element, we have

$$\gamma_P(t)=2c\alpha_P'(t)(\bar{t}-t)\frac{Q_{\alpha_P(t)}[1+56/(\bar{t}-4)]}{[(\bar{t}-4)/4]^{\alpha_P(t)+1}}. \quad (44)$$

It will be seen that the prediction is well satisfied except for $t \geqslant 0$.

$\mathrm{Re}\alpha(t)$ has its maximum at $t \approx 20\,m_\pi^2$, though we have not traced the fall of $\mathrm{Re}\alpha(t)$ in Fig. 1, since, because of the large imaginary parts of α and D, it is not correct to identify the second zero of $\mathrm{Re}D$ with the returning trajectory. From (33) we can see that if $\rho_\alpha(t)$ has its main weight in the upper part of the strip one would not expect this maximum to occur for $t<s_1/2$. Our present calculation appears to emphasize the region of the double spectral function just above threshold, so that our results cease to be correct as we enter the resonance region.

We conclude that it may be possible to "bootstrap" trajectories with some hope of obtaining the physical parameters for $t<0$, when all the trajectories are included, but there is no sign that we shall be able to obtain the correct particle masses and widths. It is likely that the presence of competing channels is important for higher angular momenta, and this possibility is being examined.[13] Also it may be necessary to iterate the potential[14] in order to obtain a better approximation to the double spectral functions.

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We would like to thank Professor G. F. Chew for innumerable discussions and suggestions throughout the course of this work. The hospitality of the Lawrence Radiation Laboratory is gratefully acknowledged.

[13] Shu-Yuan Chiu (private communication).

[14] This procedure has been discussed by B. H. Bransden *et al.*, Nuovo Cimento 30, 207 (1963), whose results are very encouraging.

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Polarization of a Decay Particle in a Two-Step Process: Application to $K^-+p \rightarrow \pi^0+\Sigma^0$, $\Sigma^0 \rightarrow \gamma+\Lambda$†‡

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The phenomenology of two-step processes of the type $A+B \rightarrow C+D$, $D \rightarrow E+F$ is studied for the particular case when among the final particles only F is observed. Formulas convenient for the computation of the polarization of F in terms of the parameters describing the production process are presented, and the connection between the polarization of F and that of D, when D is not observed, is clarified. Numerical results are obtained for the angular dependence of the Λ polarization in the process $K^-+p \rightarrow \pi^0+\Sigma^0$, $\Sigma^0 \rightarrow \gamma+\Lambda$ at a variety of incident energies.

1. INTRODUCTION

PARTICLES with spin are frequently polarized when produced in elementary-particle reactions. As is well known, measurement of this polarization provides restrictions on the values of parameters, e.g., phase shifts, used to describe the matrix element for the production process. If the particle is unstable, the polarization may often be measured from the angular distribution of the decay products and sometimes from the polarization of one of the decay particles.

Special circumstances prevail if the produced particle is the Σ^0 hyperon, e.g., in the reaction

$$K^-+p \rightarrow \pi^0+\Sigma^0. \quad (1.1)$$

The electromagnetic decay of the Σ^0, via

$$\Sigma^0 \rightarrow \gamma+\Lambda, \quad (1.2)$$

involves two more neutral particles, and of the three final particles π^0, γ, and Λ, usually only the Λ is de-

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tected, in a hydrogen bubble chamber, via the weak decay

$$\Lambda \to \pi^- + p. \quad (1.3)$$

Thus, neither the polarization nor the angular distribution of the Σ^0 is directly measurable, and the question arises as to how the observation of the decay products of the Λ may be used to gain information on the production process.

The purpose of this paper is to study aspects of the phenomenology of two-step processes of the type $A+B \to C+D$, $D \to E+F$, for the particular case when only particle F is observed. In Sec. 2 we present formulas convenient for the computation of the polarization of F in terms of the parameters describing the production process. In Sec. 3 we attempt to clarify the connection between the polarizations of F and that of D when D is not observed, a subject on which some confusion has existed in the past. Numerical results are obtained for the specific case of reactions (1.1) and (1.2) for $\bar{\eta}_\Lambda(\theta)$, the energy-averaged polarization of the Λ, for a variety of incident K-meson energies. Section 4 contains a brief concluding discussion and summary. Some calculational details are relegated to an Appendix.

2. GENERAL FORMULAS FOR $s_F(p_F)$

Consider a two-body reaction

$$A+B \to C+D, \quad (2.1)$$

followed by the decay

$$D \to E+F. \quad (2.2)$$

We assume, for the sake of simplicity of notation in this section, that particles A, C, and E have spin 0, and that particles B, D, and F have spin $\frac{1}{2}$; it will be easy to generalize the results. We are then interested in computing the four-vector polarization[1] s_F of particle F, as a function of its momentum p_F, when the momenta p_C, p_D, and p_E are not observed. Now, for fixed p_A and p_B, the process (2.1) will give rise to particles D with polarization s_D depending on p_D:$s_D = s_D(p_D)$. The decay of a beam of D particles with momentum p_D and polarization s_D will yield F particles with polarization s_F' depending on p_F as well as on p_D:$s_F' = s_F'(p_F,p_D)$. It is intuitively clear that $s_F(p_F)$ will be a weighted average of $s_F'(p_F,p_D)$, with contributions from all kinematically allowed values of p_D. We first indicate the steps leading to the general expression (2.8) for $s_F(p_F)$, which has the expected form. We then obtain the simplified formula (2.13) which holds in the c.m. system of (2.1).

