Correlation Functions for Coherent Fields*

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A factorization condition which must be satisfied by the first n correlation functions for the electromagnetic field operators has been used to define nth-order coherence. The first-order coherence condition has been shown to imply maximum fringe contrast in interference patterns. In the present paper we investigate the mathematical consequences of assuming the condition for maximum fringe contrast. By considering the correlation functions as scalar products and formulating rigorous inequalities for them we are able to show that the assumed condition in turn implies factorization of the first-order correlation function. By extending the same methods we are able to show that all of the higher order correlation functions factorize into forms similar to those required for full coherence, but differing from them through the inclusion of a sequence of constant numerical factors. These coefficients are shown to furnish a convenient description of the higher-order coherence properties of the field. Their values are presented for some typical examples. We derive a number of inequalities satisfied by the coefficients for the case of fields which possess positive-definite weight functions in the P representation. Some inequalities obeyed by the correlation functions for such fields are derived as well.

I. INTRODUCTION

TO secure a complete description of the coherence properties of an electromagnetic field it is useful to distinguish between various orders of coherence. A field which possesses mth-order coherence, for example, will exhibit particularly simple properties in measurements which detect average mth powers of field intensities or m-fold products of them. From a historic standpoint nearly all of the measurements which could be carried out in physical optics were, until recently, just measurements of quantities proportional to the average light intensity. It is natural therefore that the most familiar meaning of optical coherence, the one which describes the intensity fringes seen in a multitude of optical experiments, corresponds only to the case of first-order coherence, $m=1$.

Averages of nonlinear functions of the intensity, on the other hand, are measured either implicitly or explicitly by a number of techniques which have recently been introduced in optical experiments. Individual moments of the intensity distribution may be measured, for example, in photon-coincidence-counting experiments, or by making use of nonlinear media, while implicit measurements of the full set of moments may be made by determining sufficiently accurately the statistical distribution of photons detected by a single counter. The higher order coherence properties of the field furnish a natural basis for describing the results of such experiments.

The precise definition of the different orders of coherence is best stated in terms of a set of quantum-mechanical correlation functions for the electromagnetic field. These functions are defined as ensemble averages of normally ordered products of equal numbers of photon creation and annihilation operators. Their definition is presented in further detail in Sec. II together with a review of some of their elementary properties.

The most convenient definition of mth-order coherence, from a mathematical standpoint, takes it to correspond to a simple factorization property of the correlation functions of order up to and including m. In Ref. 1, where this definition was introduced, it was also noted that a necessary condition for mth-order coherence can be stated in terms of the absolute values of the correlation functions of order $\leq m$. For the case of first-order coherence the latter condition corresponds to the requirement of maximal contrast in the interference fringes which could be formed by superposing the fields occurring at different space-time points. In Secs. III and IV we shall show that this set of conditions is also sufficient to secure mth-order coherence. It may thus be used as an equivalent definition of mth-order coherence.

We show further that for a field which has precise first-order coherence, all of the higher order correlation functions reduce to a factorized form. The factorized correlation functions differ from the ones which would be required for coherence to all orders only by a set of real multiplicative constants. These constants furnish an extremely convenient description of the higher order coherence properties of a field which has first-order coherence. They are measurable through their proportionality to the probability per unit [time]^m for detecting m photons in coincidence by means of m ideal photodetectors. The values the constants take on are illustrated for the cases of several special fields.

In Sec. V we derive a number of more specific results for a class of quantum fields which are generated by radiation sources whose behavior may be described as predetermined. These fields are characterized by the fact that their density operator possesses a positive-definite weight function $P(\{\alpha_j\})$ in the representation which is diagonal in the eigenstates of the annihilation

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[1] R. J. Glauber, Phys. Rev. 130, 2529 (1963).

operators.[2] For this class of fields we show that the set of coefficients we have mentioned earlier forms a monotonically increasing sequence, and we derive some rigorous inequalities governing its rate of increase. We derive in addition certain upper and lower bounds for the higher order correlation functions. As a last result we show that for this class of fields the combination of first- and second-order coherence is sufficient to assure coherence to all orders.

The results of Sec. V are derived only for fields with positive-definite P functions. Those are, in fact, precisely the quantum fields which may be described in a natural way as possessing classical analogs. For the case of each of the relations derived in Sec. V we give examples which show that these relations do not hold for quantum fields of more general type. The analysis thus serves to illustrate the physical meaning of the fact that a much greater variety of states is available to quantized fields than to classical ones.

