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Forward Scattering Amplitude and Univalent Functions

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Starting with the relativistic crossing-symmetric forward scattering amplitude, we constructed previously a function $g(E)$ of energy E which is both analytic and univalent (*schlicht*) in the upper-half energy plane. In this paper we exploit the univalence of $g(E)$ to obtain information on the analytic properties of the forward-scattering amplitude. Various inequalities satisfied by $g(E)$ are derived, making use of powerful theorems on univalent functions. In particular, we have established several theorems which relate the asymptotic behavior of the phase of $g(E)$ to that of $|g(E)|$ itself. We have also obtained several inequalities for $g(E)$ which may be useful in an experimental test of the consequences of local field theory. We start with only those properties of the forward scattering amplitude that have already been proved in axiomatic local field theory. The only extra assumption used that has not yet been proved in field theory is the physical assumption that the forward scattering amplitude does not become relatively real in the high-energy limit.

I. INTRODUCTION

DURING the last decade many efforts in high-energy physics have been devoted to the study of the analytic properties of scattering amplitudes and to the related problems of their asymptotic behavior at high energies. These analytic properties are usually expressed in terms of dispersion relations and, in some cases such as pion-nucleon scattering at fixed scattering angle inside the Lehmann ellipse, they were shown to follow from the formalism of Lehmann, Symanzik, and Zimmermann. More recently, Hepp[1] has shown that these relations also follow rigorously from the Wightman axioms of local field theory. Thus, the validity of dispersion relations seems to be deeply rooted in any reasonable local field theory. Therefore, any disagreement between experiment and the relations implied by these analytic properties would be extremely serious for local field theory.

Even though the dispersion relations in principle contain all information that has been established, they are neither the only tool nor necessarily the best tool available to test analyticity or to study the possible asymptotic behavior of scattering amplitudes. In a recent paper[2] we have pointed out how certain theorems of geometric function theory do provide alternative and in some cases more powerful techniques.

In the course of this and subsequent works, it has become clear that the most powerful theorems of geometric function theory apply to functions that are not only analytic but also univalent (or *schlicht*) in a certain domain (the upper-half energy plane in our case). In general there is no guarantee from field theory that the forward scattering amplitude is a regular univalent function of the energy variable. However, as was pointed out by the present authors, such univalent functions can be constructed easily from the scattering amplitude.[3]

In Sec. II we construct, starting from the crossing-symmetric forward scattering amplitude, a function $g(E)$ [see (13)] which is univalent in the upper-half energy plane. We also establish that only two subtractions are needed in writing down the forward dispersion relation. This is a consequence of local field theory and the additional physical assumption that the forward-scattering amplitude does not become relatively real as the energy goes to infinity. A similar result was first obtained in Ref. 2, but under more restrictive assumptions. In Sec. III, theorems on univalent functions are used to obtain several useful inequalities satisfied by $g(E)$.

The relation between the asymptotic behavior of $\mathrm{Re}g(E)/\mathrm{Im}g(E)$ and that of $|g(E)|$ is studied in detail in Sec. IV. The main tool used there is the theorem of Ahlfors on the mapping of strips by univalent functions. Several theorems are proved which give upper and lower bounds for $|g(E)|$ as $E\to+\infty$ under specified assumptions about the asymptotic behavior of $\mathrm{Re}g(E)/\mathrm{Im}g(E)$ or $\mathrm{Re}g(E)$.

In Sec. V we assume the validity of the Froissart bound and study the implications of the theorems of Sec. IV on the asymptotic behavior of the forward scattering amplitude $f(E)$. We show, for example, that if $\mathrm{Re}f(E)\leq 0$ (i.e., repulsive amplitude) for $E>E_0$, then the total cross section is bounded by a constant as $E\to+\infty$. It also turns out that the Froissart bound restricts severely the possible asymptotic behavior of $\mathrm{Re}g(E)/\mathrm{Im}(E)$, namely, $\mathrm{Re}g(E)/\mathrm{Im}g(E)$ is essentially bounded by $C/\ln E$ which goes to zero as $E\to+\infty$.

Finally we show in Sec. VI how the theorem of Ahlfors can be used to derive two inequalities for the function $g(E)$ which may be useful for an experimental

* Work supported in part by the U. S. Office of Naval Research.
[1] K. Hepp, Helv. Phys. Acta **37**, 639 (1964).
[2] N. N. Khuri and T. Kinoshita, Phys. Rev. **137**, B720 (1965).
[3] N. N. Khuri and T. Kinoshita, Phys. Rev. Letters **14**, 84 (1965).

test of analyticity. Only those quantities that can be determined from the experimental data obtained over a finite energy range appear in these inequalities. In the absence of accurate data it seems hard to decide which of these inequalities, or others proposed earlier,[3] is the best for the purpose of testing analyticity and local field theory. We discuss briefly the kind of behavior of data on $\mathrm{Re}f(E)$ and $\mathrm{Im}f(E)$ that will favor the use of each inequality. Actual analysis will have to wait for the full data.

Appendix A contains a more detailed discussion of the theorems of Meiman given in Ref. 2. It is essentially an expanded version of footnote 23 of that paper. In Appendix B, considerations of Sec. II on the Greenberg-Low bound are repeated making use of the techniques of univalent functions developed in Sec. IV. Several lemmas on $g(E)$ are proved in Appendix C.

