PHYSICAL REVIEW VOLUME 140, NUMBER 3B 8 NOVEMBER 1965

Some General Features of the Bootstrap Theory of Octet Enhancement*

ROGER F. DASHEN AND STEVEN C. FRAUTSCHI†
California Institute of Technology, Pasadena, California
(Received 28 June 1965)

Some general features of the boostrap theory of octet enhancement, which can be understood without detailed calculations, are discussed. These features include: (i) the connection of this theory to the vector-mixing theory of symmetry breaking advocated by Sakurai, and to the tadpole theory of Coleman and Glashow; (ii) an understanding of why it is representations of low multiplicity that are dynamically emphasized in symmetry breaking; (iii) a demonstration that the theory remains valid when a number of assumptions made in previous applications are dropped.

I. INTRODUCTION

RECENTLY, the bootstrap theory of octet enhancement was proposed[1–3] as a general approach to understanding the pattern of $SU(3)$ symmetry breaking. Dynamical calculations[3] indicated that the theory does work in detail for the best established bootstrap: The reciprocal bootstrap connecting the $J=\frac{1}{2}^+$ octet B and $J=\frac{3}{2}^+$ decimet Δ.

In the present paper we wish to point out some further general features of the bootstrap theory of octet enhancement which can be understood without detailed calculations. These features include the connection of this theory with various other approaches to symmetry breaking, an understanding of why it is representations of low multiplicity that are dynamically emphasized in symmetry breaking, and a demonstration that the theory remains valid when a number of assumptions made in our previous applications are dropped.

In Sec. II, the question is raised, why various approaches to symmetry breaking all lead to similar patterns of octet dominance. Specifically, we discuss the connection of our theory with the vector-mixing theory of Sakurai *et al.*[4,5] and the theory of Coleman and Glashow[6] which connects the symmetry-breaking pattern to 0^+ mesons. All of these approaches are capable of yielding universal predictions linking medium-strong, electromagnetic, and weak symmetry breaking. We find that the bootstrap theory of octet enhancement and the 0^+ meson theory of octet enhancement are definitely compatible—in fact, the latter implies the former, though the converse does not seem to be true. Sakurai's vector-mixing theory, on the other hand, is distinct from the other approaches.

In Sec. III, close relations are uncovered between the ordinary crossing matrix and the A matrix used in the bootstrap theory of octet enhancement. Starting from the known tendency of the ordinary crossing matrix to favor representations of low multiplicity,[7,8] one can then understand why the A matrix is largest for representations of low multiplicity in symmetry breaking. This connection fits in well with, although it does not entirely explain, the observation that low multiplicities dominate both in $SU(3)$ hadron states (**8** and **10** definitely observed thus far) and symmetry breaking (**8**). Presumably, our general approach could also be applied to $SU(6)$, and here again the tendency to low multiplicities is observed (states **35** and **56** definitely observed, symmetry breaking dominated by low representations such as **35**).

Finally, in Sec. IV, it is shown that the bootstrap theory of octet enhancement remains a valid approach when a number of assumptions made in our previous applications are dropped. Specifically, it is not necessary to assume a unique $SU(3)$-symmetric bootstrap solution for the starting point, or to assume rapidly converging dispersion integrals with no Castillejo-Dalitz-Dyson (CDD) parameters, or to ignore the effects of second and higher order in the mass and coupling shifts. The first two of these complications can be accommodated in the "driving term" of the theory, and the third provides an additional term. Section IV concludes with some discussion of why present methods are too crude to determine whether the driving term vanishes.

II. COMPARISON WITH OTHER APPROACHES

In the bootstrap theory of octet enhancement, it is convenient to start with $SU(3)$ symmetry for the strongly interacting particles. Linear deviations of the masses (δM) and couplings (δg) are then considered. These deviations may depend on terms extraneous to the strong interactions (for example, electromagnetic mass shifts depend on photon exchange); such terms are labeled driving terms D. But in bootstrap theory they must also depend on the hadron mass and coupling shifts δM and δg in a self-consistent way. Equations of type

$$\begin{aligned}\delta M_i &= \textstyle\sum_j A_{ij}^{MM}\delta M_j+\sum_j A_{ij}^{Mg}\delta g_j+D_i^M,\\ \delta g_i &= \textstyle\sum_j A_{ij}^{gM}\delta M_j+\sum_j A_{ij}^{gg}\delta g_j+D_i^g\end{aligned}\qquad(1)$$

* Work supported in part by the U. S. Atomic Energy Commission. Prepared under Contract AT(11-1)-68 for the San Francisco Operations Office, U. S. Atomic Energy Commission.
† Alfred P. Sloan Foundation Fellow.

[1] R. E. Cutkosky and P. Tarjanne, Phys. Rev. **132**, 1355 (1963).
[2] R. Dashen and S. Frautschi, Phys. Rev. Letters **13**, 497 (1964).
[3] R. Dashen and S. Frautschi, Phys. Rev. **137**, B1331 (1965).
[4] J. Sakurai, Phys. Rev. **132**, 434 (1963).
[5] L. Picasso, L. Radicati, J. Sakurai, and D. Zanello, Nuovo Cimento **37**, 187 (1965).
[6] S. Coleman and S. Glashow, Phys. Rev. **134**, B671 (1964).
[7] R. Capps, *Proceedings of the International Conference on High-Energy Physics, Dubna, 1964* (Atomizdat, Moscow, 1965).
[8] D. Neville, Phys. Rev. Letters **13**, 118 (1964).

result; here, A_{ij} is $SU(3)$ symmetric, the linear violations being confined to the factors δM, δg, and D. If A has an eigenvalue near unity, then the mass and coupling shifts associated with the eigenvector of this eigenvalue feed on themselves and are enhanced. Since A is independent of the perturbation, the enhanced eigenvector can apply universally to strong, electromagnetic, and weak (parity-conserving) symmetry breaking. For octet dominance, the eigenvalue near unity must correspond to octet symmetry breaking.

