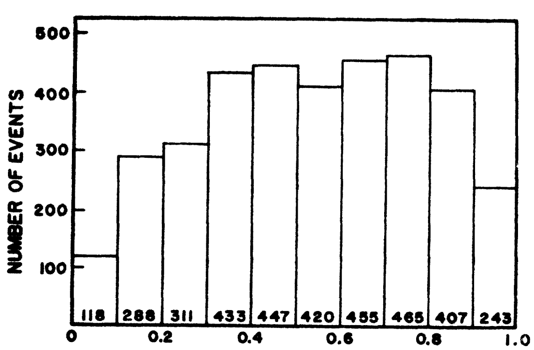

FIG. 1. Histogram of the π^- energy distribution for 3587 τ^+ decays. The number of events in each division is indicated.

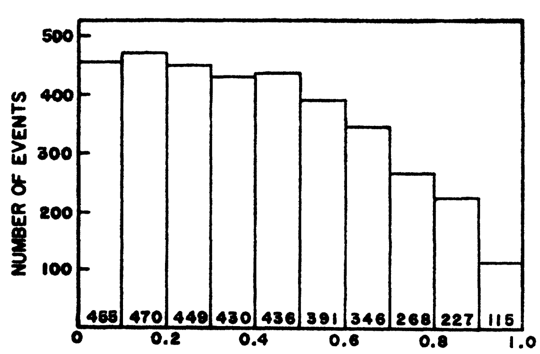

FIG. 2. Histogram of the distribution $T_1 - T_2$, where T_1 and T_2 are the positive pion energies, for 3587 τ^+ decays. The number of events in each division is indicated.

[43] P. Stamer, T. Huetter, E. L. Koller, S. Taylor, and J. Grauman, Phys. Rev. 138, B440 (1965); E. L. Koller, S. Taylor, T. Huetter, and P. Stamer, *ibid.* 129, 1381 (1963).

stopping τ decays are excluded, the sample is quite free of geometrical bias. One possible source of scanning bias, despite the distinctive appearance of stopping τ decays in emulsion, is the possibility of the scanner missing an event with a very short secondary pion. The other two tracks are then very nearly colinear, and may be mistaken for a coincidental crossover track. Each event recorded as a stopping kaon with heavy secondary and a crossover track was re-examined. About five of these were found to be τ decays. The scanning efficiency for locating τ endings is believed to have been high, particularly since the scanning was not only for τ's but for K endings with associated Dalitz pairs, for rare K-decay modes, etc.

Another possible bias is misidentification of secondary charge. Steep π^+ tracks with forward decays may be mistaken for π^- tracks which end in zero-prong stars, and π^- tracks either with a scatter $\sim 600\,\mu$ from the end, or ending in a one-prong star with the prong $\sim 600\,\mu$ long, may be mistakenly identified as $\pi \rightarrow \mu$ decays. All secondary tracks with energy less than 12 MeV were carefully re-examined in connection with another experiment,[44] 1706 tracks in total. Among these, 6 π^+ were found to be misidentified as π^-, and 7 π^- were misidentified as π^+. This comprises a rate of misidentification of 0.8%. Recalculation of the spectrum including the new values caused negligible corrections.

It was found that exclusion of the events in the few plates nearest the top and bottom of the stack did not change the degree of contamination of the sample with "incomplete" events. The stack was large enough so that secondary tracks could not leave the stack from the sides.

III. TREATMENT OF EXPERIMENTAL DATA

The final-state kinematics in K or $\eta \rightarrow 3\pi$ decays is completely described by two independent variables, for example the Lorentz-invariant variables $(S_3 - S_0)$ and $(S_1 - S_2)$, where

$$S_i = (P_0 - P_i)^2,$$

or, in the rest system of the decaying particle,

$$S_i = (M - m_i)^2 - 2MT_i,$$

and

$$3S_0 = S_1 + S_2 + S_3 = \sum_i (M - m_i)^2 - 2MQ.$$

P_0 and M are the four-momentum and mass of the decaying particle, P_i, m_i, and T_i are the four-momentum, mass, and kinetic energy of the ith pion, respectively, and Q is the sum of the pion kinetic energies. S_3 refers to the unlike pion in τ and τ' decays, and to the π^0 in $K_2^0 \rightarrow \pi^0 + \pi^+ + \pi^-$ and $\eta \rightarrow \pi^0 + \pi^+ + \pi^-$ decays. For convenience, the variables $Y = -3(S_3 - S_0)/2MQ$ and $X = -\sqrt{3}(S_1 - S_2)/4MQ$ are introduced.[45]