II. CONSTRUCTION OF UNIVALENT FUNCTIONS FROM THE FORWARD SCATTERING AMPLITUDE

We shall show in this section how one can construct from the forward scattering amplitude a function which is regular and univalent in the upper-half energy plane. For the sake of concreteness we limit ourselves to pion-nucleon scattering. We shall denote by E the total energy of the incident pion in the laboratory system, and by $f_\pm(E)$ the forward scattering amplitudes for $\pi^\pm p$ scattering, respectively. We shall be concerned exclusively with the symmetric amplitude $f(E)$ defined as follows:

$$f(E)=\tfrac{1}{2}[f_+(E)+f_-(E)]-\text{nucleon pole terms.} \quad (1)$$

As is well known from axiomatic field theory, $f(E)$ has the following properties: (i) $f(E)$ is analytic in E and regular in the cut E plane with cuts running from $-\infty$ to $-\mu$ and from μ to $+\infty$; (ii) $f(E-i0)=f^*(E+i0)$; (iii) $f(-E-i0)=f(E+i0)$; (iv) unitarity requires, besides other properties, that $\mathrm{Im}f(E+i0)$ should be positive on the cut $E>\mu$ and negative on the cut $E<-\mu$. In general the discontinuity $\mathrm{Im}f(E+i0)$ will be a tempered distribution. Thus it is necessary to regularize it over a small interval of values of E. We shall assume that this averaging is already done and $\mathrm{Im}f(E+i0)$ is continuous on the real E axis.

It has been customary to *assume* that the scattering amplitude $f(E,\cos\theta)$, where θ is the scattering angle in the center-of-mass system, is subject to the condition (v) $|f(E,\cos\theta)|<C|E|^N$ for $E\to\infty$ for any $\cos\theta$ inside the Lehmann ellipse. Recently, Hepp[1] has shown that this property can be *proved* within the framework of Wightman axioms of local field theory. It follows from the unitarity condition and (v) that

$$|f(E)|<C|E|^2(\ln|E|)^2 \quad (2)$$

as $E\to\infty$ in all directions in the E plane. This property was derived for real E by Greenberg and Low.[4] It is generalized to the case $|E|\to\infty$ making use of the Phragmén-Lindelöf theorem.[5]

The conditions (i)–(v) and (2) are enough to insure the validity of the dispersion relation for $f(E)$ with at most three subtractions. However, if one adds to these conditions the physical requirement that $\mathrm{Im}f/\mathrm{Re}f$ should not tend to zero as $E\to+\infty$, we can show, as in theorem 1 given below, that only two subtractions are needed. We wish to stress that the requirement $\mathrm{Im}f/\mathrm{Re}f\not\to 0$ as $E\to+\infty$ has not yet been proved to be a consequence of axiomatic field theory. Nevertheless, it seems to be a reasonable feature of a theory which has an infinite number of open inelastic channels as $E\to+\infty$. We shall now prove the following theorem:

Theorem 1. If $f(E)$ satisfies the conditions (i)–(v), and if there is a positive number α such that

$$|\mathrm{Im}f(E)/\mathrm{Re}f(E)|\geq\tan\alpha\pi,\quad 0<\alpha<\tfrac{1}{2}, \quad (3)$$

holds for sufficiently large real E, then the limit

$$\lim_{E\to+\infty}\int_\mu^E \frac{f(E')}{E'^3}dE'$$

does not diverge.

Proof. We consider the function $\phi(E)$ defined by

$$\phi(E)=\int_\mu^E \frac{f(E')-f(0)}{E'^3}dE', \quad (4)$$

where the path of integration is taken in the upper-half E plane. As is easily seen, the function $\phi(E)$ has the following properties: (a) $\phi(E)$ is analytic in the upper-half E plane; (b) $\mathrm{Im}\phi(E)$ increases monotonically for real $E\geq\mu$; (c) from the Greenberg-Low bound (2) we have $|\phi(E)|\leq C(\ln|E|)^3$ for large $|E|$; (d) $\phi(E)$ has no zero in the upper-half E plane outside some fixed semicircle; and, finally, (e) for real positive E

$$\begin{aligned}\mathrm{Re}\phi(-E+i0)&=\mathrm{Re}\phi(E+i0),\\ \mathrm{Im}\phi(-E+i0)&=-\mathrm{Im}\phi(E+i0)+\tfrac{1}{2}\pi f''(0).\end{aligned} \quad (5)$$

Now, since $\mathrm{Im}\phi(E)$ is positive and monotonically increasing for $E>\mu$, $\lim_{E\to+\infty}\mathrm{Im}\phi(E)$ is either finite and well defined or infinite. We shall show that the second possibility leads to a contradiction with the Greenberg-Low bound (2).

We assume that $\lim_{E\to+\infty}\mathrm{Im}\phi(E)=\infty$. If we form the function

$$G(E)=-1/\phi(E), \quad (6)$$

we find that $G(E)$ is analytic outside some semicircle in the upper half E plane as is seen from (d). Also we have

$$\lim_{E\to\infty}G(E)=0 \quad (7)$$

[4] O. W. Greenberg and F. E. Low, Phys. Rev. **124**, 2047 (1961).
[5] See Ref. 2.

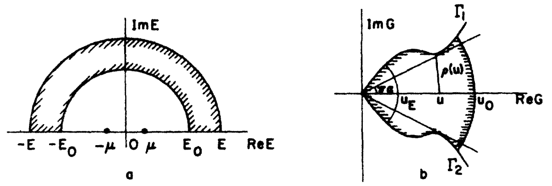

FIG. 1. (a) represents a domain in the E plane whose boundary consists of semicircles of radius E_0 and E and line segments (E_0,E) and $(-E_0,-E)$. (b) represents a domain in the G plane which is obtained from the domain (a) by the mapping $G(E)$.

by our assumption on $\phi(E)$. For large enough real E, we obtain from (3) and (4) the inequality

$$|\mathrm{Im}G(E)/\mathrm{Re}G(E)|\geq\tan\pi\alpha. \tag{8}$$

This holds regardless of whether $\int_\mu^\infty|\mathrm{Re}f(E')/E'^3|dE'$ converges or diverges, as long as $\mathrm{Im}\phi(E)\to+\infty$ as $E\to+\infty$.