Now our estimates[3] of the A matrix connecting B and Δ mass shifts, which yielded eigenvalues of A near one for octet symmetry breaking, with associated eigenvectors in good agreement with the data, were made utilizing a specific S-matrix formalism.[9,10] Equation (1) is a general relation for linear symmetry breaking, however, and other methods may equally well be used to estimate A and D.[11] In particular, Eq. (1) does not imply that no elementary particles exist.

Since Eq. (1) always applies, we shall discuss the vector-mixing theory in terms of it, a procedure which will facilitate comparisons between vector-mixing theory and our bootstrap theory. To simplify the comparison (and also the subsequent comparison with Coleman-Glashow theory), we shall assume in the present section that no arbitrary subtraction parameters exist.[12] This restriction will be dropped in Sec. IV, where we shall see that the formalism on which the comparisons are based remains valid.

In vector-mixing theory, one begins by noticing that in the absence of $SU(3)$ symmetry breaking, the $J=1^-$ octet and singlet would be degenerate or nearly so. No other well-established multiplets with a common J^P have this property. It follows from elementary quantum mechanics that even a small symmetry-breaking matrix element, connecting the octet and singlet states, can lead to large mixing of the states since they are so close together.[13,14] In terms of Eq. (1), the coupling connecting the singlet and octet states (call it δg_{18}) tends to have both a large driving term (D_{18}) and large bootstrap effects (A_{18j}). The other coupling shifts and mass shifts then receive large contributions from $A_{i18}\delta g_{18}$. Since the octet-singlet mixing transforms like an octet, all the large terms it induces transform like octets. If there are no other large terms, the ratios of δg_i and δM_i to δg_{18} are simply A_{i18} and apply universally to strong, electromagnetic, and weak symmetry breaking[15] (recall that A is independent of the perturbation).

Thus the vector-mixing theory can lead to the same general conclusions as the bootstrap theory of octet enhancement.[16] How, then, can one distinguish between the two theories?

The two theories can be distinguished only by differences which appear in detailed calculations. As an example of such differences, the coupling δg_{18} is expected to have a dominant or at least an important effect on the B and Δ mass shifts in the vector-mixing approach, whereas other terms are found to dominate in the calculations of B and Δ mass shifts by the bootstrap approach.[2,3] In conclusion, everyone agrees that vector mixing exists, but not on whether the whole pattern of $SU(3)$ violations flows from it, and detailed calculations employing the bootstrap method suggest that other terms are at least as important as the vector mixing effects.

Coleman and Glashow[6] have proposed a quite different approach involving a 0^+ octet.[17] Technically, they use the tadpole method, which is similar to expressing the $SU(3)$ violation in terms of a spurion, assigning the spurion a variable mass m, writing a dispersion relation in the spurion mass,[18] and then evaluating the dispersion relation at the "physical value" $m=0$. For parity-conserving $SU(3)$ violations, the spurion has $J^P=0^+$, and if 0^+ hadrons of low mass exist they may dominate the dispersion relation just as the vector mesons dominate the dispersion relations for electromagnetic form factors. In this case, the $SU(3)$ violations represented by the spurion-hadron couplings will come

[9] R. Dashen and S. Frautschi, Phys. Rev. **135**, B1190 (1964).

[10] R. Dashen and S. Frautschi, Phys. Rev. **137**, B1318 (1965).

[11] For example, M. Suzuki, Progr. Theoret. Phys. (Kyoto) **31**, 222 (1964), has used the conventional self-mass bubble diagrams of field theory to study spontaneous mass splitting (i.e., calculate elements of A^{MM}) in the B octet.

[12] The assumption implies that the denominator function D does not increase like a positive power of energy at large energies, as explained in Sec. III of Ref. 3. This condition greatly improves the convergence of the dispersion integrals used in our method of calculating A_{ij}. The absence of subtraction parameters also relieves us from the necessity of considering point interactions among four particles, and thus makes possible unambiguous separation of the bootstrap into terms with exchange singularities in t or u and terms with poles in s, as implied in Fig. 1.

[13] That is,

$$\delta\psi_1=\frac{\langle 1|H_{\rm pert}|2\rangle}{E_2-E_1}\psi_2$$

is large if (E_2-E_1) is small enough.

[14] Of course, the same phenomenon must occur in our S-matrix formalism, and the details are as follows. Either the octet and singlet are initially degenerate or not. If not, Eq. (23) of Ref. 10 is appropriate. Here, δg_{18} can be written as $D_8^{-1}(W)$ times an integral, all evaluated at the singlet mass $W=M_1$. Since $D_8(W)$ has its zero at $W=M_8\approx M_1$, the factor $D_8^{-1}(M_1)$ makes δg_{18} large. If the octet and singlet are initially degenerate, on the other hand, Sec. III of Ref. 10 is appropriate. The perturbation must first be diagonalized, which in general leads to a large octet-singlet mixing.

[15] That this feature holds for the dominant terms in vector-mixing was first deduced from an analysis of self-energy diagrams by L. Picasso, L. Radicati, J. Sakurai, and D. Zanello, Nuovo Cimento **37**, 187 (1965).

[16] In fact, since vector mixing enhances the matrix elements A_{18j}, it might even give A an eigenvalue near unity. This possibility of a bootstrap effect in vector mixing was pointed out by J. J. Sakurai, Phys. Rev. Letters **9**, 472 (1962).

[17] These 0^+ bosons are not to be confused with the "Goldstone bosons," which might appear in a field theory with a degenerate vacuum.

[18] In the mathematics of field theory, the idea of dispersing in a spurion mass can be stated as follows. We assume a local Hamiltonian density $\mathcal{H}'(x)$ which describes the symmetry-breaking mechanism. The mass shift of particle a can then be obtained from the matrix element $\langle a|\mathcal{H}'(x)|a\rangle=\langle \bar{a}a|\mathcal{H}'(x)|0\rangle$. Now we imagine a scalar particle, the spurion, which couples weakly to the hadrons, and we take the vertex $\bar{a}a\to$ spurion to be $\int\langle\bar{a}a|\mathcal{H}'(x)|0\rangle e^{iq\cdot x}d^4x$, where q^ν is the spurion momentum. The dispersion relation for this vertex as a function of q^2 is what we mean by a "dispersion relation in the mass of a spurion."