The differential decay probability may be written

$$\omega(X,Y)dXdY \propto |M(X,Y)|^2 C(X,Y)\phi(X,Y)dXdY,$$

where $\phi(X,Y)$ is the invariant phase space for the decay, $C(X,Y)$ is a factor to include final-state Coulomb effects, and $M(X,Y)$ is the matrix element for the decay. The factor $C(X,Y)$ applied in this work is that given by Dalitz,[46] which in the nonrelativistic limit reduces to that calculated by Lomon.[47]

The Y dependence of the decays is examined by plotting

$$\frac{N_i(Y)}{N_{\mathrm{tot}} C\phi_i(Y)} \approx \int_{\Delta Y_i} \int_{X(\Delta Y_i)} \omega(X,Y)dXdY \Bigg/ \int_{\Delta Y_i} \int_{X(\Delta Y_i)} C(X,Y)\phi(X,Y)dXdY$$

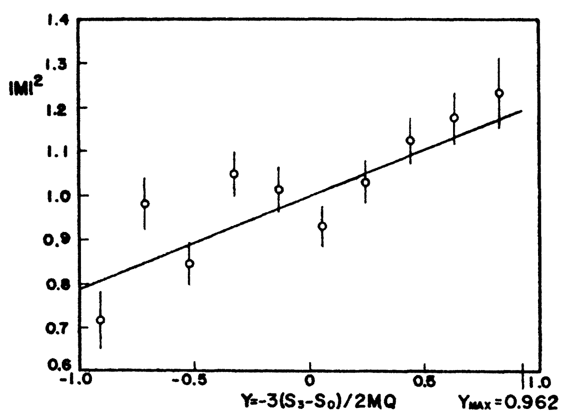

FIG. 3. Dependence of the matrix element squared on unlike pion energy. Ohe solid line is the fitted curve $|M|^2 \propto 1 + \alpha MQY/m_\pi^2$, with $\alpha = 0.11 \pm 0.02$.

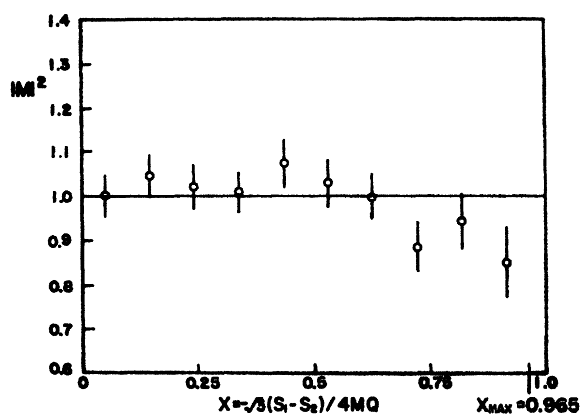

FIG. 4. Dependence of the matrix element squared on like pion energy. The solid line is the zero-slope fitted curve.

[44] S. Taylor, E. L. Koller, T. Huetter, P. Stamer, and J. Grauman, Phys. Rev. Letters **14**, 745 (1965).

[45] In the case of τ decay, the variables X and Y are identical with the Dalitz variables $x = \sqrt{3}(T_1 - T_2)/2Q$ and $y = (3T_3 - Q)/Q$. See, for example, R. H. Dalitz, Phil. Mag. **44**, 1068 (1953).

[46] R. H. Dalitz, Proc. Phys. Soc. (London) **A69**, 527 (1956); see footnote on page 537.