II. CORRELATION FUNCTIONS AND COHERENCE CONDITIONS

We begin the description of field correlations by introducing the familiar operator representation for the quantized electric-field vector

$$\mathbf{E}(\mathbf{r},t)=i\sum_k(\tfrac{1}{2}\hbar\omega_k)^{1/2}\times\{a_k\mathbf{u}_k(\mathbf{r})e^{-i\omega_kt}-a_k^\dagger\mathbf{u}_k^*(\mathbf{r})e^{i\omega_kt}\}. \quad (2.1)$$

In this expression the index k labels the normal modes of the field. Their number is denumerable if we think of the field as enclosed in a finite volume. The functions $\mathbf{u}_k(\mathbf{r})$ form an orthonormal set of vector mode functions and the ω_k are the corresponding frequencies. The operators a_k and $a_k^\dagger$ are the photon annihilation and creation operators for the kth mode. They obey the familiar commutation relations

$$[a_k,a_{k'}]=[a_k^\dagger,a_{k'}^\dagger]=0\,,$$
$$[a_k,a_{k'}^\dagger]=\delta_{kk'}. \quad (2.2)$$

The field operator (2.1) consists of a positive-frequency part

$$\mathbf{E}^{(+)}(\mathbf{r},t)=i\sum_k(\tfrac{1}{2}\hbar\omega_k)^{1/2}a_k\mathbf{u}_k(\mathbf{r})e^{-i\omega_kt} \quad (2.3)$$

and a negative-frequency part $\mathbf{E}^{(-)}(\mathbf{r},t)$ which is its Hermitian adjoint. These complex field operators describe the absorption and emission, respectively, of a single photon at the point $\mathbf{r}$ and time t. Their commutators are easily found from those of the a_k and $a_k^\dagger$.

The state of the field is described by a state vector, denoted by $|i\rangle$, or more generally by a density operator $\rho=\{|i\rangle\langle i|\}_{\mathrm{av}}$, where the average is taken over an ensemble appropriate to the way in which the system is prepared. We can express the expectation value of an operator $\mathcal{O}$ as $\{\langle i|\mathcal{O}|i\rangle\}_{\mathrm{av}}=\mathrm{tr}\{\rho\mathcal{O}\}$. The operator ρ is Hermitian; it is positive-definite and its trace is equal to unity.

In terms of these quantities the nth-order correlation function for the electromagnetic field is defined as

$$\begin{aligned}G^{(n)}{}_{\mu_1\cdots\mu_n,\mu_{n+1}\cdots\mu_{2n}}(\mathbf{r}_1t_1,\cdots\mathbf{r}_nt_n,\mathbf{r}_{n+1}t_{n+1}\cdots\mathbf{r}_{2n}t_{2n})\\=\mathrm{tr}\{\rho E_{\mu_1}{}^{(-)}(\mathbf{r}_1,t_1)\cdots E_{\mu_n}{}^{(-)}(\mathbf{r}_n,t_n)\\\times E_{\mu_{n+1}}{}^{(+)}(\mathbf{r}_{n+1},t_{n+1})\cdots E_{\mu_{2n}}{}^{(+)}(\mathbf{r}_{2n}t_{2n})\}. \quad (2.4)\end{aligned}$$

To write this and other tensor functions of several space-time variables more compactly we shall introduce a simple abbreviation. We let the variable x_j stand for the combination of the arguments $\mathbf{r}_j$, t_j and the polarization index μ_j. Then the correlation function (2,4) becomes simply $G^{(n)}(x_1\cdots x_n,x_{n+1}\cdots x_{2n})$. More generally, any function in which p variables x occur is to be interpreted as possessing p vector indices when written out more explicitly.

The detailed ways in which the correlation functions may be measured have been discussed elsewhere.[1,3] For our present purposes it is sufficient to note that $G^{(1)}(x,x)$ is proportional to the average counting rate of an ideal photodetector recording photons with a specific polarization at a specific space-time point. The functions $G^{(n)}(x_1\cdots x_n,x_n\cdots x_1)$ are related in a similar way to the delayed n-fold coincidence rate in an experiment with n ideal counters.[1,3] The function $G^{(1)}$ with arguments $x_2\neq x_1$ furnishes a basis for the description of different kinds of interference experiments, and its higher order analogs can be used to describe combined interference and coincidence experiments.

A field is said to have mth-order coherence[1] if there exists a single function $\mathcal{E}(x)$ such that for all arguments x_j and for all $n\leq m$ the correlation functions factorize according to the scheme

$$G^{(n)}(x_1\cdots x_n,x_{n+}\cdots x_{2n})=\prod_{j=1}^{n}\mathcal{E}^*(x_j)\mathcal{E}(x_{j+n}). \quad (2.5)$$

It follows immediately from this definition that the absolute value of $G^{(n)}$ obeys the relation

$$|G^{(n)}(x_1\cdots x_n,x_{n+1}\cdots x_{2n})|^2=\prod_{j=1}^{2n}G^{(1)}(x_j,x_j). \quad (2.6)$$

That conditions of this type are necessary for coherence can also be understood from the physical meaning of the functions involved. For $n=1$, Eq. (2.6) contains the statement that the interference patterns obtained by superposing the fields from two different points, have the greatest possible contrast. The higher order conditions (2.6) relate similarly the quantities which are measured in combined interference and coincidence experiments, and are thus more directly accessible to

[2] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).