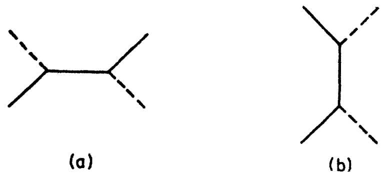

FIG. 1. (a) Typical exchange diagram. (b) Typical direct-channel pole diagram.

out proportional to the corresponding (0^+ hadron)-hadron couplings. Clearly, if the 0^+ hadrons form an octet, this theory enhances the *octet* symmetry violations, and does so in a universal fashion applicable to strong, electromagnetic, and (parity-conserving) weak symmetry breaking.

Technically, both a field theory of tadpoles and the similar method of dispersing in the spurion mass go off the mass shell, since they involve a transition from the vacuum or spurion state (physical mass necessarily zero) to a 0^+ state with nonzero mass. In the bootstrap theory of octet enhancement, by contrast, one works on the mass shell. We do not wish to dwell on this distinction here, but we *would* like to avoid the complications it may introduce into comparisons of the two theories of octet enhancement. Accordingly, we have recast the description of how a 0^+ octet can influence symmetry breaking into a form which stays on the mass shell, and which makes comparison with the bootstrap theory of octet enhancement easy.

Our on-the-mass-shell description of 0^+ octet emission is an application of a general formalism which will be described in detail elsewhere; for present purposes we shall merely outline the approach. We begin with an ordinary $SU(3)$-symmetric bootstrap, represented diagrammatically in Fig. 1 by a typical exchange diagram [Fig. 1(a)] which, with other exchange diagrams, provides a potential capable of producing bound states or resonances [Fig. 1(b)] in the direct channel. Now suppose the reaction represented by Fig. 1 proceeds with emission of a hypothetical, weakly interacting 0^+ particle. The particle is to be coupled in all possible ways to the set of diagrams 1(a) and 1(b), as in Figs. 2(a)–(f). Because of its assumed weak interaction, virtual emission and reabsorption of the 0^+ particle can be ignored.

The various couplings of the 0^+ to the exchange diagrams 1(a), portrayed in Figs. 2(a), (b), (c), can be represented by a vector δg^i, $i=1\cdots N$. The same couplings reappear in the direct channel diagrams, Figs. 2(d), (e), (f). The diagrams 2(a), (b), (c) constitute a perturbation (0^+ emission) on the original potential, while 2(d), (e), (f) represent the resulting perturbation on the original direct-channel pole term. The effect of the new couplings δg_j in the exchange channels on the new direct-channel couplings δg_i can be represented by

$$\delta g_i = X_{ij}\delta g_j. \tag{2}$$

If the set of couplings δg_i is a physically permissible perturbation on the strong interaction bootstrap, X must have an eigenvalue equal to one. This is by no means a trivial condition to satisfy; generally, X is a matrix with eigenvalues differing from one. Evidently, Eq. (2) can be satisfied, however, if our hypothetical 0^+ particle has the same *mass* as a 0^+ hadron, and the same *ratios* of couplings to the various hadrons as the 0^+ hadron (of course, we are taking the *over-all* coupling strength weak). If the hypothetical 0^+ has *nearly* the same mass and coupling ratios as a 0^+ hadron, X should have an eigenvalue *near* unity on continuity grounds.

Now what has all this to do with octet enhancement? Suppose an octet of 0^+ hadrons with low mass m_H does exist. Then the X matrix for a weakly interacting 0^+ has a unit eigenvalue at mass $m_{0^+}=m_H$. If m_H is sufficiently small, continuation of the hypothetical 0^+ mass to zero leaves X nearly unchanged; it retains an eigenvalue near unity, and the associated eigenvector still gives coupling ratios close to those of the 0^+ hadron. At this point the hypothetical 0^+, though massless, still carries off momentum and energy.

Next we take a second limit in which the 0^+ no longer carries off momentum or energy *in any Lorentz frame*.[19] We are interested in the case where X still has an eigenvalue near unity after the double limit is taken, and the associated eigenvector does not vanish. This eigenvector will give coupling ratios similar to the monopole couplings of the 0^+ hadron (in field theory, these are the couplings with no derivatives; all other 0^+ couplings vanish with the four-momentum).

At this point we make the desired contact with the problem of $SU(3)$ symmetry breaking in the original bootstrap, for emission of a 0^+ particle carrying no mass, energy, or charge is not observable. If the 0^+ couples like a neutral member of an octet, its emission is physically equivalent to spontaneous violation of $SU(3)$, occur-

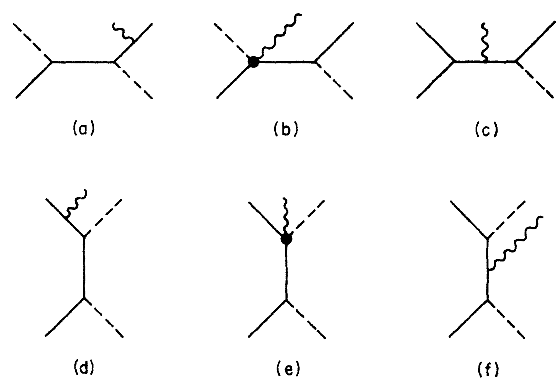

FIG. 2. Examples of diagrams in which a weakly interacting 0^+ particle (wavy line) couples to the bootstrap diagrams of Fig. 1. In the first three diagrams, the 0^+ couples to an: (a) external line, (b) vertex, and (c) internal line of an exchange diagram. In (d), (e), and (f) it makes corresponding couplings to a direct channel pole diagram.

[19] To define this limit more precisely, consider first the reaction $A+S\rightarrow B+C$, where A, B, and C are hadrons and S is a spurion. Define the Mandelstam variables $s=(p_A+p_S)^2$ and $t=(p_C-p_S)^2$. The limit we want is $s\rightarrow m_A^2$, $t\rightarrow m_C^2$, which implies $p_S=0$ as well as $m_S^2=0$. In a reaction involving four or more hadrons $i=1\cdots n$ plus the spurion, the corresponding limit is $s_i\equiv(p_i+p_S)^2\rightarrow m_i^2$.