Let us now consider the mapping of the region shown in Fig. 1(a), where $E\gg E_0\gg\mu$, into the G plane by $G(E)$. The image would look something like the picture shown in Fig. 1(b). The images of the segments (E_0,E) and $(-E_0,-E)$ will lie, respectively, above and below the two straight lines through the origin making angles $\pm\pi\alpha$ with the real G axis. The point u_E is the farthest intersection of the image of the large semicircle with the real G axis and u_0 is the nearest intersection of the image of the smaller semicircle with the real G axis. The inequality of Nevanlinna,[6] or the more precise formula (A6) due to Hersch,[7] now gives

$$\int_{u_E}^{u_0}\frac{du}{\rho(u)}\geq\tfrac{1}{2}\ln\left(\frac{E}{E_0}\right), \tag{9}$$

where $\rho(u)$ is the shortest distance from the point u on the real G axis to the image of the segment (E_0,E). It is clear from (5) and our assumption that $\mathrm{Im}\phi(E)\to\infty$ as $E\to\infty$ that the images of the segments (E_0,E) and $(-E_0,-E)$ will be approximately symmetrical with respect to the real G axis. We now have

$$\rho(u)\geq u\sin\pi\alpha, \quad u_E\leq u\leq u_0.$$

Using this and (9) we obtain

$$|u_E|\leq C|E|^{-\frac{1}{2}\sin\pi\alpha}.$$

Since $u_E=G(|E|e^{i\gamma})$ for some $\gamma(0\leq\gamma\leq\pi)$, and since $G=-1/\phi$, we finally arrive at

$$|\phi(|E|e^{i\gamma})|\geq C'|E|^{\frac{1}{2}\sin\pi\alpha}. \tag{10}$$

For large enough E this contradicts the bound (c) on ϕ obtained from the Greenberg-Low bound. We have therefore to conclude that $\lim_{E\to\infty}\mathrm{Im}\phi(E)$ cannot be infinite. It then follows that $\int_\mu^\infty|\mathrm{Re}f(E')|E'^{-3}dE'$ must also converge since $|\mathrm{Im}f/\mathrm{Re}f|\geq\tan\pi\alpha$ for large real E. Q.E.D.

[6] See Ref. 2, Appendix.

[7] J. Hersch, Commentarii Mathematici Helvetici **29**, 301 (1955).

The theorem just proved is weaker than the similar theorem stated in Ref. 2, where we showed that $|f(E)|\leq CE^{2-(\alpha/2)}$ for large enough E. However, to obtain this strong improvement of Greenberg-Low bound, we had to make the extra assumption that $f(E)$ does not have violent oscillation as $E\to\infty$. Such an assumption is not needed in theorem 1 here. The result of theorem 1 is equivalent to a theorem recently obtained by Jin and MacDowell.[8] Somewhat more refined version of theorem 1 will be given in Appendix B.

From the assumptions (i)–(v) and theorem 1 it follows that $f(E)$ satisfies the twice subtracted dispersion relation

$$f(E)-f(0)=\frac{2E^2}{\pi}\int_\mu^\infty dE'\frac{\mathrm{Im}f(E')}{E'(E'^2-E^2)}. \tag{11}$$

Making use of this representation we can easily show that the function $h(E)$ defined by

$$h(E)=\frac{f(E)-f(0)}{E} \tag{12}$$

has the properties: (α) $h(E)$ is regular for $\mathrm{Im}E>0$ and continuous for $\mathrm{Im}E\geq0$; (β) $\mathrm{Im}h(E)>0$ for $\mathrm{Im}E>0$ [namely $h(E)$ is a Herglotz function]; (γ) $h(i\lambda)$, λ real and positive, is purely imaginary; (δ) $\mathrm{Re}h(-E+i0)=-\mathrm{Re}h(E+i0)$ and $\mathrm{Im}h(-E+i0)=\mathrm{Im}h(E+i0)$ for real E. Thus, if we consider the mapping of the upper-half E plane by the function $h(E)$, the image will lie in the upper-half h plane as is seen from (β). On the other hand, the conditions (i)–(v) do not guarantee that such a mapping is one-to-one. Fortunately, however, it is not difficult to construct functions from $h(E)$ that have such a property. One such function is $g(E)$, defined by

$$g(E)=\int_0^E\frac{h(E')}{E'}dE', \quad \mathrm{Im}E\geq0, \tag{13}$$

where the path of integration is taken to lie entirely in the upper-half E plane. The integral in (13) is convergent at $E'=0$ since $f'(0)=0$ and $f(E)$ is regular near $E=0$.

To show the univalence of $g(E)$ we first note the following properties: (1) $g(E)$ is regular in $\mathrm{Im}E>0$ and continuous in $\mathrm{Im}E\geq0$; (2) $\mathrm{Im}g(E)>0$ if $\mathrm{Im}E>0$; (3) $g'(E)\neq0$ everywhere in $\mathrm{Im}E>0$; (4) $\mathrm{Re}g(-E+i0)=-\mathrm{Re}g(E+i0)$, $\mathrm{Im}g(-E+i0)=\mathrm{Im}g(E+i0)$ for all real E; (5) for real $E>\mu$, $\mathrm{Im}g(E+i0)$ is non-negative and increases monotonically along the positive real axis; (6) $\mathrm{Re}g(E)$ is nonnegative and increases monotonically in the interval $0\leq E\leq\mu$; (7) $g(i\lambda)$, for real positive λ, is purely imaginary and increases monotonically with λ.