[3] R. J. Glauber, *Quantum Optics and Electronics, Les Houches 1964*, edited by C. deWitt, A. Blandin, and C. Cohen-Tannoudji (Gordon and Breach Science Publishers, Inc., New York, 1965), p. 63. Some analogous results have also been obtained by P. L. Kelley and W. H. Kleiner, Phys. Rev. **136**, A316 (1964).

experimental test than the factorization conditions (2.5). It is therefore interesting to examine whether the conditions (2.6) for all $n \leq m$ are also sufficient for mth-order coherence, i.e., whether they imply the mathematically more useful statement (2.5). In the next section we shall prove that this is indeed the case.

Similar problems have been considered by Parrent[4] and by Mandel and Wolf[5] for a classical theory of coherence. They treated only the first-order scalar case and discussed a correlation function defined as a time average rather than an instantaneous ensemble average. Their definitions thus limit their treatments to fields which are statistically stationary in time. The physical meanings of their definitions of coherence, which they restrict to quasimonochromatic fields, are substantially different from ours. We shall not go into these approaches here, but we may note that they also led to a factorization theorem for the first-order correlation function, which is similar in part to the one we find for $G^{(1)}$.

III. CORRELATION FUNCTIONS AS SCALAR PRODUCTS

The correlation functions defined in the last section are of the general form $\mathrm{tr}\{\rho A^\dagger B\}$, where A and B are certain products of annihilation operators. Since ρ is a positive definite Hermitian operator, we can show that the form $\mathrm{tr}\{\rho A^\dagger B\}$, which we denote for brevity by (A,B) fulfills the familiar axioms of a scalar product

$$(A, \lambda B+\mu C)=\lambda(A,B)+\mu(A,C)\,, \tag{3.1a}$$

$$(A,B)=(B,A)^*\,, \tag{3.1b}$$

$$(A,A)\geq 0\,. \tag{3.1c}$$

From these axioms, one easily derives[6] a generalization of Schwarz's inequality, which states that for any two operators A and B, we have

$$(A,A)(B,B)\geq |(A,B)|^2. \tag{3.2}$$

The only difference between Eqs. (3.1) and the usual definition of a scalar product is that we allow the possibility that $(A,A)=0$ for $A\neq 0$. As a consequence of Eq. (3.2), we see that, for an operator A with $(A,A)=0$ and an arbitrary operator B, we have $(A,B)=(B,A)=0$, or, more explicitly,

$$\mathrm{tr}\{\rho A^\dagger B\}=\mathrm{tr}\{\rho B^\dagger A\}=0\,.$$

If, in particular, the space in which the operators are defined has a denumerable basis $|q_n\rangle$, $n=1, 2, 3\cdots$, we can choose for B the operator $|q_n\rangle\langle q_m|$. Then it follows that the nm matrix element of $A\rho$ and of $\rho A^\dagger$ must vanish for all n and m, a condition which implies the operator relations

$$A\rho=\rho A^\dagger=0\,. \tag{3.3}$$

This result can be applied to the case in which the two members of the inequality (3.2) become equal. If A and B are operators which satisfy the relation

$$(A,A)(B,B)=|(A,B)|^2 \tag{3.4}$$

and $(B,B)\neq 0$, we clearly have

$$\left(A-\frac{(B,A)}{(B,B)}B,\; A-\frac{(B,A)}{(B,B)}B\right)=0\,.$$

Application of Eq. (3.3) then shows that the density operator obeys the identities

$$\left[A-\frac{(B,A)}{(B,B)}B\right]\rho=\rho\left[A^\dagger-\frac{(A,B)}{(B,B)}B^\dagger\right]=0\,. \tag{3.5}$$

If, for example, we let $A=E^{(+)}(x_1)$ and $B=E^{(+)}(x_2)$, we see that the scalar product (A,B) is simply the first-order correlation function $G^{(1)}(x_1,x_2)$. The Schwarz inequality (3.2) for this case is

$$G^{(1)}(x_1,x_1)G^{(1)}(x_2,x_2)\geq |G^{(1)}(x_1,x_2)|^2. \tag{3.6}$$

The condition that the two members of this inequality be equal,

$$G^{(1)}(x_1,x_1)G^{(1)}(x_2,x_2)=|G^{(1)}(x_1,x_2)|^2, \tag{3.7}$$

is the condition for maximum fringe contrast noted in Eq. (2.6), for $n=1$. The restriction that it imposes on the density operator may be found by noting that Eq. (3.7) takes the form of Eq. (3.4). Hence, if we choose x_0 to be a coordinate for which

$$\mathrm{tr}\{\rho E^{(-)}(x_0)E^{(+)}(x_0)\}=G^{(1)}(x_0,x_0)\neq 0 \tag{3.8}$$

and let $x_2=x_0$, we find from Eq. (3.5) the relations

$$E^{(+)}(x_1)\rho=\frac{G^{(1)}(x_0,x_1)}{G^{(1)}(x_0,x_0)}E^{(+)}(x_0)\rho\,, \tag{3.9a}$$

$$\rho E^{(-)}(x_1)=\frac{G^{(1)}(x_1,x_0)}{G^{(1)}(x_0,x_0)}\rho E^{(-)}(x_0)\,. \tag{3.9b}$$

These relations must hold as identities for all x_1; they imply rigorous restrictions on the density operator which will be discussed further in a forthcoming paper.[7] For the present, however, we will confine ourselves to discussing the restrictions they imply upon the form of the correlation functions.