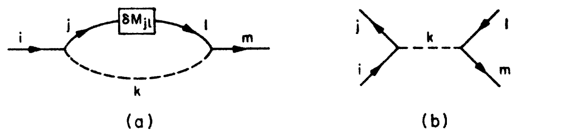

FIG. 3. (a) Diagram representing the group theoretical structure of $A_{im,jl}^{M(B)M(B\,\mathrm{ext})}$, the effect of a shift in external baryon mass on the baryon mass shift. The $J=\frac{1}{2}^+$ baryon octet B is represented by a solid line, the $J=0^-$ meson octet Π by a dashed line. (b) Diagram representing $V_{im,jl}$, the Π exchange potential in $B\bar{B}$ scattering.

ring with octet transformation properties but without 0^+ emission.

Now, consistent emission of weakly coupled 0^+ mesons requires X to have a unit eigenvalue, while spontaneous breakdown of $SU(3)$ in the linear approximation requires A to have a unit eigenvalue in the original bootstrap. More generally, one can show[20] that independently of whether 0^+ emission is consistent, there is an exact correspondence between terms of the X and A matrices in the limit of vanishing 0^+ mass and four-momentum: Figures 2(a) and (d) correspond to external mass terms in A, 2(c) to exchange mass terms in A, 2(b) to coupling shifts in the exchange, and 2(f) and (e) to mass and coupling shifts in the direct channel. Thus, in this limit, X and A have the same eigenvalues and eigenvectors. If an octet of light 0^+ hadrons exists, resulting in an eigenvalue of X near one in the limit of vanishing 0^+ mass and four-momentum, then A also exhibits an eigenvalue near one for octet symmetry violations.[21] In other words, if there are 0^+ mesons light enough to have a large effect on the pattern of symmetry breaking, the bootstrap theory of octet enhancement gives the pattern equally well; the two theories describe the same reality from different points of view.

It would be most interesting if one could turn the argument around, and deduce from a near octet instability (eigenvalue of A_8 near one) that an octet of 0^+ hadrons must exist. Unfortunately, we do not see how to do this. It is true that X and A share the same eigenvalues at vanishing 0^+ momentum and energy, and an eigenvalue of X near one is compatible with, indeed suggestive of, an eigenvalue passing through unity at some nearby point $m_{0^+}=m_H$, but we cannot predict for sure that m_H is small or even that this point exists. Moreover, a unit eigenvalue of X does not necessarily imply the existence of a hadron.

To summarize: Our version of the Coleman-Glashow theory, and the bootstrap theory of octet enhancement are compatible and related despite differences in appearance. The former implies the latter, and the latter at least raises the betting odds on the former.

[20] A proof and more precise statement of these relationships will appear in a separate paper.

[21] A connection between 0^+ emission and spontaneous symmetry breakdown has previously been established by M. Suzuki, Progr. Theoret. Phys. (Kyoto) **31**, 1073 (1964), who used a different method and worked to order g^2 in the strong coupling. Our present result holds to all orders in the strong coupling.

III. A RELATION BETWEEN THE A MATRIX AND THE ORDINARY CROSSING MATRIX

In the Coleman-Glashow view, octet enhancement is directly related to the existence of light 0^+ hadrons, which they postulate as a feature of the $SU(3)$-symmetric strong interactions. We have just seen that the Coleman-Glashow approach is related to the bootstrap theory of octet enhancement. Therefore, it is not surprising to find that a connection also exists between the later theory and $SU(3)$-symmetric strong interactions.

The connection is especially easy to see in the case of "external mass" contributions to the A matrix [Eq. (1)] of bootstrap theory. Consider the contribution to the baryon (B) mass shift from the external baryon mass shift in the ΠB bootstrap (where Π represents the octet of pseudoscalar mesons). The bootstrap calculation does not employ the bubble diagram of field theory for the mass shift, but the group theory factors are the same as in a bubble diagram,[1,10] so we can display the point we wish to make in the bubble diagram of Fig. 3(a). Here, $i=1\cdots 8$, $k=1\cdots 8$, etc., are $SU(3)$ indices. Letting g_{ijk} represent the coupling strength of $B_iB_j\Pi_k$, we can read the dependence on these indices from the diagram:

$$\delta M_{im}=C\sum_{jkl} g_{ijk}g_{lmk}\delta M_{jl}\,, \tag{3}$$

where only C depends on detailed dynamical considerations. Comparing Eqs. (3) and (1), we see that

$$A_{im,jl}^{M(B)M(B\,\mathrm{ext})}=C\sum\nolimits_k g_{ijk}g_{lmk}\,. \tag{4}$$

Now, in $SU(3)$-symmetric strong interactions, the potential for $B\bar{B}$ scattering due to Π exchange is given by [Fig. 3(b)]:

$$V_{im,jl}=C'\sum\nolimits_k g_{ijk}g_{lmk}\,. \tag{5}$$

Here, C' is a dynamical factor including the Π exchange pole, and the product of couplings is proportional to the crossing matrix. The point we wish to make is that $V_{im,jl}$ is proportional to $A^{M(B)M(B\,\mathrm{ext})}$.[22,23] Thus, $A^{M(B)M(B\,\mathrm{ext})}$

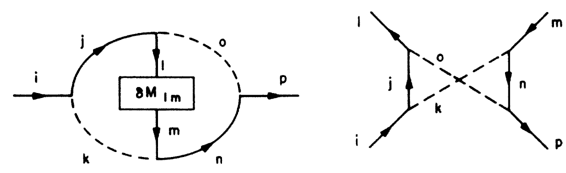

FIG. 4. (a) Diagram representing the group theoretical structure of $A_{ip,lm}^{M(B)M(B\,\mathrm{exch})}$, the effect of a shift in exchanged baryon mass on the baryon mass. (b) Diagram representing $V_{ip,lm}$, the 2Π exchange potential in $B\bar{B}$ scattering.

[22] The relation between the two cases can be seen topologically by cutting open Fig. 3(a) between j and l.

[23] I. Gerstein and M. Whippman (University of Pennsylvania report) have independently noticed the connection between elements of the A matrix and crossing matrix.