Let M_1 and M_2 denote the Feynman amplitudes for the processes (2.1) and (2.2), respectively. Then we may write

$$M_1 = \bar{u}_D O_1 u_B$$

and

$$M_2 = \bar{u}_F O_2 u_D,$$

where O_1 and O_2 are formal scalars under proper Lorentz transformations, constructed from the relevant four-momenta and Dirac matrices. The symbols u_B, u_D, and u_F denote Dirac spinors, with $u_B = u(p_B,t_B)$, etc., where $u(p,t)$ is defined by

$$\Lambda u(p,t) = u(p,t), \quad i\gamma_5 t u(p,t) = u(p,t).$$

Here Λ is the positive energy projection operator

$$\Lambda = (p+m)/2m$$

and t is the pseudo-four-vector labeling the spin state.

The differential cross section for reaction (2.1), when the spin of D is not observed, is given by

$$d\sigma_1 = (m_B/2\mathfrak{F})(\mathrm{Tr}\rho_1)D_1, \quad (2.3)$$

where

$$\mathfrak{F} = [(p_A \cdot p_B)^2 - m_A^2 m_B^2]^{1/2}$$

and D_1 is proportional to the conventional Lorentz-invariant volume element in phase space:

$$D_1 = (2\pi)^{-2}\delta(p_C + p_D - p_A - p_B)(m_D/2E_C E_D)\, d\mathbf{p}_C\, d\mathbf{p}_D.$$

The matrix ρ_1 in Eq. (2.3) is defined by

$$\rho_1 = \Lambda_D O_1 \rho_B O_1' \Lambda_D, \quad (2.4)$$

where ρ_B is the spin density matrix of the target particle B, related to the polarization four-vector s_B of B by

$$\rho_B = \tfrac{1}{2}(1 + i\gamma_5 s_B)\Lambda_B.$$

In Eq. (2.4) we have introduced the abbreviation

$$O' = \gamma^0 O^\dagger \gamma^0$$

for any 4×4 matrix O.[2] The polarization four-vector s_D of the outgoing particle D is given by

$$s_D{}^\mu = \mathrm{Tr}(-i\gamma_5\gamma^\mu \rho_1)/\mathrm{Tr}\rho_1,$$

and the corresponding spin density matrix ρ_D is

$$\rho_D = \tfrac{1}{2}(1 + i\gamma_5 s_D)\Lambda_D.$$

The decay (2.2) of D with momentum p_D and polarization s_D gives rise, with a rate $d\Gamma_2$, to an F particle with three-momentum in the interval $d\mathbf{p}_F$. The rate is given by

$$d\Gamma_2 = (m_D/E_D)\int' (\mathrm{Tr}\rho_2)D_2.$$

Here, and henceforth, a prime on the integral sign

[1] L. Michel and A. Wightman, Phys. Rev. 98, 1190 (1955).

[2] Our notation is the same as that of S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Row, Peterson and Company, Evanston, Illinois, 1961).

indicates that $\mathbf{p}_F$ is to be kept fixed during the integration. D_2 is defined by

$$D_2=(2\pi)^{-2}\delta(p_E+p_F-p_D)(m_F/2E_EE_F)\,d\mathbf{p}_E\,d\mathbf{p}_F\,,$$

and ρ_2 by

$$\rho_2=\Lambda_F O_2\rho_D O_2'\Lambda_F\,.$$

The polarization four-vector s_F' of F will depend on both p_F and p_D and is given by

$$s_F'^{\mu}=s_F'^{\mu}(p_F,p_D)=\mathrm{Tr}(-i\gamma_5\gamma^{\mu}\rho_2)/\mathrm{Tr}\rho_2\,.$$

We now consider the two-step process. The differential cross section for producing particle F in the interval $d\mathbf{p}_F$ when p_D is measured, is

$$d\sigma'=d\sigma_1(d\Gamma_2/\Gamma_2)\,,$$

where Γ_2 is the total decay rate, for D, moving with momentum p_D. Thus

$$\Gamma_2=(m_D/E_D)\Gamma_D\,,$$

where Γ_D is the decay rate in the rest frame of D. When D, or equivalently E, is not observed, $d\sigma'\to d\sigma$ with

$$d\sigma=(m_B/2\mathfrak{F}\Gamma_D)\int'(\mathrm{Tr}\rho_2)(\mathrm{Tr}\rho_1)D_2D_1\,. \tag{2.5}$$

The F-particle polarization four-vector $s_F=s_F(p_F)$, when D is not observed, may be obtained by computing $d\sigma(t_F)$, the cross section for producing F with momentum in the interval $d\mathbf{p}_F$ and the spin state labeled by the pseudo-four-vector t_F. The sum on the intermediate spin states of D must then be performed on the product M_2M_1 coherently, i.e., before taking the absolute square. The result is

$$d\sigma(t_F)=\tfrac{1}{2}(1-t_F\cdot s_F)d\sigma\,,$$

where

$$s_F=s_F(p_F)=\int' s_F'(p_F,p_D)\,\mathrm{Tr}\rho_2\,\mathrm{Tr}\rho_1 D_2D_1 \times\left[\int'\mathrm{Tr}\rho_2\,\mathrm{Tr}\rho_1D_2D_1\right]^{-1}. \tag{2.6}$$

Equations (2.5) and (2.6) have been written in such a way that they remain valid, for example, for a three-body decay of particle D, if the phase-space factor D_2 is replaced by the corresponding factor appropriate for three-body decay. If the unobserved decay particles have spin, $\mathrm{Tr}\rho_2$ should be replaced by

$$\sum\mathrm{Tr}\rho_2\,,$$

where $\sum$ denotes a sum over a complete set of orthogonal spin states for these particles.

For the two-body decay (2.2), we may infer from the definitions of ρ_2 and $d\Gamma_2$, that

$$\mathrm{Tr}\rho_2=(2\pi m_D\Gamma_D/m_Fq)W_F\,, \tag{2.7a}$$

where q is the magnitude of the F-particle three-momentum in the rest system of D and the invariant W_F is the angular distribution of F in the same system, normalized to unity for $s_D=0$. The general form of W_F is

$$W_F=1+\rho' s_D\cdot p_F\,, \tag{2.7b}$$

where ρ' is a constant. If parity is conserved in the decay, then necessarily $\rho'=0$, and $W_F=1$.