If we apply the identities (3.9a, 3.9b) to the definition

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[4] G. B. Parrent, J. Opt. Soc. Am. **49**, 787 (1959). For some related results see also C. L. Mehta, E. Wolf, and A. P. Balachandran (to be published).

[5] L. Mandel and E. Wolf, J. Opt. Soc. Am. **51**, 815 (1961).

[6] See, e.g., R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience Publishers, Inc., New York, 1953), Vol. I, p. 2. For the case that A and B are products of photon annihilation operators, (the case to which we will apply the theorem) a proof of (3.2) is contained in the Appendix of Ref. 1.

[7] U. M. Titulaer and R. J. Glauber (to be published).

of $G^{(1)}(x_1,x_2)$ we find the relation

$$\mathrm{tr}\{\rho E^{(-)}(x_1)E^{(+)}(x_2)\}=\frac{G^{(1)}(x_1,x_0)G^{(1)}(x_0,x_2)}{[G^{(1)}(x_0,x_0)]^2}\times\mathrm{tr}\{\rho E^{(-)}(x_0)E^{(+)}(x_0)\}.$$

The correlation function, in other words, obeys the functional equation

$$G^{(1)}(x_1,x_2)=\frac{G^{(1)}(x_1,x_0)G^{(1)}(x_0,x_2)}{G^{(1)}(x_0,x_0)} \tag{3.10}$$

for all x_1 and x_2 and all x_0 for which $G^{(1)}(x_0,x_0)\neq 0$.

Let us now define the complex function $\mathcal{E}(x)$ as

$$\mathcal{E}(x)=G^{(1)}(x_0,x)[G^{(1)}(x_0,x_0)]^{-1/2}. \tag{3.11}$$

With this definition we can formulate our result as follows: There exists a complex function $\mathcal{E}(x)$ such that for all x_1 and x_2 we have

$$G^{(1)}(x_1,x_2)=\mathcal{E}^*(x_1)\mathcal{E}(x_2), \tag{3.12}$$

which is exactly the condition (2.5) for first-order coherence. Since it is obvious that this factorization condition in turn implies the condition (3.7) we see that the two conditions are equivalent.

The definition (3.11) of $\mathcal{E}(x)$ would seem to imply a dependence of the way in which the form (3.12) factorizes on the arbitrary choice of the reference point x_0. We shall show, however, that if condition (3.12) holds for all pairs of arguments this dependence can only be a trivial one. Let us suppose that $G^{(1)}(x_1,x_2)$ has a second factorized form in which $\mathcal{E}(x)$ is replaced by $\mathcal{E}'(x)$. Then we have the relations (valid for all x_1 and x_2)

$$\mathcal{E}'^*(x_1)\mathcal{E}'(x_2)=\mathcal{E}^*(x_1)\mathcal{E}(x_2)$$

or

$$\mathcal{E}'^*(x_1)/\mathcal{E}^*(x_1)=\mathcal{E}(x_2)/\mathcal{E}'(x_2).$$

From the second form of this identity we see that there must exist a constant λ such that $\mathcal{E}'(x)=\lambda\mathcal{E}(x)$. Furthermore, from the first form of the identity we see that $|\lambda|^2=1$. For first-order coherent fields a change of the point x_0 can thus only correspond to multiplying $\mathcal{E}(x)$ by a phase factor. Such a phase factor clearly cancels in the calculation of any correlation function.

IV. APPLICATION TO HIGHER ORDER CORRELATION FUNCTIONS

Although Eq. (3.7) is a condition imposed only on the first-order correlation function, its consequences include a remarkable sequence of identities which must be obeyed by the higher order correlation functions as well. These identities allow us to reduce the $G^{(n)}$ for arbitrary arguments to a standard form. To derive those identities we begin by letting x_0 again be a point for which $G^{(1)}(x_0,x_0)\neq 0$. We then observe that the operators $E^{(+)}(x_j)$ all commute with one another, as do the $E^{(-)}(x_j)$. If we now apply each of the identities (3.9a) and (3.9b) n times to the definition of the nth-order correlation function given by Eq. (2.4), we find

$$\mathrm{tr}\{\rho E^{(-)}(x_1)\cdots E^{(-)}(x_n)E^{(+)}(x_{n+1})\cdots E^{(+)}(x_{2n})\}=\prod_{j=1}^{n}\frac{G^{(1)}(x_j,x_0)}{G^{(1)}(x_0,x_0)}\frac{G^{(1)}(x_0,x_{j+n})}{G^{(1)}(x_0,x_0)}\times\mathrm{tr}\{\rho[E^{(-)}(x_0)]^n[E^{(+)}(x_0)]^n\}.$$