[47] Earle L. Lomon, Phys. Rev. **108**, 458 (1957).

versus Y, where $N_i(Y)$ is the number of events in the interval ΔY_i, N_{tot} is the total number of events, and $C\phi_i(Y)$ is the corresponding "Coulomb-corrected" Lorentz-invariant phase space. Then $N_i(Y)/N_{tot}C\phi_i(Y)$ is proportional to $|M(X,Y)|^2$ averaged over ΔY_i and the corresponding values of X. These data are presented in Fig. 3. The data are normalized such that the weighted mean ordinate is 1.0. Similarly, the X dependence of the distribution is examined by plotting $N_i(X)/N_{tot}C\phi_i(X)$ versus X. These data are presented in Fig. 4.

IV. COMPARISON WITH THEORY

A. Linear-Matrix-Element Theory

It has been proposed that the matrix element in $K\rightarrow 3\pi$ decays may be expanded as a power series in X and Y.[21] Owing to the Bose statistics of the final-state pions, the expansion may contain only even powers of X. Neglecting higher order terms in the expansion,

$$|M_j(X,Y)|^2 \propto 1+\alpha_j(M_jQ_j/m_\pi^2)Y\,,$$

where j is τ, τ', or K_2^0, and m_π is the charged-pion mass. If the final-state pions are in a pure $T=1$ state, the relationship $\alpha_{\tau'}=-2\alpha_\tau$ follows.[21,23,48] This is consistent with the $\Delta T=\frac{1}{2}$ rule, but does not rule out admixtures of $\Delta T=\frac{3}{2}$ in the decay interaction, since the $T=1$ state is accessible through either $\Delta T=\frac{1}{2}$ or $\frac{3}{2}$.[23,48] It has been shown that the π^0 energy spectrum in $K_2^0\rightarrow\pi^0+\pi^++\pi^-$ decays is identical with the π^+ energy spectrum in τ' decay, if a $\Delta T=\frac{1}{2}$ rule is operative.[23,48] It follows that $\alpha_{K_2^0}=-2\alpha_\tau$ under the $\Delta T=\frac{1}{2}$ rule. However, this is a weak test of the rule, since any mechanism which leads to the *same* $T=1$ final state for both the charged and neutral decay will satisfy the condition on the slopes.

A weighted least-squares fit of the function $1+\alpha_\tau MQY/m_\pi^2$ to the normalized reduced π^--spectrum data of the present experiment gives a value $\alpha_\tau=0.11\pm0.02$. The χ^2 value is 18.5, corresponding to a χ^2 probability of $\sim 2\%$ for eight degrees of freedom. Although the χ^2 value is rather high, there is no real evidence for a quadratic term in the expansion, since the deviations of the experimental points from the linear fit are "scattered" rather than systematic.[49] (See Fig. 3.)

The X dependence of the events is well fitted by a zero-slope straight line, with a χ^2 value of 12.5. The χ^2 probability for 9 degrees of freedom is $\sim 20\%$. Although there seems to be some suggestion of "shape" to the data, there is no statistically significant evidence for higher order terms in the X dependence of the matrix element. (See Fig. 4).

The results of this experiment are in agreement with those of other experiments on[1,2] τ^+ and[3,4] τ^-. Table II contains the values of α_τ found by other experimenters, along with values of $\alpha_{\tau'}$, $\alpha_{K_2^0}$, and α_η from various experiments.

Combining the value of α_τ for the present experiment with that obtained by Smith *et al.*,[4] $\alpha_\tau=0.12\pm0.02$, in an analysis of a compilation of 3205 $\tau^\pm$ decay events, the combined value $\alpha_\tau=0.115\pm0.015$ is obtained.