Eqs. (2.6) and (2.5) may be rewritten, using Eqs. (2.7) and (2.3) as

$$s_F(p_F)=\int' s_F'(p_F,p_D)W_Fd\sigma_1D_2\Big/\int' W_Fd\sigma_1D_2\,, \tag{2.8}$$

$$d\sigma=(2\pi m_D/m_Fq)\int' W_Fd\sigma_1D_2\,. \tag{2.9}$$

We now consider the simplification of Eqs. (2.8) and (2.9) in the c.m. system of reaction (2.1).

In the c.m. system of the production reaction, D_2 reduces to

$$(8\pi^2)^{-1}m_F|\mathbf{p}_F|E_E^{-1}\delta(E_D-E_E-E_F)dE_Fd\Omega_F\,,$$

so Eq. (2.8) becomes

$$s_F(p_F)=\int' s_F'(p_F,p_D)\sigma_1W_Fd\tau\Big/\int'\sigma_1W_Fd\tau\,, \tag{2.10}$$

where

$$d\tau=E_E^{-1}\delta(E_D-E_E-E_F)d\Omega_D\,, \tag{2.11}$$

and

$$\sigma_1\equiv(d\sigma_1/d\Omega_D)_{\mathrm{c.m.}}\,.$$

Equation (2.9) reduces similarly to

$$d\sigma/dE_Fd\Omega_F=(m_D|\mathbf{p}_F|/4\pi q)\int'\sigma_1W_Fd\tau\,. \tag{2.12}$$

To proceed further, we note that considerable simplification is achieved by introducing, as polar axis for the integration over the angles of particle D, not the axis defined by the incident beam $\hat{p}_A$ but rather the axis defined by the momentum of particle F. Thus, on defining χ and φ as polar and azimuthal angles of $\mathbf{p}_D$ with respect to $\mathbf{p}_F$, so that

$$d\Omega_D=\sin\chi\,d\chi\,d\varphi\,,$$

we may use the δ function in Eq. (2.11) to carry out the integration over χ. Since

$$E_E=(\mathbf{p}_F^2+\mathbf{p}_D^2-2|\mathbf{p}_F||\mathbf{p}_D|\cos\chi+m_E^2)^{1/2}\,,$$

so that for fixed $|\mathbf{p}_F|$ and $|\mathbf{p}_D|$

$$\sin\chi d\chi=E_EdE_E/(|\mathbf{p}_F||\mathbf{p}_D|)\,,$$

Eq. (2.11) becomes

$$d\tau=(|\mathbf{p}_F||\mathbf{p}_D|)^{-1}d\varphi\,.$$

Thus, Eqs. (2.10) and (2.12) become, respectively,

$$s_F(p_F)=\int_0^{2\pi} s_F'(p_F,p_D)\sigma_1 W_F\, d\varphi \Big/ \int_0^{2\pi} \sigma_1 W_F\, d\varphi \quad (2.13)$$

and

$$d\sigma/dE_F d\Omega_F=\frac{1}{2\pi(E_F^{(+)}-E_F^{(-)})}\int_0^{2\pi}\sigma_1 W_F\, d\varphi. \quad (2.14)$$

In the last equation we have introduced $E_F^{(+)}$ and $E_F^{(-)}$, the maximum and minimum values of E_F in the c.m. system, which are given by

$$E_F^{(\pm)}=[E_D(m_D^2+m_F^2-m_E^2)/2m_D^2]\pm[q|\mathbf{p}_D|/m_D].$$

3. APPLICATION TO $\Sigma^0\to\gamma+\Lambda$

Equation (2.13) of the preceding section may be used to obtain an explicit formula for $s_F(p_F)$ in terms of parameters describing the production reaction and parameters describing the relation between $s_F'(p_F,p_D)$ and $s_D(p_D)$, determined by the dynamics of the decay. The general analysis is given in the Appendix. Here we shall consider specifically the case of Σ^0 production followed by

$$\Sigma^0\to\gamma+\Lambda.$$

Then, for either relative parity of Σ^0 and Λ,

$$s_\Lambda'(p_\Lambda,p_\Sigma)=\alpha_0(-s_\Sigma\cdot p_\Lambda)\left(p_\Sigma-\frac{m_\Sigma^2+m_\Lambda^2}{2m_\Lambda^2}p_\Lambda\right), \quad (3.1)$$

with

$$\alpha_0=-4m_\Lambda m_\Sigma/(m_\Sigma^2-m_\Lambda^2)^2.$$

When the Σ^0 is not observed, we get, from Eqs. (2.13) and (3.1),

$$s_\Lambda^0(p_\Lambda)=0, \quad (3.2)$$

and

$$\mathbf{s}_\Lambda(p_\Lambda)=\eta_\Lambda\hat{x}, \quad (3.3a)$$

where the direction of polarization $\hat{x}$ is defined by

$$\hat{x}=\mathbf{p}_A\times\mathbf{p}_\Lambda/|\mathbf{p}_A\times\mathbf{p}_\Lambda|,$$

and the magnitude of polarization (up to a sign), is

$$\eta_\Lambda=2\pi\sin\theta_\Lambda\xi_\Lambda\Big/\int_0^{2\pi}\sigma_1(u)\, d\varphi, \quad (3.3b)$$

with

$$\xi_\Lambda=2(\pi)^{-1}\alpha_0|\mathbf{p}_\Lambda||\mathbf{p}_\Sigma|\sin^2\chi\int_0^{2\pi}\xi_\Sigma(u)\sin^2\varphi\, d\varphi. \quad (3.3c)$$