By using the definition (3.11) of the field $\mathcal{E}(x)$ we can reformulate this result as

$$G^{(n)}(x_1\cdots x_n,x_{n+1}\cdots x_{2n})=g_n\prod_j \mathcal{E}^*(x_j)\mathcal{E}(x_{j+n}) \tag{4.1}$$

with

$$g_n=G^{(n)}(x_0\cdots x_0,x_0\cdots x_0)[G^{(1)}(x_0,x_0)]^{-n}. \tag{4.2}$$

The quantities g_n may be regarded as constants in Eq. (4.1) since they are independent of $x_1\cdots x_{2n}$. It is easy to show that the g_n cannot actually depend on the choice of x_0 either. To see this we need only note that $G^{(n)}$ is independent of x_0 and recall that in the last section we showed that products of the fields $\mathcal{E}(x)$ such as the one occurring in Eq. (4.1) are independent of x_0 as well. The g_n are simply a set of constants determined only by the state of the field; it is evident from Eq. (4.2) that they are real and non-negative.

We have shown that for any field which possesses first-order coherence, the higher order correlation functions must factorize into the forms given by Eq. (4.1). These forms are quite similar in structure to those which define higher-order coherence [cf. Eq. (2.5)], and differ from them only through the inclusion of the factors g_n. It is clear, from the assumption of first-order coherence, e.g., from Eq. (3.12), that $g_1=1$, but the g_n for $n\neq 1$ only assume the value unity for special choices of fields. If, for example, the field is one for which the conditions (2.6) on the absolute value of $G^{(n)}$ are fulfilled for $n\leq m$, then we see that $|g_n|^2=1$ for $n\leq m$. Since the g_n are real and positive we must in fact have $g_n=1$ for $n\leq m$. The conditions (2.6), in other words, are sufficient to require that the correlation functions fall into precisely the form needed for mth-order coherence. The case of full coherence corresponds, of course, to $g_n=1$ for all n.

It is possible at present to generate electromagnetic fields which possess the property of first-order coherence to an excellent approximation, i.e., fields for which the factorization condition (3.12) holds for quite large space-time separations of x_1 relative to x_2. Since for such fields the constants g_n should play an important role in the description of higher order coherence properties, we shall make a number of comments on the values they may take on.

From our definition of g_n and the discussions in Refs. 1 and 3 it is clear that g_n is proportional to the probability per unit [time]^n of detecting n photons with an idealized photodetector. If the density operator is such that the number of photons in the field cannot exceed some value M, then, of course, $g_n=0$ for $n\geq M$. In particular, if exactly M photons are present in a single mode of the field, we easily find from the commutation rules of the mode amplitudes that

$$g_n=M\,[M^n(M-n)!]^{-1}\quad\text{for}\quad n\leq M. \tag{4.3}$$

A particularly important case for which the coefficients g_n are known is formed by the fields generated by chaotic sources, a class which includes thermal ones. For such fields, it is shown in Ref. 2 that

$$g_n=n!. \tag{4.4}$$

In the next section we shall demonstrate that for a category of fields which may be thought of as generated by classical sources, the coefficients g_n always form a monotonically increasing sequence.

We conclude this section with an observation about photon coincidence experiments such as those performed by Hanbury Brown and Twiss. When the separation of the two counters used in such experiments is quite small compared to the coherence length of the field (and when their relative delay time is small compared to the coherence time) the number g_2-1 furnishes a measure[1,3] of the difference between the observed and the accidental coincidence rates. The quantity analogous to g_2-1 in classical electromagnetic theory is proportional to the variance of the energy density of the field and is intrinsically positive. From the preceding discussion, however, it is clear that g_2-1 is by no means necessarily positive [e.g., for the case (4.3) it is not]. The Hanbury Brown–Twiss effect assumes the particular form observed for natural light sources because of the particular statistical mixtures of quantum states which such sources tend to produce. Artificial sources could in principle produce fields with coincidence rates smaller than the product of the individual counting rates.