Bisi *et al.*,[20] have fitted the reduced π^+ energy spectrum for 1874 τ'-decay events with a linear-squared matrix element, using an error analysis similar to that used in the present work, and have obtained $\alpha_{\tau'}=-0.40\pm0.07$. Using these values, one obtains

$$\alpha_{\tau'}/\alpha_\tau=-3.5\pm0.6$$

compared to the predicted ratio of -2. The 1792 τ' events analyzed by Kalmus *et al.*,[19] were fitted with a linear matrix element, rather than linear-reduced spectrum. However, their value for $\alpha_{\tau'}$ is consistent with the value obtained by Bisi *et al.*

Luers *et al.*[22] have fitted the reduced π^0 energy spectrum for 83 $K_2^0\rightarrow\pi^0+\pi^++\pi^-$ with a linear function and obtain $\alpha_{K_2^0}=-0.32\pm0.07$. Using the combined α_τ value and their value

$$\alpha_{K_2^0}/\alpha_\tau=-2.8\pm0.7$$

as compared to the predicted ratio -2.

All of the $K\rightarrow 3\pi$ reduced spectra are well fitted by linear functions. The ratio $\alpha_{K_2^0}/\alpha_\tau$ is in agreement with the predictions of linear-matrix-element theory, a final $T=1$ state, and the $\Delta T=\frac{1}{2}$ rule. However, this is only a weak test of the rule. The ratio of $\alpha_{\tau'}/\alpha_\tau$ is about 2.5 standard deviations from that predicted by the theory, and further data are needed to clarify this situation. However, it is felt that the data on $K\rightarrow 3\pi$ spectra, viewed as a whole, are consistent with linear-matrix-element theory, a pure $T=1$ final state, and the $\Delta T=\frac{1}{2}$ rule.

B. The Pion Pole Model

Barton and Rosen[27] have considered a model in which the decays $\eta\rightarrow\pi^0+\pi^++\pi^-$ and $K\rightarrow 3\pi$ both proceed predominantly through a one-pion intermediate state, the pion pole model.[50] Then the decay amplitudes for the η and the various K modes are just different isotopic projections of the same $T=1$ function, apart from a constant factor depending on the mechanism whereby the single-pion state is reached. The matrix element is expanded in the invariant variables, and neglecting quadratic and higher order terms, the relationship $\alpha_{\tau'}=\alpha_\eta$ holds, at least insofar as the $K-\eta$ mass difference can be neglected in the structure of the interactions. Since the predictions for the ratios of the α's in the various $K\rightarrow 3\pi$ states are identical with those given under Sec. A above, the relationship $\alpha_\eta=-2\alpha_\tau$ holds

[48] G. Barton, C. Kacser, and S. P. Rosen, Phys. Rev. **130**, 783 (1963).

[49] Smith *et al.*, Ref. 4, have included a quadratic term in the expansion; the error in the coefficient of the quadratic term is as large as the coefficient itself.

[50] For further information on the pole model see, for example, S. Hori, S. Oneda, S. Chiba, and A. Wakasa, Phys. Letters **5**, 399 (1963); E. Eberle and S. Iwao, *ibid.* **6**, 238 (1963); Riazuddin and A. Zimmerman, Phys. Rev. **135**, B1211 (1964); S. Oneda, Y. S. Kim, and L. M. Kaplan, Nuovo Cimento **34**, 655 (1964).

TABLE II. Summary of published values of the parameter α for the linear matrix element theory in $\tau^{\pm}$, τ', K_2^0 and η decays. The unlike pion-reduced spectra (the π^0 in η and K_2^0 decay) are fitted with either the function $|M|^2 \propto 1+\alpha MQY/m_\pi^2$ or $|M|^2 \propto (1+\alpha MQY/2m_\pi^2)^2$, designated by L and S, respectively, in column 3. If the $\Delta T=\frac{1}{2}$ rule is valid in $K\rightarrow 3\pi$ decays and the decays $K\rightarrow 3\pi$ and $\eta\rightarrow 3\pi$ have the *same* $T=1$ final state, the relationships $2\alpha_\tau = -\alpha_{\tau'} = -\alpha_{K_2^0} = -\alpha_\eta$ should hold.