χ is the angle between $\mathbf{p}_\Sigma$ and $\mathbf{p}_\Lambda$, and

$$\cos\chi=(2E_\Sigma E_\Lambda-m_\Sigma^2-m_\Lambda^2)/2|\mathbf{p}_\Sigma||\mathbf{p}_\Lambda|.$$

The quantities σ_1, ξ_Σ describe the production cross section and polarization of Σ^0 in the c.m. system, with the magnitude of Σ^0 polarization $\eta_\Sigma=\sin\theta_\Sigma\xi_\Sigma,/\sigma_1(u)$, and

$$u=\cos\theta_\Sigma=\hat{p}_A\cdot\hat{p}_\Sigma.$$

Equations (3.2) and (3.3) describe the polarization of the Λ for a fixed p_Λ. Experimentally, it may be measured from the up-down asymmetry, with respect to the Λ production plane, in the distribution of π^- from the parity-nonconserving decay $\Lambda\to\pi^-+p$. If the number of events is small, a more accurately measurable quantity may be the up-down asymmetry, averaged over the range of energy of the Λ at the same angle, or perhaps averaged over both the energy and the angle, weighted with the corresponding cross section. In these cases the relevant theoretical average quantities are, respectively,

$$\bar{\eta}_\Lambda(\theta_\Lambda)=\int_{E_\Lambda^{(-)}}^{E_\Lambda^{(+)}}\eta_\Lambda\frac{d\sigma}{dE_\Lambda d\Omega_\Lambda}dE_\Lambda\Big/\int_{E_\Lambda^{(-)}}^{E_\Lambda^{(+)}}\frac{d\sigma}{dE_\Lambda d\Omega_\Lambda}dE_\Lambda \quad (3.4)$$

and

$$\langle\bar{\eta}_\Lambda\rangle=\int\int_{E_\Lambda^{(-)}}^{E_\Lambda^{(+)}}\eta_\Lambda\frac{d\sigma}{dE_\Lambda d\Omega_\Lambda}dE_\Lambda\, d\Omega_\Lambda\Big/\int\int_{E_\Lambda^{(-)}}^{E_\Lambda^{(+)}}\frac{d\sigma}{dE_\Lambda d\Omega_\Lambda}dE_\Lambda d\Omega_\Lambda. \quad (3.5)$$

If we now represent σ_1 and ξ_Σ as power series in u [or in Legendre polynomials $P_n(u)$] the integrations in (3.4) and (3.5) may be carried out. For example, with

$$\sigma_1=\sum_{n=0}^{\infty}a_n u^n, \quad (3.6a)$$

and

$$\xi_\Sigma=\sum_{n=0}^{\infty}b_n u^n, \quad (3.6b)$$

we obtain the following results for $\langle\bar{\eta}_{\Lambda},\rangle$ retaining only terms with $n\leq 3$:

$$\langle\bar{\eta}_\Lambda\rangle=\frac{\pi}{4}\left[C_1\left(b_0+\frac{b_2}{4}\right)-C_2\frac{b_2}{4}\right]\Big/\left(a_0+\frac{a_2}{3}\right), \quad (3.7)$$

where the coefficient C_1 is given by (we denote $|\mathbf{p}|$ by p, from now on)

$$C_1=K\left[\frac{m_\Sigma^4-m_\Lambda^4}{4}\left(\frac{3}{p_\Sigma^2}+\frac{2}{m_\Lambda^2}\right)-\left(\frac{m_\Sigma^4+m_\Lambda^4+4m_\Sigma^2m_\Lambda^2}{4p_\Sigma^2}+m_\Lambda^2\right)\ln\left(\frac{m_\Sigma}{m_\Lambda}\right)^2\right],\quad (p_\Sigma\geqslant p_c), \tag{3.8a}$$

and

$$C_1=K\left[\frac{E_\Sigma}{2m_\Lambda^2p_\Sigma}(m_\Sigma^4+m_\Lambda^4+4m_\Sigma^2m_\Lambda^2)-\left(\frac{m_\Sigma^4+m_\Lambda^4+4m_\Sigma^2m_\Lambda^2}{4p_\Sigma^2}+m_\Lambda^2\right)\ln\left(\frac{E_\Sigma+p_\Sigma}{E_\Sigma-p_\Sigma}\right)\right],\quad (p_\Sigma\leqslant p_c). \tag{3.8b}$$

Here $K=-2m_\Lambda^3m_\Sigma/(m_\Sigma^2-m_\Lambda^2)^3$, and p_c is the minimum value of p_Σ for which the decay Λ in $\Sigma^0\to\gamma+\Lambda$ all lie in a forward cone with respect to $\mathbf{p}_\Sigma$: $p_c=(m_\Sigma^2-m_\Lambda^2)/2m_\Lambda$. C_2 is related to C_1 via

$$\frac{C_2}{C_1}=\frac{1}{4}\left[\int_{E_\Lambda^{(-)}}^{E_\Lambda^{(+)}}p_\Lambda\sin^4\chi dE_\Lambda\bigg/\int_{E_\Lambda^{(-)}}^{E_\Lambda^{(+)}}p_\Lambda\sin^2\chi dE_\Lambda\right],$$

so that

$$C_2/C_1\leqslant\tfrac{1}{4}(\sin^2\chi)_{\max}.$$

The expressions (3.8a) and (3.8b) may be expanded as power series in the parameter $\delta=(m_\Sigma-m_\Lambda)/m_\Sigma$, and, for (3.8b), also in $p_\Sigma/E_\Sigma\delta(\leq 1)$. Thus,

$$C_1=-\tfrac{1}{3}[1-\delta^2((m_\Sigma^2/5p_\Sigma^2)+\tfrac{3}{10})]+O(\delta^3),\quad (p_\Sigma\geqslant p_c), \tag{3.9a}$$

and

$$C_1=-\tfrac{1}{3}(p_\Sigma/E_\Sigma\delta)[1-\tfrac{1}{5}(p_\Sigma/E_\Sigma\delta)^2+\tfrac{1}{2}\delta+\tfrac{3}{10}(p_\Sigma/E_\Sigma\delta)^2\delta]+O(\delta^2),\quad (p_\Sigma\leqslant p_c). \tag{3.9b}$$