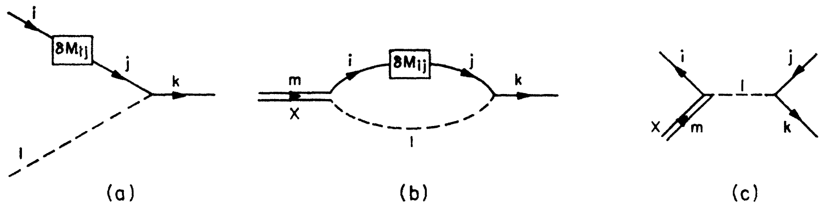

FIG. 5. Diagram representing the group theoretical structure of (a) $A_{ilk,ij}^{g(BB\Pi)M(B\,\mathrm{ext})}$, (b) $A_{mk,ij}^{g(X8)M(B\,\mathrm{ext})}$, a part of $A^{g(BB\Pi)M(B\,\mathrm{ext})}$, where $B_i\Pi_l$ is in the $SU(3)$ state X_m (X=1, 8, 8′, 10, $\overline{\mathbf{10}}$, 27). (c) $V_{mk,ij}$, the Π exchange potential for the (fictitious) reaction $X+\bar{B}\rightarrow B+\bar{B}$.

is large for just those states which have a large potential in $B\bar{B}$ scattering with Π exchange. To make this statement more useful, let us transform to the representation in terms of definite $SU(3)$ multiplets. It is well known, and follows from Eq. (5), that the potential connects the singlet state of *im* only to the singlet state of *jl*, **8** only to **8**, **27** only to **27**, that all **27** states have the *same* potential, and so forth. Thus the convenience of this representation: the potential can be characterized in terms of the few numbers V_1, $V_8(8_S\rightarrow 8_S)$, $V_8(8_S\rightarrow 8_A)$, $V_8(8_A\rightarrow 8_A)$, $V_{27}\cdots$. Similarly, the A matrix connects singlet mass shifts only to singlet shifts, **8** only to **8**, A_{27} is independent of *which* **27** state is shifted, and so forth. These properties of the A matrix are well known, but the connection to the crossing matrix throws a new light on why they have to be that way. Now from the proportionality between Eqs. (4) and (5), one deduces that

$$\frac{A_1}{V_1}=\frac{A_{27}}{V_{27}}=\frac{A_8(8_S\rightarrow 8_S)}{V_8(8_S\rightarrow 8_S)}=\frac{A_8(8_S\rightarrow 8_A)}{V_8(8_S\rightarrow 8_A)}\cdots. \quad (6)$$

If the potential is larger for **8** scattering than for **27** scattering, the corresponding element of A_8 is larger than A_{27}.

The reader can readily verify that any other external mass term in A is similarly proportional to a crossing matrix in some strong-interaction scattering process.[24] The proportionality constant in any specific case—even its sign—may be hard to obtain, but this information is not needed to see the pattern. The pattern for strong interactions is that the crossing matrix favors **8** channels over **27** channels and other higher channels. The empirical evidence for this is simply the predominance of octets among the observed low-mass states. Theoretical understanding of the pattern is furnished by the work of Capps[7] and Neville,[8] who show that for scattering of a given set of particles with a given exchange, the crossing matrix definitely and generally favors low-multiplicity states. From the close relation between crossing matrices and external mass contributions to the A matrix, it follows that the latter will generally be larger for octet mass shifts than for shifts with higher multiplicity.

The A matrix also contains "exchanged mass" terms and "coupling shift" terms. The group-theoretical content of an exchanged mass term can again be represented by a kind of bubble diagram[1,10]; in Fig. 4(a) we have diagrammed the contribution of the exchanged B mass shift to the B mass in a ΠB bootstrap. It has the structure

$$A_{ip,lm}^{M(B)M(B\,\mathrm{exch})}=C\sum_{jkno} g_{ijk}g_{jlo}g_{mnk}g_{npo}. \quad (7)$$

We compare Eq. (7) with the crossing matrix for a 2Π exchange contribution to the $B\bar{B}$ potential [Fig. 4(b)]:

$$V_{ip,lm}=C'\sum_{jkno} g_{ijk}g_{jlo}g_{mnk}g_{npo}. \quad (8)$$

Once again, the A matrix is seen to be proportional to a specific crossing matrix, so the previous arguments apply to exchange as well as external mass terms.

Turning to coupling shifts, one finds that the contribution of an "external B mass shift" to a coupling shift $\delta g^{BB\Pi}$, for example, has a group theoretical content represented by Fig. 5(a). It is useful to relabel the couplings δg_{ilk} in terms of $SU(3)$ multiplets, such as 8 for $k=1\cdots 8$ and X for il (X=1, 8, 8′, 10, $\overline{\mathbf{10}}$, 27).[25] The calculation of the relabeled coupling δg_{8X} has the same group theoretical content as a bubble diagram connecting B to a (fictitious) multiplet X [Fig. 5(b)], and one easily finds that the couplings in the bubble diagram are proportional to the crossing matrix of Fig. 5(c) for the reaction $X+\bar{B}\rightarrow B+\bar{B}$.

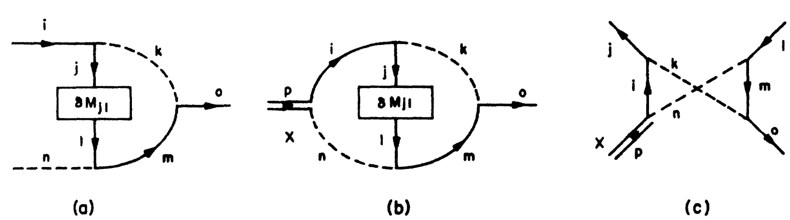

FIG. 6. Diagram representng the group theoretical structure of (a) $A_{ino,jl}^{g(BB\Pi)M(B\,\mathrm{exch})}$, (b) $A_{po,jl}^{g(X8)M(B\,\mathrm{exch})}$, a part of $A^{g(BB\Pi)\,MB(\mathrm{exch})}$, where $B_i\Pi_n$ is in the $SU(3)$ state X_p, (c) $V_{po,jl}$, the 2Π exchange potential for the reaction $X+\bar{B}\rightarrow B+\bar{B}$.