Experiment	Reference	Function	α
899 τ^{+} [a]	2	L	0.14 ± 0.02
3587 τ^+	our	L	0.11 ± 0.02
1347 τ^-	3	L	0.15 ± 0.02
948 τ^-	4	L	0.11 ± 0.04
83 K_2^0	22	L	-0.32 ± 0.08
1874 τ'^+	20	L	-0.40 ± 0.07
		S	-0.36 ± 0.09
1792 τ'^+	19	S	-0.32 ± 0.03
109 η	25	S	-0.26 ± 0.04
274 η	26	S	-0.24 ± 0.03

[a] 400 of these events are from Ref. 1.

under the pion pole model. However, since the same predictions follow for any model in which the K and η have the *same* $T=1$ final state, this is a weak test for the pion pole model.

Combining the value for α_η quoted by Crawford *et al.*,[25] on the basis of 109 η decays and that of Foster *et al.*,[26] with 274 η decays (see Table II), the combined value $\alpha_\eta = -0.25\pm0.025$ is obtained. Using this value for α_η, together with the combined α_τ obtained above

$$\alpha_\eta/\alpha_\tau = -2.2\pm0.35$$

as compared with the predicted ratio -2. The ratio α_η/α_τ is in good agreement with the predictions of the pion pole model.

C. *S*-wave Dipion Resonance

Several authors[51–55] have found evidence for the existence of a π-π resonance at an energy of about 400 MeV, consistent with the assignment of the quantum numbers $T=J=0$. Brown and Singer[56] have formulated a model based on the existence of such a resonance, the σ, in order to explain the apparent enhancement of the three-pion decay mode in the η meson. These authors have extended the model to include both $\eta\rightarrow 3\pi$ and $K\rightarrow 3\pi$ decays.[30] As in the pion-pole model, the final-state pions must be in a pure $T=1$ state, which is consistent with the $\Delta T=\frac{1}{2}$ rule, but does not rule out $\Delta T=\frac{3}{2}$.

The theory predicts that the reduced π^- spectrum in $K^+\rightarrow \pi^+ + \pi^+ + \pi^-$ decay is given by

$$F(T_3) \propto \frac{1}{\phi}\left[\frac{1}{h}\ln\frac{(h+\phi)^2+4B^2}{(h-\phi)^2+4B^2} + \frac{1}{B}\tan^{-1}\left(\frac{4B\phi}{h^2-\phi^2+4B^2}\right)\right],$$

where ϕ is the phase space and

$$h = M - 3m_\pi - 2A - T_3\,,$$
$$A = [(M-m_\pi)^2 - m_r^2]/2M\,,$$
$$B = m_r\Gamma_r/2M\,.$$

Here m_r and Γ_r are the parameters of the theory, the mass and full width of the resonance.

The function $F(T_3)$ for various values of the parameters was compared to the normalized experimental data in a 10-division χ^2 test. $F(T_3)$ was normalized such that the mean ordinate was 1.0, and since the function is nearly linear for the range of parameters under consideration, the integrals over the divisions were approximated by the ordinate of the function at the midpoint of the division. Since $F(T_3)$ is a slowly varying function of the resonance parameters, there is a large range of m_r and Γ_r for which a reasonable fit is obtained. A contour plot of constant χ^2 for the parameters m_r and Γ_r is shown in Fig. 5. The contours $\chi^2=20$ and 23 are shown; these correspond roughly to one and two standard deviations from minimum χ^2, respectively. The minimum χ^2 value is 16.1, with a χ^2 probability corresponding to 7 degrees of freedom of $\sim$3%. The best-fit values of the parameters are approximately $m_r=340$ MeV, $\Gamma_r=90$ MeV. The data, fitted with the optimum parameter spectrum, are shown in Fig. 6. In addition,