Since, for the case at hand, $\delta\sim0.08$, we have, for $p_\Sigma/E_\Sigma\leqslant\delta$,

$$C\approx-\tfrac{1}{3}(p_\Sigma/E_\Sigma\delta);$$

for $p_\Sigma=p_c$, to 1% accuracy, $C_1=-4/15$; and, for $p_\Sigma\geqslant p_c$, $C_1\to-\frac{1}{3}$, e.g., for $p_\Sigma>2p_c$,

$$C_1\approx-\tfrac{1}{3}, \tag{3.10}$$

to within 5%.

To clarify some confusion which seems to have existed in the past on the relation between the Λ and the Σ^0 polarizations it is instructive to compare Eq. (3.7) with $\langle\eta_\Sigma\rangle$, the magnitude of the Σ^0 polarization averaged over θ_Σ:

$$\langle\eta_\Sigma\rangle=\int\eta_\Sigma\sigma_1d\Omega_\Sigma\bigg/\int\sigma_1d\Omega_\Sigma.$$

On using (3.6a) and (3.6b) with $n\leq3$, we get

$$\langle\eta_\Sigma\rangle=\frac{\pi}{4}\left(b_0+\frac{b_2}{4}\right)\bigg/\left(a_0+\frac{a_2}{3}\right).$$

Thus, if $C_2\ll C_1$, one finds

$$\langle\bar\eta_\Lambda\rangle/\langle\eta_\Sigma\rangle\approx C_1,$$

and if also $p_\Sigma\gg p_c$, from (3.10)

$$\langle\bar\eta_\Lambda\rangle\approx-\tfrac{1}{3}\langle\eta_\Sigma\rangle. \tag{3.11}$$

On the other hand, it is definitely not true that

$$\bar\eta_\Lambda(\theta)\approx-\tfrac{1}{3}\eta_\Sigma(\theta); \tag{3.12}$$

i.e., the (energy-averaged) polarization of the Λ emitted at angle θ, with respect to the incident beam, is in general not related to the polarization of a Σ^0 emitted at the same angle by an angle-independent factor. It is true (and a source of confusion) that, for *fixed* $\mathbf{p}_\Sigma$, the polarization of the Λ *averaged* over $\mathbf{p}_\Lambda$, is $-\frac{1}{3}$ times the polarization of the Σ^0, but this statement, which is independent of the production process, can not be used to infer (incorrect) Eq. (3.12) or Eq. (3.11), the validity of which depends on conditions such as $C_2\ll C_1$.

Of course, if Eq. (3.12) is used only to infer Eq. (3.11) by integration over θ, no appreciable numerical error is ultimately made, provided that the conditions for the validity of (3.11) are satisfied. For the reaction $K^-+p\to\pi^0+\Sigma^0$ studied by Watson, Ferro-Luzzi, and Tripp[3] in the range $(p_K)_{\rm lab}=300$ to 500 MeV/c, one finds $C_2/C_1\sim1\%$, and $p_\Sigma>p_c\sim80$ MeV/c. Using (3.9a) we find less than 5% deviation from (3.11) in this region.

To illustrate the use of Eq. (3.4) we have computed $\bar\eta_\Lambda(\theta)$ for $(p_K)_{\rm lab}=350$, 400, and 450 MeV/c using the results of the phenomenological analysis of Watson *et al.* Figure 1 shows $\bar\eta_\Lambda(\theta)$ versus θ, for solutions I and V of these authors, corresponding respectively to even and odd relative Σ-Λ parity. [The Σ-Λ parity is of course now known to be definitely even[4]; an odd-parity solution has been included to indicate the sensitivity of $\bar\eta_\Lambda(\theta)$ to the assumed parity.] It is clear that the measurement of $\bar\eta_\Lambda(\theta)$ can provide important restrictions on any phase-shift analysis of the initial reaction and can more or less replace direct measurement of $\eta_{\Sigma^0}(\theta)$. Of course, if enough events are available still more information concerning the production reaction would in principle be gained from a determination of $\eta_\Lambda(\theta,E_\Lambda)$. We hope that the results of this section will prove useful in the phenomenological analysis of Σ^0 production reactions.

[3] M. B. Watson, M. Ferro-Luzzi, and R. D. Tripp, Phys. Rev. **131**, 2248 (1963).

[4] H. Courant *et al.*, Phys. Rev. Letters **10**, 409 (1963).