[8] Y. S. Jin and S. W. MacDowell, Phys. Rev. **138**, B1279 (1965). Note that footnote 3 of this reference was written without recognition of footnote 23 of Ref. 2. Note also that part of the footnote 3 of this reference which goes beyond footnote 23 of Ref. 2 and Appendix A of this paper is in our opinion not correct.

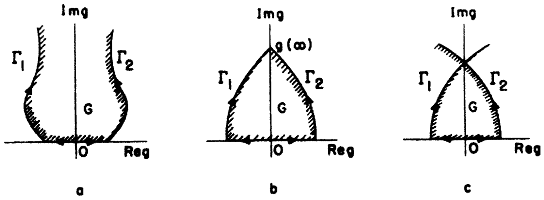

FIG. 2. Schematic drawings of the domain G. (a) represents the case $g(\infty)=\infty$. (b) represents the case $g(\infty)=$finite. (c) is an example of an impossible case.

The property (2) follows from (β) as is seen by choosing the straight line connecting 0 and E as the path of integration in (13). The property (3) also follows from (β) if we note that $g'(E)=h(E)/E$. Finally, (6) is proved using the dispersion relation (11).

As is seen from the property (2), the function $g(E)$ maps the upper-half E plane into a domain G located in the upper-half g plane. We know from (3) that this mapping is locally one-to-one everywhere in the upper-half E plane. The mapping will therefore be globally univalent and conformal if the boundary curve of G has no double points.[9] To examine this, let us denote by Γ_1 and Γ_2 the images of the negative and positive real E axis, respectively. Because of (4), Γ_2 and Γ_1 are symmetric with respect to the imaginary g axis. We know from (6) that the part of Γ_2 corresponding to $0\leq E\leq\mu$ does not intersect with itself and lies on the positive real g axis. For $E>\mu$, $g(E)$ becomes complex and the corresponding part of Γ_2 rises monotonically from the real g axis according to (5). Thus Γ_2 cannot have a double point. The same holds for Γ_1. Hence the only remaining possibility is that Γ_1 and Γ_2 have some common points. Because of the monotonicity and symmetry of Γ_1 and Γ_2 such a common point could be found only on the imaginary g axis, see Fig. 2(c). However, a configuration like this cannot take place since the mapping is everywhere locally conformal, $g'(E)\neq 0$, and since Γ_1 and Γ_2 cannot turn back towards the real g axis because of (5). Thus Γ_1 and Γ_2 have either no point in common as in Fig. 2(a) [$g(\infty)=\infty$], or only one common point $g(\infty)(<\infty)$ on the imaginary g axis as in Fig. 2(b). Thus, the boundary curve of G has no double point, which proves the univalence of $g(E)$ in the upper-half E plane. It is clear from Figs. 2(a), 2(b) that, for all finite real positive E, we have the inequality

$$\mathrm{Re}g(E+i0)>0. \tag{14}$$

One can check this inequality directly from the dispersion relation (11). If we divide both sides of (11) by E^2, integrate along the radial direction from 0 to E, and take the real part, we obtain

$$\mathrm{Re}g(E)=\frac{1}{\pi}\int_\mu^\infty dE'\frac{\mathrm{Im}f(E')}{E'^2}\ln\left|\frac{E'+E}{E'-E}\right| \tag{15}$$

for $0\leq\arg E\leq\pi$. We note that $\ln|(E'+E)/(E'-E)|>0$ for any E in the first quadrant and that $\mathrm{Im}f(E')>0$ for real $E'>\mu$. Thus, $\mathrm{Re}g(E)$ is positive for all E such that $0\leq\arg E<\pi/2$. In other words, $g(E)$ maps the first quadrant of the E plane into a domain in the first quadrant of the g plane. In particular, we obtain (14) for positive real E.

Furthermore, in the same manner, one can show that $\mathrm{Re}g(|E|e^{i\theta})$ decreases monotonically to zero as θ varies from 0 to $\pi/2$ for any fixed $|E|$. Thus the image of a large semicircle in the upper-half E plane centered at the origin will have no double point. One can further show that it does not intersect Γ_1 or Γ_2. The theorem of Ref. 9 thus guarantees again that $g(E)$ is univalent inside any semicircle centered at the origin and lying in the upper-half E plane.

Although the function $g(E)$ defined by (13) is perhaps the most useful for our purpose, it is by no means the only univalent function that can be constructed from the scattering amplitude. Another useful function will be

$$g_1(E)=\int_0^E h(E')dE',\quad \mathrm{Im}E\geq 0, \tag{16}$$

which is regular and univalent in the half-plane $\mathrm{Im}E>0$. Although $\mathrm{Im}g_1(E)$ is not positive definite in the upper-half E plane, it does not give rise to any particular difficulty. As a matter of fact, if necessary, we could also introduce functions of the form

$$g_n(E)=\int_0^E E'^{n-1}h(E')dE',\quad n=2,3,\cdots, \tag{17}$$

which are no longer univalent but rather multivalent, with a definite multiplicity, in the upper-half E plane. Since $\ln g_n(E)$ is univalent in $\mathrm{Im}E>0$, we will have no difficulty in treating $g_n(E)$ under most circumstances.

III. SOME INEQUALITIES SATISFIED BY $g(E)$

We have seen in the last section that the function $g(E)$ defined by (13) is regular, Herglotz, and univalent in the upper-half E plane and also symmetric with respect to the $\mathrm{Im}E$ axis. These properties impose some restrictions on the possible behavior of $g(E)$. In particular, univalent functions are known to satisfy various sharp inequalities. In this section we shall write down some of the inequalities satisfied by $g(E)$.