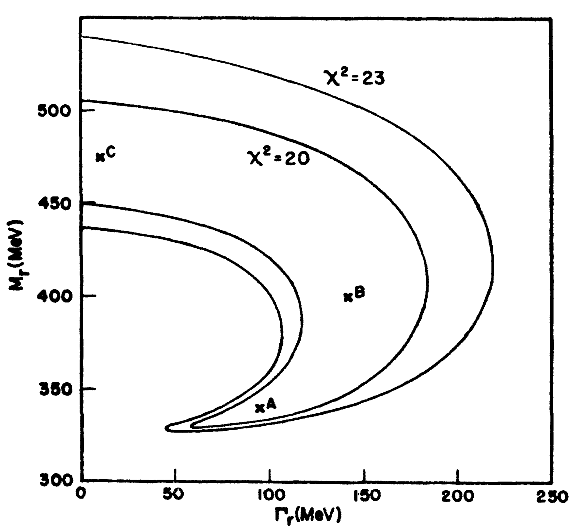

FIG. 5. Contour plot of constant χ^2 for the mass M_r and full width Γ_r of the resonance in the Brown and Singer model for τ decay. Point A corresponds to the best-fit parameters, $\chi^2=16.1$; the points A, B, and C correspond to curves A, B, and C in Fig. 6. The lines $\chi^2=20$ and $\chi^2=23$ correspond roughly to one and two standard deviations from minimum χ^2, respectively, the "allowed" range for the parameters.

[51] N. Samios, A. Bachman, R. Lea, T. Kalogeropoulos, and W. Shephard, Phys. Rev. Letters **9**, 139 (1962).
[52] J. Kirz, J. Schwartz, and R. D. Tripp, Phys. Rev. **130**, 2481 (1963).
[53] I. M. Blair, G. Conforto, C. Rubbia, G. Torelli, and E. Zavattini, Phys. Letters **11**, 79 (1964).
[54] R. Del Fabbro, M. De Pretis, R. Jones, G. Marini, A. Odian, G. Stoppini, and L. Tau, Phys. Rev. Letters **12**, 674 (1964).
[55] V. Barnes, W. Fowler, K. Lai, D. Radojicic, M. Webster, A. Bachman, P. Baumel, and R. Lea, Bull. Am. Phys. Soc. **10**, 65 (1965).
[56] Laurie M. Brown and Paul Singer, Phys. Rev. Letters **8**, 460 (1962).

spectra for other selected values of the parameters are shown. They all fit the data as well as the linear function.

As can be seen from Figs. 5 and 6, the present data cannot determine the resonance parameters with any degree of certitude. Other experimenters have fitted this theory to τ' and η decays. Kalmus *et al.*,[19] with 1792 τ' events find $m_r=337\pm4$ MeV, $\Gamma_r=87\pm9$ MeV, Crawford *et al.*,[25] find $m_r=392\pm9$ MeV, $\Gamma_r=88\pm15$ MeV on the basis of 109 η decays, and Foster *et al.*,[26] with 274 η decay events find $m_r=407_{-12}^{+25}$ MeV, $\Gamma_r=117\pm15$ MeV. Each of these pairs of parameters overlaps the "allowed" region on Fig. 5. Bisi *et al.*,[20] have presented a contour plot of m_r and Γ_r; their plot and Fig. 5 have the same general shape and range of the parameters.

A similar theory has been formulated by Mitra and Ray.[31] The reduced π^- spectrum in τ decay is given by

$$F(T_3)=(\eta(\chi^+)-\eta(\chi^-))/\phi\,,$$

where

$$\eta(\chi^\pm)=(4m_\pi/M)\gamma(r^2-\tfrac{1}{4}\gamma^2m_\pi^2)^{-1/2}\tan^{-1}\chi^\pm\,,$$

$$\chi^\pm=\tfrac{1}{4}(\gamma m_\pi)^{-1}(r^2-\tfrac{1}{4}\gamma^2m_\pi^2)^{-1/2}\times(MT_3+Mm_\pi-3m_\pi^2-4r^2+2m_\pi^2\gamma^2\pm M\phi)\,,$$

$$\gamma=\Gamma_r/2r\,,$$

$$r=(\tfrac{1}{4}m_r^2-m_\pi^2)^{1/2}\,.$$

In the above expressions m_r and Γ_r are the total energy and full width of the resonance.