V. FIELDS WITH POSITIVE-DEFINITE *P* FUNCTIONS

In this section we derive some relations obeyed by the correlation functions for a special class of fields corresponding to sources with predetermined behavior. In order to give a mathematical characterization of these fields we first consider a particular set of states, the coherent states $|\{\alpha_k\}\rangle$, which have been discussed in Ref. 2. These are simultaneous right eigenstates of the full set of annihilation operators and have complex eigenvalues α_k corresponding to the a_k. Thus they are also right eigenstates of the positive frequency part of the field at any point x. The latter property is expressed by the relations

$$E^{(+)}(x)|\{\alpha_k\}\rangle=\mathcal{E}(x,\{\alpha_k\})|\{\alpha_k\}\rangle, \tag{5.1a}$$

$$\langle\{\alpha_k\}|E^{(-)}(x)=\mathcal{E}^*(x,\{\alpha_k\})\langle\{\alpha_k\}| \tag{5.1b}$$

with

$$\mathcal{E}_\mu(\mathbf{r},t,\{\alpha_k\})=i\textstyle\sum_k(\frac{1}{2}\hbar\omega_k)^{1/2}\alpha_k u_{k\mu}(\mathbf{r})e^{-i\omega_k t}.$$

The latter equation illustrates our convention regarding the meaning of the variable x.

We now consider fields which can be described by means of a density operator of the form

$$\rho=\int P(\{\alpha_k\})|\{\alpha_k\}\rangle\langle\{\alpha_k\}|\textstyle\prod_k d^2\alpha_k \tag{5.2}$$

with a positive-definite weight function $P(\{\alpha_k\})$ which has, at most, δ-type singularities. In this integral the differential $d^2\alpha_k$ stands for $d\,\mathrm{Re}(\alpha_k)d\,\mathrm{Im}(\alpha_k)$. Since all our integrals will be taken over the full set of amplitude variables $\{\alpha_k\}$ we shall simply write $\boldsymbol{\alpha}$ in the remainder of this section instead of $\{\alpha_k\}$ and $d^2\boldsymbol{\alpha}$ instead of $\prod_k d^2\alpha_k$. The function $P(\boldsymbol{\alpha})$ has to be normalized so that $\int P(\boldsymbol{\alpha})d^2\boldsymbol{\alpha}=1$. Density operators of the type (5.2) can be shown[2] to furnish a description appropriate for any field generated by a prescribed charge-current distribution. They also describe the light emitted by a completely chaotic source, a model appropriate for virtually all known natural light sources.

The representation (5.2) has the pre-eminent advantage of permitting us to express the correlation functions as integrals over $P(\boldsymbol{\alpha})$:

$$G^{(n)}(x_1\cdots x_{2n})=\int P(\boldsymbol{\alpha})\prod_{j=1}^{n}\mathcal{E}^*(x_j,\boldsymbol{\alpha})\mathcal{E}(x_{j+n},\boldsymbol{\alpha})d^2\boldsymbol{\alpha}. \tag{5.3}$$

This expression for $G^{(n)}$ as an integral, rather than the more general form of scalar product considered in Sec. III, makes it possible to derive a number of inequalities which need not hold for more general quantum mechanical fields.

One such inequality can be derived by considering real-valued functions $A(\boldsymbol{\alpha})$ and $B(\boldsymbol{\alpha})$ which satisfy the relation

$$[A(\boldsymbol{\alpha})-A(\boldsymbol{\beta})][B(\boldsymbol{\alpha})-B(\boldsymbol{\beta})]\geq 0 \tag{5.4}$$

for all $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. For such functions one can easily prove the following generalization of the Tchebycheff inequality [using positive definiteness and normalization of $P(\boldsymbol{\alpha})$][8]

$$\int P(\boldsymbol{\alpha})A(\boldsymbol{\alpha})B(\boldsymbol{\alpha})d^2\boldsymbol{\alpha} \geq \int P(\boldsymbol{\alpha})A(\boldsymbol{\alpha})d^2\boldsymbol{\alpha}\int P(\boldsymbol{\beta})B(\boldsymbol{\beta})d^2\boldsymbol{\beta}. \tag{5.5}$$

Condition (5.4) may be fulfilled by choosing $A(\boldsymbol{\alpha})$ and $B(\boldsymbol{\alpha})$ each to be a power of the same positive function $|\mathcal{E}(x,\boldsymbol{\alpha})|^2$. The integral of the nth power of this expression, taken over $P(\boldsymbol{\alpha})$, is just equal to the correlation function $G^{(n)}(x\cdots x)$, with all of its $2n$ arguments set

[8] G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities* (Cambridge University Press, Cambridge, 1952), 2nd ed., pp. 43, 168.

equal. Now if we let $A(\boldsymbol{\alpha})=|\mathcal{E}(x,\boldsymbol{\alpha})|^{2m}$ and $B(\boldsymbol{\alpha})=|\mathcal{E}(x,\boldsymbol{\alpha})|^{2(n-m)}$ where $n\geq m\geq 0$, the inequality (5.5) implies the relation

$$G^{(n)}(x\cdots x)\geq G^{(m)}(x\cdots x)G^{(n-m)}(x\cdots x). \quad (5.6)$$

If both members are divided by $[G^{(1)}(x,x)]^n$, we find the inequality

$$g_n\geq g_m g_{n-m} \quad (5.7)$$

for the coefficients g_n. If we let $m=1$, in particular, and recall that $g_1=1$, we see that

$$g_n\geq g_{n-1}, \quad (5.8)$$

or the constants g_n form a monotonically increasing sequence. We note in particular $g_2\geq 1$, which means that, for the class of fields under consideration, the coincidence rate in a Hanbury Brown–Twiss experiment always exceeds or equals the product of the individual counting rates as the separation of the detectors approaches zero.