For this purpose it is convenient to introduce the new variable z defined by

$$z=\frac{E-i\lambda}{E+i\lambda},\quad \lambda>0. \tag{18}$$

This function maps the upper-half E plane into a unit circular disk, $|z|<1$, in the z plane with the point $E=i\lambda$ going into the origin $z=0$. We also define, for

[9] E. C. Titchmarsh, *The Theory of Functions* (Oxford University Press, New York, 1939), 2nd ed., p. 201.

fixed λ, the function

$$\varphi(z)\equiv\frac{g(E)-g(i\lambda)}{2i\lambda g'(i\lambda)}. \tag{19}$$

Here $g'(i\lambda)=dg(E)/dE|_{E=i\lambda}$ and is not zero for $\lambda>0$ as was shown in Sec. II. Thus $\varphi(z)$ is regular and univalent in the unit disk $|z|<1$ and its power series expansion at $z=0$ has the normalized form

$$\varphi(z)=z+a_2z^2+a_3z^3+\cdots. \tag{20}$$

Since $g(i\lambda)$ is purely imaginary while $g'(i\lambda)$ is purely real, $\varphi(z)$ is real if and only if z is real, $|z|<1$, and has the symmetry property $\varphi^*(z)=\varphi(z^*)$ inside the disk $|z|<1$. For such univalent functions the coefficients of the power series in (20) satisfy the inequalities[10]

$$|a_n|\leq n,\quad n=2,3,\cdots. \tag{21}$$

This puts upper bounds on all derivatives of $g(E)$ at $E=i\lambda$ which depend only on λ and $g'(i\lambda)$. Although it is unlikely that the bounds (21) are of direct practical use, except for the case $n=2$, they might be useful in some theoretical considerations.

In the following we shall concentrate on the properties of $g(E)$ that follow from the inequality $|a_2|\leq 2$. One of the direct consequences of this property is Koebe's theorem.[11] This theorem states that, if $\varphi(z)$ is regular and univalent for $|z|<1$ and has the normalized form (20), then the image of the disk $|z|<1$ must cover at least a disk of radius $\frac{1}{4}$ in the φ plane. In other words,

$$|\varphi(e^{i\theta})|\geq\tfrac{1}{4},\quad 0\leq\theta\leq 2\pi. \tag{22}$$

This result can be translated for the function $g(E)$ as follows: we draw in the g plane a straight line parallel to the Reg axis through the point $g(i\lambda)$ on the Img axis. As is shown in Appendix C, this line will intersect with the boundary curve Γ_2 of the domain G at one point which we denote by $g(E_\lambda)$ (see Fig. 3). Here E_λ is real, positive, and determined uniquely by the equality

$$\mathrm{Im}g(E_\lambda+i0)=|g(i\lambda)| \tag{23}$$

for any given positive real λ. For this choice of E_λ, $g(E_\lambda)-g(i\lambda)$ is purely real. Thus, using Koebe's theorem and (19), we obtain the inequality

$$\mathrm{Re}g(E_\lambda+i0)\geq(1/2\lambda)|f(i\lambda)-f(0)|. \tag{24}$$

Since (22) is valid for all univalent functions normalized by (20), it is quite likely that (24) is not the best possible inequality that applies to the specific function $g(E)$. In fact, just taking account of the symmetry property of $g(E)$, we can easily improve (24) by a factor of 2 and obtain

$$\mathrm{Re}g(E_\lambda+i0)\geq(1/\lambda)|f(i\lambda)-f(0)|. \tag{25}$$

This follows from a theorem of Szegö.[11] Applied to the function $\varphi(z)$, this theorem asserts that of two points lying on the same straight line in the φ plane going through $\varphi=0$ and on opposite sides of $\varphi=0$, neither of which belonging to the map of the unit disk $|z|<1$ by $\varphi=\varphi(z)$, one at least must be at a distance not less than $\frac{1}{2}$ from the origin $\varphi=0$. For the function $g(E)$ this means that either $\mathrm{Re}g(E_\lambda+i0)$ or $-\mathrm{Re}g(-E_\lambda+i0)$ must be larger than $|f(i\lambda)-f(0)|/\lambda$. Since $\mathrm{Re}g(E_\lambda+i0)=-\mathrm{Re}g(-E_\lambda+i0)$ because of the symmetry, we must have (25).[12]

The inequality (25) might be useful in an experimental test of the analytic properties of $f(E)$, as was discussed in Ref. 3.

Another consequence of the univalence of $\varphi(z)$ is given by the following theorem[13]: If $\varphi(z)$ is univalent in $|z|<1$ and has the form (20), we have for $|z|=r$, $0<r<1$, the inequalities

$$\begin{gathered}\frac{r}{(1+r)^2}\leq|\varphi(z)|\leq\frac{r}{(1-r)^2},\\ \frac{1-r}{(1+r)^3}\leq|\varphi'(z)|\leq\frac{1+r}{(1-r)^3},\\ \frac{1-r}{r(1+r)}\leq\left|\frac{\varphi'(z)}{\varphi(z)}\right|\leq\frac{1+r}{r(1-r)}.\end{gathered} \tag{26}$$

This theorem gives upper and lower bounds for $g(E)$ and $g'(E)$ for all E such that $\mathrm{Im}E>0$. These bounds of course depend on $g(i\lambda)$ and $g'(i\lambda)$. They are useful in obtaining estimates of $g(E)$ for complex E.