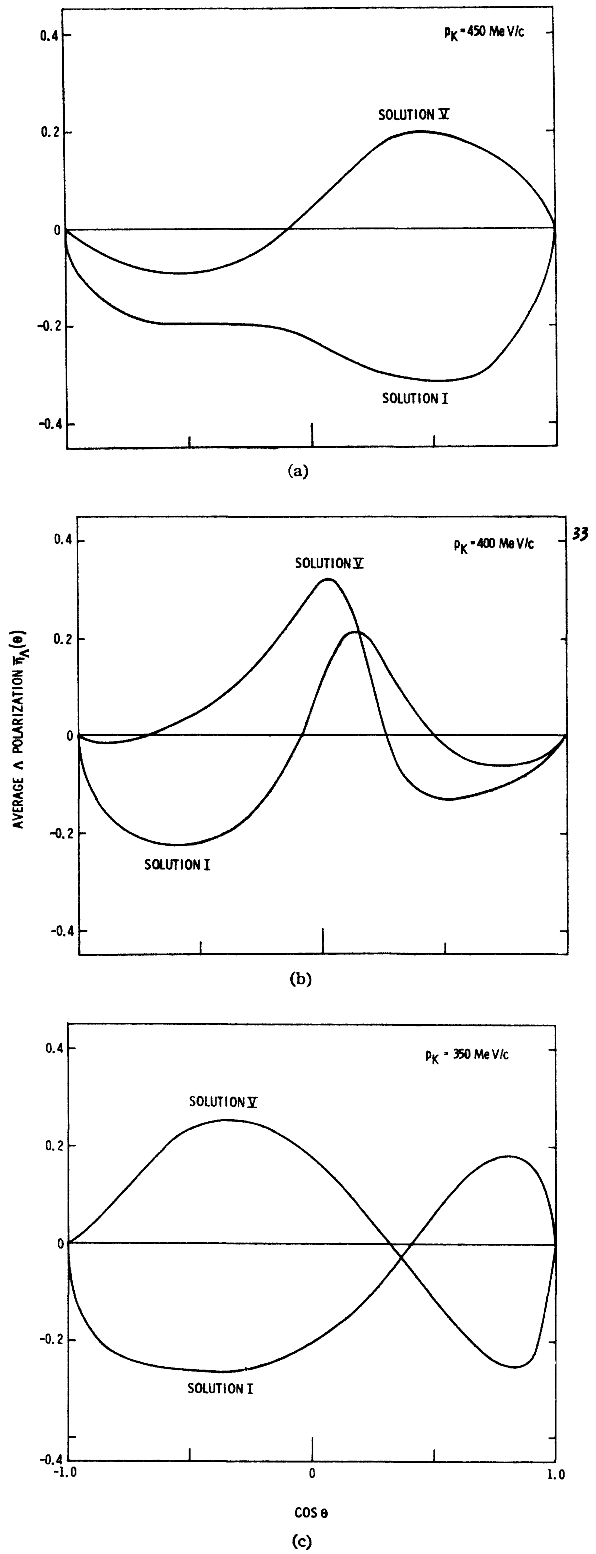

FIG. 1. The energy-averaged Λ polarization $\bar{\eta}_\Lambda(\theta)$ versus $\cos\theta$, in the K^--p c.m. system, computed at $(p_K)_{\text{lab}}=450$, 400, and 350 MeV/c. Solutions I and V refer to two choices of phase shifts obtained by Watson *et al.* (Ref. 3).

4. SUMMARY AND DISCUSSION

In Sec. 2 a two-step process of the type $A+B\to C+D$, $D\to E+F$ was studied in some detail for the case of spin-0 A, C, and E particles and spin-$\frac{1}{2}$ B, D, and F particles. The four-vector polarization $s_F(p_F)$ of particle F, when none of the particles C, D, or E is observed, was expressed as a covariant integral, Eq. (2.8). It should be emphasized that Eq. (2.8) is valid whatever the spins of the other particles may be provided that:

(i) $d\sigma_1$ is interpreted as the differential cross section for producing particle D in the element of solid angle $d\Omega_D$ with particles A and B in given states of polarization;

(ii) $s_F(p_F,p_D)$ is the polarization four-vector of F resulting from the decay of D, with momentum p_D, and in the state of polarization (statistical spin state) arising from the production process; and

(iii) W_F is the angular distribution of the F particle resulting from the same decay as seen in the rest system of the D particle.

If the spin of F is greater than one-half, $s_F(p_F)$ and $s_F'(p_F,p_D)$ in Eq. (2.8) may be replaced by the corresponding polarization tensors of higher order. The simple modifications necessary when D decays into, say, three particles, were also indicated in Sec. 2. For a two-particle decay, considerable simplification was achieved by evaluating $s_F(p_F)$ in the c.m. system of the production reaction and choosing as integration variable the azimuthal angle ϕ of p_D with p_F as polar axis. The resulting equations [Eqs. (2.13) and (2.14)] are especially convenient for calculational purposes. General formulas expressing $s_F(p_F)$ in terms of the parameters of the production process, when F and D have spin $\frac{1}{2}$, are given in the Appendix.

The case of Σ^0 production followed by $\Sigma^0\to\gamma+\Lambda$ was considered in detail in Sec. 3. It was shown that although the ubiquitous factor $-\frac{1}{3}$ does not relate $\bar{\eta}_\Lambda(\theta)$ to $\eta_\Sigma(\theta)$, the relation $\langle\bar{\eta}_\Lambda\rangle=-\frac{1}{3}\langle\eta_\Sigma\rangle$ is approximately correct for $p_\Sigma>(m_\Sigma-m_\Lambda)/c\approx 75$ MeV/c, mainly as a result of the relatively small value of $\delta=(m_\Sigma-m_\Lambda)/m_\Sigma$.

Measurement of $\bar{\eta}_\Lambda(\theta)$ will provide additional restrictions on any proposed phase-shift analysis of the reactions $K^-+p\to\pi^\pm+\Sigma^\mp$, related to $K^-+p\to\pi^0+\Sigma^0$ by charge independence. The predicted values of $\bar{\eta}_\Lambda(\theta)$ for $p_K=350$, 400, and 450 MeV/c, using solutions I and V of Watson *et al.*, are shown in Fig. 1.

In conclusion, it should be pointed out that Eqs. (2.13) or (3.4) may be useful for two-step reactions in which more than one of the final momenta can be measured, e.g., if charged particles (or resonances) are produced, but the number of events is small. A summation over the other momenta is then advantageous for improving the statistics.