Another sequence of inequalities can be derived from the Schwarz inequality expressed in the form which applies to integrals of complex functions,

$$\int P(\boldsymbol{\alpha})|A(\boldsymbol{\alpha})|^2 d^2\boldsymbol{\alpha}\int P(\boldsymbol{\beta})|B(\boldsymbol{\beta})|^2 d^2\boldsymbol{\beta} \geq \left|\int P(\boldsymbol{\alpha})A^*(\boldsymbol{\alpha})B(\boldsymbol{\alpha})d^2\boldsymbol{\alpha}\right|^2. \quad (5.9)$$

If we now substitute $A(\boldsymbol{\alpha})=[\mathcal{E}(x,\boldsymbol{\alpha})]^{n-m}[G^{(1)}x,x)]^{-n}$ and $B(\boldsymbol{\alpha})=[\mathcal{E}^*(x,\boldsymbol{\alpha})]^m[\mathcal{E}(x,\boldsymbol{\alpha})]^n$ and recall the definition (4.2) of g_n, we see that

$$g_{n-m}g_{n+m}\geq g_n^2. \quad (5.10)$$

This inequality shows that

$$\tfrac{1}{2}(\ln g_{n+m}+\ln g_{n-m})\geq \ln g_n, \quad (5.11)$$

or $\ln g_n$ is a convex function[9] of n. The convexity property permits us to show that for $n\geq l\geq 0$ and $m\geq 0$ we have[9]

$$l\ln g_{n+m}+m\ln g_{n-l}\geq (l+m)\ln g_n. \quad (5.12)$$

If we let $l=n-1$ in this relation and write $p=n+m$, then by recalling that $g_1=1$ we find the inequality

$$g_p\geq g_n^{(p-1)/(n-1)} \quad \text{for} \quad p\geq n. \quad (5.13)$$

This relation sets lower limits to the rapidity with which the g_p increase. It shows furthermore that, for any field which does not possess second-order coherence, i.e., for which $g_2>1$, the g_p increase without bound as $p\to\infty$.

We turn next to the consideration of some bounds on the values of the correlation functions of the form $G^{(n)}(x_1\cdots x_n,x_n\cdots x_1)$, i.e., on the values of the n-fold coincidence counting rates. For this purpose we can use the inequality

$$\prod_{j=1}^{n}\int P(\boldsymbol{\alpha})|\mathcal{E}(x_j,\boldsymbol{\alpha})|^{2n}d^2\boldsymbol{\alpha}\geq\left[\int P(\boldsymbol{\alpha})\prod_{j=1}^{n}|\mathcal{E}(x_j,\boldsymbol{\alpha})|^2 d^2\boldsymbol{\alpha}\right]^n,$$

which is a simple consequence of the Hölder inequality,[10] as stated for integrals. When expressed in terms of correlation functions this relation becomes

$$\prod_{j=1}^{n}G^{(n)}(x_j\cdots x_j)\geq[G^{(n)}(x_1\cdots x_n,x_n\cdots x_1)]^n. \quad (5.14)$$

For fields with first-order coherence the inequality reduces to an equality.

A slightly more general inequality can be derived when we combine Eq. (5.14) with the inequality

$$G^{(n)}(x_1\cdots x_n,x_n\cdots x_1)G^{(n)}(x_{n+1}\cdots x_{2n},x_{2n}\cdots x_{n+1}) \geq |G^{(n)}(x_1\cdots x_{2n})|^2$$

which has been derived from the Schwarz inequality in Ref. 1. In this way we find

$$\prod_{j=1}^{2n}G^{(n)}(x_j\cdots x_j)\geq|G^{(n)}(x_1\cdots x_{2n})|^{2n}. \quad (5.15)$$

This relation leads, for example, to a simple property of linearly polarized fields which are invariant under space-time translations. For these fields the $G^{(n)}(x_j\cdots x_j)$ are independent of x_j as long as the vector index it specifies corresponds to the direction of polarization. The inequality (5.15) then implies for all x_j

$$G^{(n)}(x_j\cdots x_j)\geq|G^{(n)}(x_1\cdots x_{2n})|, \quad (5.16)$$

i.e., the absolute value of $G^{(n)}$ reaches its maximum when the $2n$ arguments x_j are all equal.