Finally we should like to quote a theorem due to Seidel and Walsh[14]: If $\psi(z)$ is regular and univalent in $|z|<1$, the first derivative $\psi'(z)$ satisfies the relation

$$\lim_{z\to e^{i\alpha}}|\psi'(z)|(1-|z|)^{1/2}=0 \tag{27}$$

for all points $e^{i\alpha}$ of the circumference $|z|=1$ with the

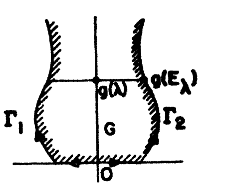

FIG. 3. Definition of $g(E_\lambda)$. $g(\lambda)$ on the Img axis should read $g(i\lambda)$

[10] W. K. Hayman, *Multivalent Functions* (Cambridge University Press, Cambridge, 1958), p. 14.

[11] See, for instance, G. M. Golusin, *Interior Problems of the Theory of Schlicht Functions*, translated by T. C. Doyle, A. C. Schaeffer, and D. C. Spencer (U. S. Office of Naval Research, Washington, 1947), p. 9.

[12] The formula (25) may be improved further if more detailed information on $g(E)$ is available. For this purpose a generalization of Szegö's theorem (Ref. 11, p. 9) will be useful. See also G. M. Golusin, *Geometrische Funktionentheorie* (VEB Deutscher Verlag der Wissenschaften, Berlin, 1957), p. 143.

[13] See, for instance, Ref. 10, p. 4.

[14] W. Seidel and J. L. Walsh, Trans. Am. Math. Soc. **52**, 128 (1942).

exception of at most a set of measure zero, where the limit is taken in any angle less than π with vertex in $e^{i\alpha}$ and bisected by the radius joining $z=0$ with $z=e^{i\alpha}$. Furthermore, in any such angle the above limit is uniform. This theorem shows that, almost everywhere on the unit circle $|z|=1$, $|\psi'(z)|$ is substantially smaller than the upper bound given by (26). It will be obvious that there is a close relation between this theorem and lemma 3 given in Appendix C.

IV. THEOREMS ON THE ASYMPTOTIC BEHAVIOR OF $g(E)$ FOR LARGE E

In this section we shall study the asymptotic behavior of $g(E)$ with particular emphasis on the relation between the asymptotic behavior of $\text{Re}g(E)/\text{Im}g(E)$ and that of $|g(E)|$. The theorems proved below will be used in the next section to study the asymptotic behavior of $f(E)$.

One can study the asymptotic properties of $g(E)$ making use of the theorems of Meiman which were discussed in detail in Ref. 2. However, since $g(E)$ is univalent in the upper-half E plane, we have available to us more powerful tools such as the theorem of Ahlfors discussed below. In this approach there is an additional advantage in that some of the assumptions made in Ref. 2 can be eliminated or simplified.

If we impose no restriction on the behavior of $\text{Re}g(E)/\text{Im}g(E)$, all we know about the asymptotic behavior of $g(E)$ is that

$$|g(E)|<C|E|(\ln|E|)^2,$$

which follows from the Greenberg-Low bound (2). Throughout this section, however, we shall assume as in the last sections that the condition (3) of theorem 1 is satisfied by $f(E)$. Then the integral $\int^E f(E')E'^{-3}dE'$ is convergent as $E\rightarrow+\infty$. From this it follows that

$$|g(E)/E|\leq\text{constant}$$

as $|E|\rightarrow\infty$.

Let us start by a short discussion of Ahlfors' theorem[15] We consider a simple (*schlicht*) domain D in the z plane $(z=x+iy)$ which is simply connected and symmetric with respect to the x axis. Let $Z_1=X_1+iY_1$ and $Z_2=X_2+iY_2$ be the points on the boundary curve of D with the smallest and largest real part, respectively. For any x, $X_1<x<X_2$, the vertical line $\text{Re}z=x$ will have one or more intersections with D, each of which is bisecting D into two disconnected parts. Under our assumption on D, there is one intersection which crosses the x axis. This line segment we denote by θ_x and its length by $\theta(x)$; see Fig. 4(a). The line segment θ_x divides D into two disconnected parts in such a way that X_1 and X_2 belong to different parts. We require that $\theta(x)$ is a continuous function of x for $X_1<x<X_2$ except at some isolated points.

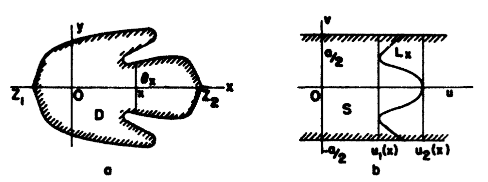

FIG. 4. The domain D of the z plane in (a) is mapped by $w(z)$ onto the strip S of the w plane in (b).

[15] R. Nevanlinna, *Eindeutige Analytische Funktionen* (Springer-Verlag, Berlin, 1953), 2nd ed., p. 93.