$F(T_3)$ for Mitra and Ray's theory is fitted to the data of the present experiment in the same manner as that of Brown and Singer above. A contour plot of constant χ^2 for the parameters is shown in Fig. 7. The best fit parameters are approximately $m_r=335$ MeV and $\Gamma_r=65$ MeV; the minimum χ^2 value is 15.8, corresponding to a χ^2 probability of $\sim3\%$ with 7 degrees of freedom. The data, fitted with the best value parameter spectrum is presented in Fig. 8. The spectra for two other values are

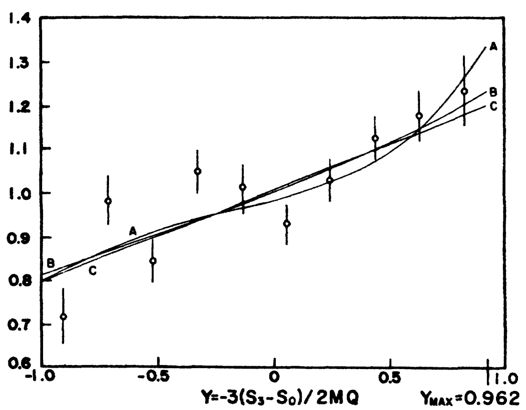

FIG. 6. Energy dependence of the reduced π^- energy spectrum for the Brown and Singer model for τ decay. Curve A is for the best-fit parameters, $M_r=340$ MeV and $\Gamma_r=90$ MeV. Curve B is for the parameters $M_r=400$ MeV and $\Gamma_r=140$ MeV, curve C is for $M_r=475$ MeV and $\Gamma_r=10$ MeV. Curves A, B, and C correspond to points A, B, and C on Fig. 5.

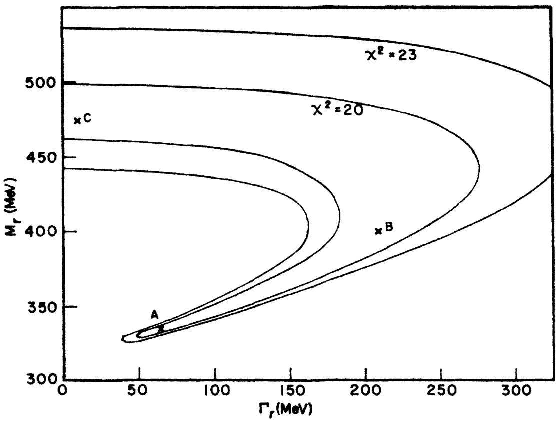

FIG. 7. Contour plot of constant χ^2 for the total energy M_r and full width Γ_r of the resonance in the Mitra and Ray model for τ decay. Point A corresponds to the best-fit parameter, $\chi^2=15.8$; the points A, B, and C correspond to curves A, B, and C in Fig. 8. The lines $\chi^2=20$ and $\chi^2=23$ correspond roughly to one and two standard deviations from minimum χ^2, respectively, the "allowed" range for the parameters.

also shown; as in the other resonance model the fit is reasonable for all three values.

Several authors have found other experimental evidence for a resonance with parameters near the region required in the resonance models. Samios *et al.*[51] have found evidence for the existence of a resonance with $T=0$ or 1 and $m_r=395\pm10$ MeV, $\Gamma_r=50\pm20$ MeV in an analysis of π^--p collisions. However, Alff *et al.*,[57] have found no evidence for a resonance in this region in the products of π^+-p collisions. Kirz *et al.*,[52] have shown the existence of a peak in the $T=0$ dipion state in the process $\pi^-+p\rightarrow\pi^++\pi^-+n$, but the peak changes posi-