For correlation functions of even order we can also state a lower limit for the coincidence rate. This may be done by substituting in the Schwarz inequality, Eq. (5.9), the functions

$$A(\boldsymbol{\alpha})=1$$

and

$$B(\boldsymbol{\alpha})=\prod_{j=1}^{n}\mathcal{E}^*(x_j,\boldsymbol{\alpha})\mathcal{E}(x_{j+n},\boldsymbol{\alpha}).$$

We then find

$$G^{(2n)}(x_1\cdots x_{2n},x_{2n}\cdots x_1) \geq|G^{(n)}(x_1\cdots x_n,x_{n+1}\cdots x_{2n})|^2. \quad (5.17)$$

An interesting property of the fields under consideration is that the combination of first- and second-order coherence implies coherence to all orders. To show this we note that as a consequence of first-order coherence the correlation functions all factorize and we need only consider the coefficients g_n which characterize the field. Second-order coherence implies $g_2=1$, which in turn implies that $G^{(2)}(x_0,x_0,x_0,x_0)=[G^{(1)}(x_0,x_0)]^2$. The latter

[9] Reference 8, p. 70.

[10] Reference 8, p. 140.

relation is the statement

$$\int P(\boldsymbol{\alpha})|\mathcal{E}(x_0,\boldsymbol{\alpha})|^4 d^2\boldsymbol{\alpha}-[G^{(1)}(x_0,x_0)]^2$$
$$=\int P(\boldsymbol{\alpha})\{|\mathcal{E}(x_0,\boldsymbol{\alpha})|^4-[G^{(1)}(x_0,x_0)]^2\}d^2\boldsymbol{\alpha}$$
$$=0,$$

which may also be written in the form

$$\int P(\boldsymbol{\alpha})\{|\mathcal{E}(x_0,\boldsymbol{\alpha})|^2-G^{(1)}(x_0,x_0)\}^2 d^2\boldsymbol{\alpha}=0. \quad (5.18)$$

If we now define the scalar product of two functions $f(\boldsymbol{\alpha})$ and $g(\boldsymbol{\alpha})$ to be

$$(f(\boldsymbol{\alpha}),g(\boldsymbol{\alpha}))=\int P(\boldsymbol{\alpha})f^*(\boldsymbol{\alpha})g(\boldsymbol{\alpha})d^2\boldsymbol{\alpha},$$

then we see that Eq. (5.18) states that the norm of the function

$$f(\boldsymbol{\alpha})=|\mathcal{E}(x_0,\boldsymbol{\alpha})|^2-G^{(1)}(x_0,x_0)$$

vanishes. This is a situation we have already encountered in Sec. III. Application of the Schwarz inequality, Eq. (3.2), shows that the scalar product of any other function $g(\boldsymbol{\alpha})$ with it vanishes. We have shown, in other words, that

$$\int P(\boldsymbol{\alpha})g^*(\boldsymbol{\alpha})\{|\mathcal{E}(x_0,\boldsymbol{\alpha})|^2-G^{(1)}(x_0,x_0)\}d^2\boldsymbol{\alpha}=0,$$

or that $|\mathcal{E}(x_0,\boldsymbol{\alpha})|^2$ can be replaced by $G^{(1)}(x_0,x_0)$ in all integrals taken over the weight function $P(\boldsymbol{\alpha})$. In particular evaluation of the correlation function $G^{(n)}(x_0\cdots x_0)$ in this way yields

$$G^{(n)}(x_0\cdots x_0)=[G^{(1)}(x_0,x_0)]^n\int P(\boldsymbol{\alpha})d^2\boldsymbol{\alpha}$$
$$=[G^{(1)}(x_0,x_0)]^n. \quad (5.19)$$

By referring to the definition of the g_n, Eq. (4.2), we see that this relation implies $g_n=1$ for all n and thus full coherence of the field in question.

As a final remark we wish to emphasize once more that the results we have derived in this section are only valid for fields which have a positive-definite $P(\boldsymbol{\alpha})$ and need not hold for more general types of fields. It is not difficult, in fact, to find among the more general fields explicit counterexamples for each of the relations proved. We note first that the inequalities (5.8) and (5.10)–(5.13) fail to hold for fields with a finite number of photons present; this can be seen for example from Eq. (4.3) when only a single mode is occupied. For any n-photon state, the inequality (5.17) is clearly disobeyed, since the left-hand side is always zero while the right-hand side is not. The inequalities (5.14)–(5.16) cannot hold for a field that is constructed as the superposition of n one-photon wave packets which have no spatial overlap. Once again, for this case, the left-hand sides of the inequalities vanish while the right-hand sides can be made different from zero. Finally, if we consider the state $2^{-1/2}|\text{vac}\rangle+2^{-1/2}|2_k\rangle$, where $|2_k\rangle$ is a state with just two photons present in a particular mode k, then this state is easily shown to have first-order coherence, and further to have g_2 equal to unity, but nevertheless to be far from fully coherent since all the g_n for $n>2$ are equal to zero.

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