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APPENDIX: DEPENDENCE OF $s_F(p_F)$ ON PARAMETERS OF THE PRODUCTION PROCESS

Consider a two-body reaction, involving particles of arbitrary spin,

$$A+B\rightarrow C+D\,, \tag{A1}$$

but with the initial particles unpolarized. The c.m. differential cross section σ_1 then depends only on the total c.m. energy W and on the c.m. scattering angle θ_D, and is conventionally written, at not too large W, as a power series (in practice, a polynomial) in $\cos\theta_D$, i.e.,

$$\sigma_1=\sigma_1(u)=\sum_{n=0}^{\infty}a_n u^n\,, \tag{A2}$$

where

$$u\equiv\cos\theta_D=\hat{p}_A\cdot\hat{p}_D\,,$$

and the a_n are functions of W only. If particle D has spin $\frac{1}{2}$, and if the reaction (A1) conserves parity, the four-vector polarization s_D has, in the c.m. system, necessarily the form

$$s_D=(0,\mathbf{s}_D)\,,$$

with the three-vector $\mathbf{s}_D$ perpendicular to the production plane. Thus $\mathbf{s}_D=\eta_D\hat{w}$, where η_D is the magnitude of the polarization (up to a sign) and $\hat{w}=(\mathbf{p}_A\times\mathbf{p}_D)/|\mathbf{p}_A\times\mathbf{p}_D|$. η_D is generally written in the form

$$\eta_D=\sin\theta_D\xi_D(u)/\sigma_1(u) \tag{A3}$$

with $\xi_D(u)$ assumed expandable as a power series in u,

$$\xi_D(u)=\sum_{n=0}^{\infty}b_n u^n\,,$$

and the b_n dependent on W only.

In order to exhibit explicitly the dependence on $\{a_n,b_n\}$ of the effective polarization $s_F(p_F)$ of the spin-$\frac{1}{2}$ particle F, arising from the decay $D\rightarrow E+F$, we consider first, in part (a), the general form of $s_F'(p_F,p_D)$, the polarization of F when p_D is measured. In part (b), this form is combined with the results of Sec. 2 to obtain an explicit formula for $s_F(p_F)$, in the c.m. system of the reaction (A1), valid for both of the two possibilities for the spin of E (0 or 1). In part (c) the result of (b) is applied to the case where E is a photon, relevant for Σ^0 production, followed by the decay $\Sigma^0\rightarrow\gamma+\Lambda$.

(a) General Form of $s_F'(p_F,p_D)$

From the definition of s_F' in Sec. 2 it follows that s_F' must transform as a four-vector under proper Lorentz transformation, and satisfy the condition

$$p_F\cdot s_F'=0\,. \tag{A4}$$

When the spin, if any, of particle E is not observed, the only available four-vectors are p_F, p_D, and the D-particle polarization four-vector s_D. The only nonvanishing and nonconstant scalar product which can be formed from these four-vectors is $p_F\cdot s_D$. However, since s_F' is the ratio of two linear inhomogeneous functions of s_D, using Eqs. (2.7a) and (2.7b) for the denominator in the expression for s_F, we see that the most general form of s_F' may be taken as

$$s_F'(p_F,p_D)=(1+\rho' s_D\cdot p_F)^{-1}[(s_D\cdot p_F)(\alpha p_D+\beta p_F) +\delta s_D+\alpha' p_D+\beta' p_F]\,, \tag{A5}$$

where α, β, δ and α', β', ρ' are constants whose value depends on the dynamics of the decay. Since $s_D\cdot p_F$ is a continuous variable, Eq. (A4) implies the pair of equations

$$\alpha p_F\cdot p_D+\beta m_F^2+\delta=0\,,$$

and

$$\alpha' p_F\cdot p_D+\beta' m_F^2=0\,,$$

relating α, β, δ and α', β', respectively.

If parity is conserved in the decay s_F' must change sign with a change in sign of s_D, so that then $\alpha'=\beta'=\rho'=0$. If the decay is electromagnetic, with E a photon, then one finds $\delta=0$ also, so that Eq. (A5) reduces to

$$s_F'(p_F,p_D)=\alpha(s_D\cdot p_F)(p_D-\lambda p_F)\,,$$

where

$$\lambda=(m_F^2+m_D^2)/2m_F^2\,. \tag{A6}$$

(b) Explicit Form of $s_F(p_F)$

It is convenient to define the azimuth ϕ of $\hat{p}_D$ with respect to $\hat{p}_F$ in such a way that $\phi=0$ when $\hat{p}_D$ lies in the plane containing $\mathbf{p}_F$ and $\mathbf{p}_A$ and $\theta_D>\theta_F$, with

$$\cos\theta_F=\hat{p}_F\cdot\hat{p}_A\,.$$

On defining a unit vector $\hat{x}$, perpendicular to the F particle-production plane, by

$$\hat{x}=(\mathbf{p}_A\times\mathbf{p}_F)/|\mathbf{p}_A\times\mathbf{p}_F|\,,$$

and a unit vector $\hat{y}$, perpendicular to p_F but in the production plane, by $\hat{y}=(\mathbf{p}_F\times\hat{x})/|\mathbf{p}_F\times\hat{x}|$, it follows that

$$\hat{p}_D=(\sin\chi\,\sin\phi)\hat{x}-(\sin\chi\,\cos\phi)\hat{y}+\cos\chi\hat{p}_F\,, \tag{A7}$$

and

$$\hat{w}=[(\cos\chi\,\sin\theta_F+\sin\chi\,\cos\theta_F\,\cos\phi)\hat{x} +(\sin\chi\,\sin\phi)\hat{p}_A\times\hat{x}]/\sin\theta_D\,. \tag{A8}$$