[24] For example, $A^{M(B)M(\Pi\,\mathrm{ext})}$ in the $B\Pi$ bootstrap model of baryons, is proportional to the crossing matrix for $\Pi+\Pi\rightarrow B+\bar{B}$.

[25] In the dispersion formalism of Dashen and Frautschi (Ref. 10), this labeling has the advantage that only D_8 and D_X appear for a specific X. Comparisons between couplings to *different* X depend on the dynamical differences between the D_X's, in addition to group theoretical factors.

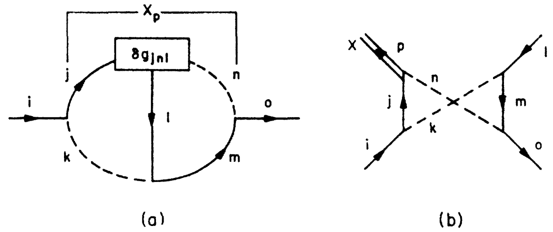

FIG 7. Diagram representing the group theoretical structure of (a) $A_{io,pl}{}^{M(B)g(X8)}$, a part of $A_{io,jnl}{}^{M(B)g(BB\Pi)}$, where $B_j\Pi_n$ is in the state X_p, (b) $V_{io,pl}$, the 2Π exchange potential for the reaction $B+\bar{B}\rightarrow X+B$.

In similar fashion, the A matrix elements $A^{gM\,\mathrm{exch}}$, A^{Mg}, and A^{gg} are all proportional to certain crossing matrices, as indicated in Figs. 6, 7, and 8. Thus all terms of A are proportional to specific crossing matrix elements, and will favor low-multiplicity representations in the same general manner as crossing matrices do.[26]

The tendency for A_8 to have larger eigenvalues than A_{27}, etc., can readily be verified in the results of detailed calculations. For example, the largest eigenvalues found by the present authors, in their study[3] of the $J=\frac{1}{2}^+$ octet and $J=\frac{3}{2}^+$ decimet mass shifts, were of magnitude 1.0 for A_8, 0.4 for A_{27}, and 0.1 for A_{64}.

Although we have located a general mechanism which makes elements of A_8 larger than elements of A_{27}, etc., the mechanism does not fix the absolute magnitude or sign of A_8, and thus falls well short of a complete explanation of octet dominance (which requires A_8 to have an eigenvalue near one). In a recent preprint, Gerstein and Whippman[23] report progress in this direction. For a large class of symmetries, assuming a static model, particles belonging to the adjoint representation, and a linear D function, they find that mass-scale invariance plus the group-theoretic ratio $A_{\text{adjoint representation}}/A_1$ ensures enhancement of mass splittings transforming like the adjoint representation [e.g., like **8** in $SU(3)$].

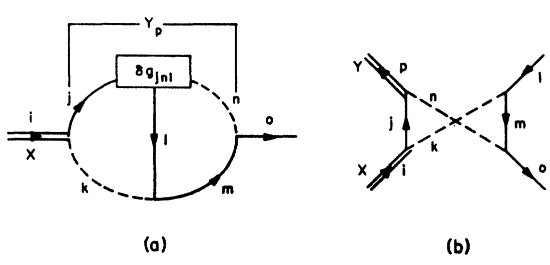

FIG. 8. Diagram representing the group theoretical structure of (a)$A_{io,pl}{}^{g(X8)g(Y8)}$, a part of $A_{jko,jnl}{}^{g(BB\Pi)g(BB\Pi)}$, where $B_j\Pi_k$ is in the state X_i, and $B_j\Pi_n$ is in the state Y_p, (b) $V_{io,pl}$, the 2Π exchange potential for the reaction $X+\bar{B}\rightarrow Y+\bar{B}$.

[26] There is also an A matrix for parity violations, relating parity-violating coupling shifts to themselves [see Ref. 3 and R. Dashen, S. Frautschi, and D. Sharp, Phys. Rev. Letters **13**, 777 (1964)]. The elements of this matrix are related to crossing matrix elements for parity-violating reactions in the same way as the elements of A^{gg} (parity conserving) are related to crossing matrix elements for parity-conserving reactions.

Our general approach can probably also be applied to $SU(6)$,[27] where hopefully it becomes "the bootstrap theory of **35** enhancement." Without doing any work, we can already see emerging the general pattern described earlier in this section. The well-identified multiplets have low multiplicity (**35**, **56**) as expected from the general results of Capps[7] and Neville[8] on crossing matrices. The observed symmetry violations are also dominated by low-multiplicity representations,[28] which is quite consistent with our general connection between crossing matrices and the A matrix for symmetry violations.[29] For example, in the *empirical* mass splittings of the **56** representation, the $SU(3)$ octet components belonging to $SU(6)$ splittings **35**, **405**, and **2695** are in the approximate ratios 40:10:1.[30] The observed mass splittings of the **35** representation admit of two possible solutions because the sign of ϕ_8-ϕ_1 mixing is experimentally uncertain; for one of the two solutions, the low representations again dominate. There is also some evidence for **35** dominance in parity-violating weak couplings.[31]

IV. GENERAL DISCUSSION OF THE DRIVING TERMS AND HIGHER ORDER EFFECTS

In previous discussions of the A matrix, a number of special assumptions have been made. A unique $SU(3)$ symmetric bootstrap solution has been assumed for the starting point, rapidly converging dispersion integrals with no "CDD parameters" have been assumed, and effects of second and higher order in the mass or coupling shifts have been ignored. In the present section, it will be shown that none of these assumptions are necessary in principle. Relaxation of the first two assumptions produces contributions to the "driving term," and higher order effects merely produce an additional term which stands along side the driving term. Thus the bootstrap theory of octet enhancement has a more general basis than may have appeared at first sight.

In our argument, the requirement of self-consistency for the strong interactions will take the following relatively noncontroversial form. Given some physical quantity x_i, such as a coupling constant or the mass of a particle, we suppose that x_i can be calculated in terms of other physical quantities x_j and *possibly* some additional parameters C. With respect to C, we have in mind parameters such as those associated with a CDD pole in a dispersion relation or the bare masses and

[27] At least if there are cases where an $SU(6)$-symmetric formulation is available as a starting point.