Let $w=u+iv=w(z)$ be a function which is regular and univalent in D and maps D conformally onto a strip S defined by $|v|<\frac{1}{2}a$ $(a>0)$ in such a way that $u(Z_1)=-\infty$ and $u(Z_2)=+\infty$. In this mapping the line segment θ_x will be mapped onto a continuous curve L_x which connects the two boundary curves $v=\pm\frac{1}{2}a$. The largest and smallest values of u on L_x are denoted by $u_2(x)$ and $u_1(x)$, respectively. [See Fig. 4(b).] The theorem of Ahlfors now states that

$$u_1(x_2)-u_2(x_1)\geq a\int_{x_1}^{x_2}\frac{dx}{\theta(x)}-4a \tag{28}$$

holds for any pair of points x_1, x_2 such that

$$\textstyle\int_{x_1}^{x_2}dx/\theta(x)>2.$$

We shall now use (28) to prove several theorems on the asymptotic behavior of $g(E)$. We recall that, for real positive E, $g(E)$ lies in the first quadrant of the g plane. Thus, insofar as $\text{Re}g(E)/\text{Im}g(E)$ does not tend to zero as $E\rightarrow+\infty$, we may characterize the asymptotic behavior of $g(E)$ by the inequality

$$\frac{\text{Re}g(E)}{\text{Im}g(E)}\geq\tan\pi\alpha,\quad 0<\alpha<\tfrac{1}{2}, \tag{29}$$

or

$$\frac{\text{Re}g(E)}{\text{Im}g(E)}\leq\tan\pi\alpha',\quad 0<\alpha'<\tfrac{1}{2}, \tag{30}$$

which holds for all real E greater than some positive E_0. We then obtain the following two theorems:

Theorem 2. If the function $g(E)$ satisfies (29) for $E>E_0$, $g(E)$ has the lower bound

$$|g(E)|\geq C(E/E_0)^{2\alpha},\quad 0<\alpha<\tfrac{1}{2}, \tag{31}$$

for all E sufficiently larger than E_0.

Proof. We define z and $w(z)$ by

$$z=\ln(E/C)-i\pi/2,\quad w(z)=\ln g(E)-i\pi/2,\quad C>0, \tag{32}$$

and apply Ahlfors' theorem to the mapping $z\rightarrow w$. The lines $\text{Re}g(E)/\text{Im}g(E)=\pm\tan\pi\alpha$ correspond to the two straight lines in the w plane which are parallel to the u axis $(w=u+iv)$ and separated by the distance $a=2\alpha\pi$. We then choose D to be the domain whose boundary curve consists of two vertical line segments with $\text{Re}z=x_1=\ln(E_1/C)$ and $\text{Re}z=x_2=\ln(E_2/C)$ and two

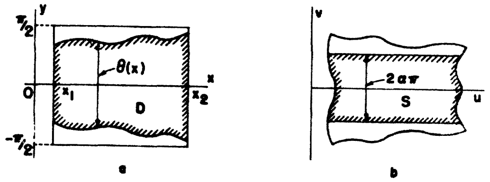

FIG. 5. Mapping of the domain D in (a) onto the strip S in (b).

Jordan curves which are maps of the two parallel lines in the w plane, mentioned above, by the inverse transformation $z=z(w)$; see Fig. 5. Obviously the length $\theta(x)$ is by definition less than π. Thus we obtain the inequality

$$u_1(x_2)-u_2(x_1)\geq 2\alpha\pi\left(\frac{x_2-x_1}{\pi}-4\right)$$
$$=2\alpha\ln(E_2/E_1)-8\alpha\pi. \quad (33)$$

This follows directly from (28). From the definition of $u_1(x_2)$ we know that

$$\ln|g(|E_2|e^{i\varphi})|\geq u_1(x_2), \quad \epsilon\leq\varphi\leq\pi-\epsilon, \quad (34)$$

where ϵ depends on $|E_2|$ and is determined by the condition $\arg g(|E_2|e^{i\epsilon})=\pi/2-\alpha\pi$. In particular, if we choose $\varphi=\pi/2$, we obtain from (33) and (34) the inequality

$$|g(i|E_2|)|\geq C'(E_2/E_1)^{2\alpha} \quad (35)$$

for $E_2\gg E_1$, where C' does not depend on E_2. This is not yet a lower bound of $|g(E)|$ for real positive E. However, from (C8) we can easily obtain

$$|g(E)|>(1/\sqrt{2})|\mathrm{Im}g(iE)| \quad (36)$$

for real positive E. Formulas (35) and (36) lead us immediately to (31). Q.E.D.

Theorem 3. If the regular univalent function $g(E)$ satisfies (30) for $E>E_0$, $g(E)$ has the upper bound

$$|g(E)|\leq C'(E/E_0)^{2\alpha'} \quad (37)$$

for all E sufficiently larger than E_0.

Proof. In this proof we still use Ahlfors' theorem but we now reverse the definition of z and $w(z)$. We put

$$z=\ln g(E)-i\pi/2, \quad w(z)=\ln(E/C)-i\pi/2, \quad (38)$$

and consider the mapping $z\rightarrow w$. The domain S of Ahlfors' theorem in the w plane is now taken to be the strip $|v|\leq\pi/2$. The domain D in the z plane is bounded by two Jordan curves which are the images of the lines $v=\pm\pi/2$ and by two vertical line segments $\mathrm{Re}z=x_1$ and $\mathrm{Re}z=x_2$. The images of the vertical line segments $\mathrm{Re}z=x_1$ and $\mathrm{Re}z=x_2$ are now curves in the w plane connecting the lines $v=\pm\pi/2$, see Fig. 6. Applying Ahlfors' theorem to this case, we obtain

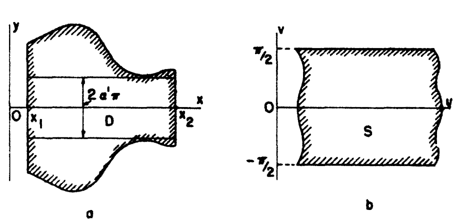

FIG. 6. Mapping of the domain D in (a) onto the strip S in (b).

$$u_1(x_2)-u_2(x_1)\geq\frac{x_2-x_1}{2\alpha'}-4\pi. \quad (39)$$

We define real and positive E_1 and E_2, $E_2\gg E_1$, by

$$\ln|g(E_1)|=x_1, \quad \ln|g(E_2)|=x_2. \quad (40)$$

By definition of u_1 and u_2 we also have

$$u_1(x_2)\leq\ln(E_2/C), \quad u_2(x_1)\geq\ln(E_1/C). \quad (41)$$

Substituting (40) and (41) in (39) we obtain (37). Q.E.D.