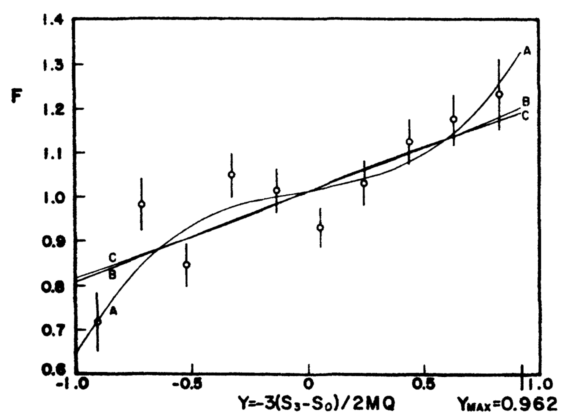

FIG. 8. Energy dependence of the reduced π^- energy spectrum for the Mitra and Ray model for τ decay. Curve A is for the best-fit parameters, $M_r=335$ MeV and $\Gamma_r=65$ MeV. Curve B is for the parameters $M_r=400$ MeV and $\Gamma_r=210$ MeV, curve C is for $M_r=475$ MeV and $\Gamma_r=10$ MeV. Curves A, B, and C correspond to points A, B, and C on Fig. 7.

[57] C. Alff, D. Berley, D. Colley, N. Gelfand, U. Nauenberg, D. Miller, J. Schultz, J. Steinberger, T. Tan, H. Brugger, P. Kramer, and R. Plano, Phys. Rev. Letters 9, 322 (1962); 9, 325 (1962).

tion with the incident pion energy. Blair *et al.*,[53] have also found evidence for a peak near 380 MeV in the invariant π-π mass squared for this reaction. Del Fabbro *et al.*,[54] find that the dipion effective-mass spectrum in the reaction $\gamma+p \rightarrow \pi^+ + \pi^- + p$ can be explained by the inclusion of an s-wave resonance with the parameters $m_r = 379 \pm 4$ MeV, $\Gamma_r = 139 \pm 13$ MeV. Barnes *et al.*,[55] have found in π^--p collisions a 3.5 standard deviation departure from phase space in the π^+-π^- effective-mass spectrum which could be explained by a resonance with $m_r \sim 400$ MeV, $\Gamma_r \sim 80$ MeV.

The present data are reasonably well fitted by the resonance models, and the range of resonance parameters found in this experiment for the Brown and Singer model are consistent with those found in other experiments on τ and τ', as cited above. While the resonance parameters proposed by Samios *et al.*,[51] could not explain τ decay with the present models, the other experimentally detected resonances quoted above agree, within statistics, with the present data.

V. SUMMARY

Comparisons between the various $K \rightarrow 3\pi$ decay spectra are reasonably consistent with a linear spectrum, a $T=1$ final state, and the $\Delta T = \frac{1}{2}$ rule. However, this is a weak test of the rule. The ratio $\alpha_{\tau'}/\alpha_\tau$ is about 2.5 standard deviations away from the predicted value, and should be investigated further. The comparison of η spectra with $K \rightarrow 3\pi$ decay spectra is consistent with the η having a predominantly $T=1$ final state, and with the predictions of the pion-pole model.

The resonance models fit the data of the present experiment reasonably well, and the range of resonance parameters determined are consistent with those found by other experimenters in τ' and η decay. However, the present data can be reasonably well fitted by a large range of parameters in the resonance models.

Since both the pion-pole model and the resonance hypothesis have "built-in" $T=1$ final states and consistency with the $\Delta T = \frac{1}{2}$ rule, the ratio of the linear terms in an expansion of the matrix element in the various $K \rightarrow 3\pi$ and $\eta \rightarrow 3\pi$ final states is fixed. Consequently, so long as the experimental data on the decay spectra can be well fitted by a linear function, the only available information will be on the validity of the $T=1$ final state and consistency with the $\Delta T = \frac{1}{2}$ rule. In fact, Prasad[58] has shown explicitly that the linear matrix element squared is compatible with a resonant π-π phase in the $T=0$ channel with roughly the same resonance parameters as those needed in the Brown and Singer theory. The answer to the question of the validity of one or the other of the models will have to wait until the higher order terms in the expansion become statistically significant.

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[58] R. Prasad, Nuovo Cimento 35, 682 (1965).