[28] M. A. B. Bég and V. Singh, Phys. Rev. Letters **13**, 418 (1964).

[29] Again the general connection favors A_{35} over A_{405}, etc., but is not strong enough to fix the over-all magnitude or sign of A_{35}.

[30] As usual, we are referring to the coefficients of matrices O_i, which transform like **35**, **405**, or **2695**, respectively, and are normalized according to the rule $\mathrm{Tr}O_iO_j=\delta_{ij}$.

[31] G. Altarelli, F. Buccella, and R. Gatto, Phys. Letters **14**, 70 (1965); K. Kawarabayashi, Phys. Rev. Letters **14**, 86 (1965); **14**, 169 (1965); P. Babu, *ibid.* **14**, 166 (1965); S. P. Rosen and S. Pakvasa, *ibid.* **13**, 733 (1964); M. Suzuki, Phys. Letters **14**, 64 (1965).

couplings of an elementary particle in the language of field theory. The strong interactions are, then, governed by equations like

$$x_i = f_i(x_1,x_2,\cdots,C). \quad (9)$$

Of course, in a true bootstrap theory, there are no "CDD pole" parameters C, but for our present purposes we are considering the more general situation.

Let us now use the fact that the strong interactions seem to obey an approximate $SU(3)$ symmetry. We can set $x^i=\bar{x}^i+\delta x^i$, where the $\bar{x}^i$ obey $SU(3)$ exactly and δx^i is a deviation from $SU(3)$. Presumably, we can also set $C=\bar{C}+\delta C$, where again $\bar{C}$ is $SU(3)$ symmetric and δC is not symmetric. We assume that the δx^i are sufficiently small so that an expansion of the f_i in powers of δx^i is useful. The expansion yields

$$\delta x_i = \sum_j A_{ij}\delta x_j + \sum_{jk} B_{ijk}\delta x_j \delta x_k + \cdots + D_i, \quad (10)$$

where

$$A_{ij} = \frac{\partial f_i}{\partial x_j}(\bar{x}_1,\bar{x}_2,\cdots \bar{C}), \quad (11)$$

$$B_{ijk} = \frac{1}{2}\frac{\partial^2 f_i}{\partial x_j \partial x_k}(\bar{x}_1,\bar{x}_2,\cdots \bar{C}), \quad (12)$$

$$D_i = (f_i(\bar{x}_1,\bar{x}_2,\cdots \bar{C}) - \bar{x}_i) + (f_i(\bar{x}_1,\bar{x}_2,\cdots \bar{C}+\delta C) - f_i(\bar{x}_1,\bar{x}_2,\cdots \bar{C})). \quad (13)$$

Here, one can immediately recognize the usual matrix A_{ij} and the driving term D_i. This time we have also kept the term in $(\delta x)^2$ as an example of higher order effects[32]; it will become clear as we proceed that all other higher order effects could also have been retained without changing the nature of the results. Note that we have *not* assumed that the strong interactions have an exact $SU(3)$-symmetric solution: i.e., we have not set $f_i(\bar{x}_1,\bar{x}_2,\cdots \bar{C})=\bar{x}_i$.

To obtain some insight into the content of Eq. (10), let us assume that A has a complete set of eigenvectors[33] f_i^ν, $k=1, 2, \cdots$ satisfying

$$\textstyle\sum_j A_{ij} f_j^\nu = a^\nu f_i^\nu. \quad (14)$$

Now A is not a symmetric matrix[3] so the f^ν's are not orthogonal, but if they are complete we can always find a reciprocal set g_i^ν, $\nu=1, 2, \cdots$ which has the property that $\sum_\nu g_i^\nu f_j^\nu = \delta_{ij}$. Then, setting

$$\delta x^\nu = \textstyle\sum_i \delta x_i g_i^\nu \quad (15)$$

and

$$D^\nu = \textstyle\sum D_i g_i^\nu, \quad (16)$$

Eq. (10) becomes

$$\delta x^\nu = [1/(1-a^\nu)](D^\nu + \sum_{\nu'\nu''} B^{\nu\nu'\nu''}\delta x^{\nu'}\delta x^{\nu''}), \quad (17)$$

$$\delta x^i = \sum_\nu \delta x^\nu f_i^\nu, \quad (18)$$

where

$$B^{\nu\nu'\nu''} = \sum_{ijk} B_{ijk} g_i^\nu f_j^{\nu'} f_k^{\nu''}. \quad (19)$$

Since our strong-interaction equations treat particles symmetrically, and the $\bar{x}$'s are $SU(3)$ symmetric, A and B must be symmetric under rotations in $SU(3)$ space. Thus in the general case, if there is an enhanced eigenvector f^ν [i.e., a^ν near one in Eq. (17)], the enhancement will correspond as usual to violations with definite $SU(3)$ transformation properties.

Let us repeat the generalizations we have been able to make without losing the octet-enhancement mechanism:

(i) One need not assume a complete bootstrap theory which has no CDD parameters.

(ii) It is not necessary that the strong interactions have a consistent $SU(3)$-symmetric solution.

(iii) Higher order terms need not be neglected. All that is required in Eq. (17), for example, is that the second-order term $\sum B^{\nu\nu'\nu''}\delta x^{\nu'}\delta x^{\nu''}$ be small compared to the *enhanced* δx^ν.

Note that, in effect, all of these generalizations only modify the driving term.

Our general formalism raises several questions of uniqueness. In the first place, if there is no $SU(3)$-symmetric solution, then the choice of $\bar{x}^i$'s is not unique. Of course, the freedom of choice is restricted by the requirement that δx^i is small; with this restriction the ambiguity in the definition of A and D is of order δx. In the second place, even if there is an $SU(3)$-symmetric solution, we are free to pick various sets of the x^i and C in Eq. (9). If only the variables x^i are changed (for example, by going from particle masses and couplings to Regge parameters), this merely corresponds to calculating A in a different basis.[34] If we pick $C_{\text{new}}=F(C_{\text{old}},x^i)$ (for example, by defining a subtraction parameter in a dispersion relation at a different energy), then both A and D may change substantially, but of course the physical results are unchanged.