We now assume explicitly that σ_1 and ξ_D depend only on $u=\cos\theta_D$, as is necessarily the case when the initial particles are unpolarized. Then

$$\int_0^{2\pi}(\sigma_1 \text{ or } \xi_D)\sin\phi\, d\phi=0\,, \tag{A9}$$

so that on substitution into Eq. (2.13) we find, using Eqs. (A7), (A8), (A9), and (2.7b), that

$$s_F{}^0=\alpha' E_D+\beta' E_F\,, \tag{A10a}$$

and

$$\begin{aligned}\mathbf{s}_F=M_0{}^{-1}[\alpha|\mathbf{p}_F|\,|\mathbf{p}_D|\sin^2\chi\sin\theta_F(L_0-L_2)\\ +\delta(\cos\chi\sin\theta_F\, L_0+\sin\chi\cos\theta_F\, L_1)]\hat{x}\\ +M_0{}^{-1}[-\alpha'|\mathbf{p}_D|\sin\chi\, M_1]\hat{y}\\ +[\alpha'|\mathbf{p}_D|\cos\chi+\beta'|\mathbf{p}_F|]\hat{p}_F\,.\end{aligned} \tag{A10b}$$

Here

$$L_n=\frac{1}{2\pi}\int_0^{2\pi}\xi_D(u)\cos^n\phi\, d\phi\,, \tag{A11a}$$

$$M_n=\frac{1}{2\pi}\int_0^{2\pi}\sigma_1(u)\cos^n\phi\, d\phi\,, \tag{A11b}$$

and the variable u is related to ϕ by

$$u=\cos\chi\cos\theta_F-\sin\chi\sin\theta_F\cos\phi\,.$$

On substituting the expansions (A2) and (A3) into Eqs. (A11a) and (A11b), one gets

$$L_n=\sum_{m=0}^{\infty}b_m K_{mn}\,,\quad M_n=\sum_{m=0}^{\infty}a_m K_{mn}\,,$$

where

$$K_{mn}=\frac{1}{2\pi}\int_0^{2\pi}u^m\cos^n\phi\, d\phi\,.$$

On expansion of u^m by the binomial theorem one finds that

$$K_{mn}=(-1)^n\sum_{j=0}^{m}(\cos\theta_F\cos\chi)^{m-j}(\sin\theta_F\sin\chi)^j K_{mn}{}^{(j)}\,,$$

where

$$\begin{aligned}K_{mn}{}^{(j)}&=[m!/j!(m-j)!][(j+n-1)!!/(j+n)!!]\,, && (j+n \text{ even})\,;\\ &=0\,, && (j+n \text{ odd})\,.\end{aligned}$$

For even n, the function K_{mn} is a polynomial of order m in $\cos\theta_F$, as well as in $\cos\chi$.

The differential production cross section, Eq. (2.14), becomes, in the present notation,

$$d\sigma/dE_F d\Omega_F=(E_F{}^{(+)}-E_F{}^{(-)})^{-1}M_0\,.$$

We now consider in more detail the case where E is a photon.

(c) Application: $D\to\gamma+F$

The most general form of O_2 (Sec. 2) when particle E is a photon is, for a parity-conserving electromagnetic decay, and for, say, even relative parity of D and F, $O_2=$ (const) $\boldsymbol{k}\,\boldsymbol{\epsilon}$, where $k=p_D-p_F$ is the photon momentum and ϵ is the four-vector photon polarization, with $\epsilon\cdot k=0$. From the definition of s_F' one then obtains, on calculation of the traces, summed over the γ polarization, the result[5,6]

$$s_F'(p_F,p_D)=\alpha_0(s_D\cdot p_F)(p_D-\lambda p_F)\,, \tag{A12}$$

where λ is defined by Eq. (A6) and $\alpha_0=-4m_Dm_F/(m_D{}^2-m_F{}^2)^2$. The same result holds for odd relative parity of D and F, in which case $O_2\to\gamma_5 O_2$.

Equation (A5) therefore holds with $\alpha=\alpha_0$, $\beta=-\lambda\alpha_0$, and $\delta=\alpha'=\beta'=\rho'=0$, so that from Eqs. (A10a) and (A10b) one obtains $s_F{}^0(p_F)=0$ and $\mathbf{s}_F(p_F)=\eta_F\hat{x}$. Here η_F, the magnitude of the polarization, is given by $\eta_F=\sin\theta_F\xi_F/M_0$ and $\xi_F=\alpha_0|\mathbf{p}_F|\,|\mathbf{p}_D|\sin^2\chi(L_0-L_2)$, with[7]

$$\cos\chi=(2E_FE_D-m_F{}^2-m_D{}^2)/2|\mathbf{p}_F|\,|\mathbf{p}_D|\,.$$

These results give Eqs. (3.2) and (3.3) of the text.

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[5] In agreement with the result of L. Michel and H. Rouhaninejad, Phys. Rev. **122**, 242 (1961). [Equation (A.21) of this reference is incorrect since the kinematical and dynamical weight factors present in our Eq. (2.8) are missing.]

[6] In the rest system of the D particle Eq. (A12) reduces to

$$\mathbf{s}_F'=-K_+(\mathbf{s}_D\cdot\hat{q})\hat{q},\quad s'_F{}^0=-K_-\mathbf{s}_D\cdot\hat{q},$$

where $\hat{q}$ is a unit vector in the direction of the F particle momentum and $K_\pm=(m_D{}^2\pm m_F{}^2)/2m_Dm_F$. Unless $m_F=m_D$, $\mathbf{s}_F'$ differs from the three-vector polarization $\mathbf{P}$ obtained from a noncovariant treatment [R. Gatto, Phys. Rev. **109**, 610 (1958)] which is $\mathbf{P}=-(\mathbf{s}_D\cdot\hat{q})\hat{q}$. However, the (invariant) degree of polarization $(-s_F{}^2)^{1/2}$ is equal to $|\mathbf{s}_D\cdot\hat{q}|$, which coincides with $|\mathbf{P}|$.

[7] When $m_E\neq 0$, the numerator in the expression for $\cos\chi$ contains an additional $m_E{}^2$ term.