For an actual calculation of A, some specific approach to strong interaction dynamics is required, and a specific approach should avoid most of the ambiguities of the general case. In particular, the approach based on the N/D method with the assumption of rapidly converging dispersion integrals, which we have used in our applications, would seem to be natural only if there are no

[32] Our treatment here differs from the discussion of Sec. VI, Ref. 3, where we chose to incorporate some of the nonlinear effects directly into A. The present treatment displays the preservation of the octet enhancement pattern more clearly.

[33] For some nonsymmetric matrices with degenerate eigenvalues, the eigenvectors do not form a complete set. In this case, however, Eq. (10) can still be solved and continues to exhibit properties similar to those of Eq. (17).

[34] In a very precise calculation where cutoffs are not employed, the choice of basis could be important; for example, the use of Regge parameters would avoid spurious divergences which might arise from using parameters of the $J=\frac{3}{2}^+$, πN state and other high spin states.

"CDD" parameters, and in this case A is unique except for possible corrections of order δx.

We now turn to the role of driving terms in the strong interactions. The A matrix can explain the *pattern* of $SU(3)$ violation in the strong interactions but not its *origin*; in particular, one wonders whether or not the driving terms exist. According to Eq. (13), nonvanishing driving terms could come from two sources. First, the strong interactions could contain some nonsymmetric CDD parameters. In this category we can include the possibility that there are "very strong" interactions, which are $SU(3)$ symmetric and are completely determined by the bootstrap while the violation of $SU(3)$ is due to a "medium strong" interaction which introduces an asymmetric CCD parameter δC. An example is Ne'eman's suggestion[35] that $SU(3)$ is broken by an "elementary" vector meson coupled to strangeness. The second possibility is simply that strong-interaction dynamics does not have an exactly $SU(3)$ symmetric solution. This would produce a driving term $D^i=f_i(\bar{x}_1,\bar{x}_2,\cdots\bar{C})-\bar{x}_i\neq 0$. In a pure bootstrap theory, this would be a reasonable situation (in other words, spontaneous breakdown need *not* imply $D=0$); also in a field theory, it could result from a degenerate vacuum.

Alternatively, it is possible that the driving terms might vanish for strong violations of $SU(3)$. In this case, the equations governing the strong interactions should have two solutions, one $SU(3)$ symmetric with $x^i=\bar{x}^i$ [since we want $f(\bar{x})=\bar{x}$ to ensure $D=0$], and the other[36] with $x^i=\bar{x}^i+\delta x$. This situation seems rather unattractive, and we would like to stress that it is not a necessary consequence of a pure bootstrap theory.

We conclude with a short discussion of the possibility that $D=0$ in a pure bootstrap theory (i.e., one with no C's). One way to find out if the driving terms are, in fact, zero is to calculate A_{ij} and B_{ijk} in models and see if there are solutions to (10) with $D=0$. Unfortunately, although approximate bootstrap models allow us to determine whether A has eigenvalues near one, it is much more difficult to verify or disprove the more precise condition $D=0$. In any existing model, only a few parameters, such as masses or couplings, are matched self-consistently in the input or output; other parameters appear *only* as input (cutoff parameters, for example) or output (for example, the behavior of phase shifts at energies other than the resonance one is concentrating upon). As a result of these unbootstrapped parameters, existing models are by no means fully consistent and thus fail to provide firm information on whether D would vanish in a fully consistent theory. A simple way to exhibit the lack of consistency is to observe that existing models violate the basic requirement that the bootstrap cannot set a scale of mass.

To see how this goes in a typical model, consider the bootstrap which contains a single octet of vector mesons. We need only consider the mass shifts δM_i, $i=1\cdots 8$; including coupling shifts would not change the situation. In such a model, Eq. (10) with $D=0$ takes the form

$$\delta M_i=\sum_j A_{ij}\delta M_j+\sum_{jk} B_{ijk}\delta M_j\delta M_k \qquad (10')$$

and we know that A can be diagonalized by converting from the particle masses δM_j to the mass shifts δM_1, δM_8, and δM_{27} which correspond to violations which transform like the **1**, **8**, and **27** representations of $SU(3)$. [Since we are here concerned only with strong violations of $SU(3)$, we suppress the subscript n which referred to an axis in $SU(3)$ space.] Now suppose that A_8 is near one but A_{27} is far from unity so that we obtain an octet violation of $SU(3)$. In that case, Eq. (17) will be approximated by

$$\delta M_1\approx 1/(1-A_1)[B_{111}(\delta M_1)^2+B_{188}(\delta M_8)^2], \qquad (20)$$

$$\delta M_8\approx\frac{1}{1-A_8}[B_{818}\delta M_1\delta M_8+B_{888}(\delta M_8)^2], \qquad (21)$$

$$\delta M_{27}\approx 0, \qquad (22)$$

where we have assumed that the second-order terms containing δM_{27} are small. Evidently, there is no difficulty in satisfying Eq. (21) for δM_8; the trouble comes with the requirements of scale in Eq. (20) for δM_1. Since a bootstrap does not determine a scale of mass, Eq. (20) must have a solution for $\delta M_i=\lambda M$, where M is the symmetric mass. This condition requires that $A_1=1$ and $B_{111}=0$. Therefore, Eq. (20) can be satisfied for nonzero $(\delta M_8)^2$ only if $B_{188}=0$.

Unfortunately, a bootstrap model which is supposed to contain only a single octet of vector mesons always ends up containing a fixed unbootstrapped mass associated either with a cutoff or a "fixed pole" left cut. The fixed mass spoils scale invariance; thus the model does not ensure $A_1=1$, $B_{111}=B_{188}=0$ and one cannot tell if a solution with $D=0$ really exists. Similarly, the ΠB bootstrap model of the B octet and Δ decimet contains fixed masses associated with the unbootstrapped Π's and with the implicit cutoff of B and Δ exchange.

[35] Y. Ne'eman, Phys. Rev. **134**, B1355 (1964).

[36] Note that Eq. (17) with $D=0$ cannot have a solution with small but nonzero δx unless at least one of the a^ν is near unity; thus enhancement by the factor $(1-a^\nu)^{-1}$ seems to be a necessary condition for $D=0$.