It would be useful to compare the results obtained so far with those that could have been obtained by using Meiman's theorem of Ref. 2. One should note that it is not necessary here to assume that the boundary curves Γ_1 and Γ_2 do not intersect, since this is already guaranteed by univalence. Also we do not need here any regularity assumption on the boundary curves Γ_1 and Γ_2 beyond the very general ones needed for Ahlfors' theorem. Furthermore, the above theorems hold for *all* E greater than some fixed energy ($\gg E_0$) whereas Meiman's theorem gives under similar assumptions a result like (31) which is valid only for *some* E. It is also important to note that the power of E in (31) is better by a factor of 4 than what one would have obtained by applying Meiman's theorem to the function $-1/g(E)$. It is not possible to improve this power further.[15]

As is obvious from (31) and (37), the lower and upper bounds for $|g(E)|$ have similar energy dependence. If $\mathrm{Re}g/\mathrm{Im}g\sim\tan\pi\alpha$ and $\alpha=\alpha'$, and if one makes certain specific assumptions on the smoothness of the boundary curves, then the two bounds approach each other as $E\rightarrow+\infty$ and (31) or (37) gives the actual asymptotic expression for $g(E)$.[16]

Theorem 2 is useful if $\mathrm{Re}g(E)$ and $\mathrm{Im}g(E)$ both tend to infinity as $E\rightarrow+\infty$ in such a way that (29) is satisfied. If $\mathrm{Re}g(E)$ grows less rapidly or is bounded as $E\rightarrow+\infty$, the condition (29) is no longer convenient since we cannot choose a positive α. In such cases it is better to give different characterization of the asymptotic behavior of $\mathrm{Re}g(E)/\mathrm{Im}g(E)$. For instance, one could still get a useful result if $\mathrm{Re}g/\mathrm{Im}g$ satisfies an inequality like (29) in which the constant α is replaced by a function $\alpha(E)$ which decreases *monotonically* to zero as $E\rightarrow+\infty$. Then the argument of theorem 2 still applies and we obtain, for sufficiently large E, the inequality

$$|g(E)|>C\exp[2\alpha(E)\ln(E/E_0)]. \quad (42)$$

[16] S. E. Warschawski, Trans. Am. Math. Soc. 51, 280 (1942).

In particular we have the following corollary to theorem 2:

Corollary. If $g(E)$ satisfies the inequality

$$\mathrm{Re}g(E)/\mathrm{Im}g(E)>C/(\ln E)^a,\quad 0<a<1, \tag{43}$$

for $E>E_0$, then

$$|g(E)|>C'[\ln(E/E_0)]^\gamma \tag{44}$$

for large enough E. Here γ is greater than any positive number.

If $a\geq 1$ in (43), the methods discussed so far do not give any useful information. The case $a=1$ will be treated by a different method in Sec. V.

To handle the situation where

$$\frac{\mathrm{Re}g(E)}{\mathrm{Im}g(E)}\leq\frac{C}{(\ln E)^a},\quad a\geq 1,\quad E>E_0, \tag{45}$$

we shall make use of Koebe's theorem. We first note that, if we choose $\lambda=E$, E being real and positive, in (19) and (22), we obtain

$$|g(E)-g(iE)|>\tfrac{1}{2}E|g'(iE)|\,.$$

Combining this with (C8), we find

$$\mathrm{Re}g(E)>(1/2\sqrt{2})E|g'(iE)|\,. \tag{46}$$

On the other hand, from (C8) and (45) we derive

$$\left(1-\frac{C}{(\ln E)^a}\right)\mathrm{Im}g(E)<\mathrm{Im}g(iE)\,. \tag{47}$$

Inequalities (46) and (47) lead us to

$$\left|\frac{g'(iE)}{g(iE)}\right|<\frac{2\sqrt{2}}{1-C(\ln E)^{-a}}\frac{1}{E}\frac{\mathrm{Re}g(E)}{\mathrm{Im}g(E)}<\frac{2\sqrt{2}C}{E((\ln E)^a-C)}<\frac{C'}{E(\ln E)^a} \tag{48}$$

for sufficiently large E, where C' is a suitably chosen finite constant. Integrating both sides of (48) from E_0 to E, we therefore obtain

$$\left|\frac{g(iE)}{g(iE_0)}\right|<\exp\left\{\int_{E_0}^{E}\frac{C'dE}{E(\ln E)^a}\right\}. \tag{49}$$

For $a>1$ this gives

$$\left|\frac{g(iE)}{g(iE_0)}\right|<\exp\left\{\frac{C'}{1-a}[(\ln E)^{1-a}-(\ln E_0)^{1-a}]\right\}. \tag{50}$$

Now we know from (47) that $\mathrm{Im}g(E)$ is bounded as $E\to+\infty$ if $\mathrm{Im}g(iE)$ is bounded. Using (45) and (50), we therefore obtain:

Theorem 4. If $g(E)$ satisfies (45) for $E>E_0$, $a>1$, then $|g(E)|$ is bounded as $E\to+\infty$ and $\mathrm{Re}g(E)\to 0$ in that limit.

For the case $a=1$ we have from (49) the upper bound

$$|g(E)|\leq C''(\ln E)^{C'}, \tag{51}$$

where C' is defined in (48).

Another way of characterizing the asymptotic behavior for the case $\alpha=0$ is to consider the situation where for all $E>E_0$

$$\mathrm{Re}g(E)\geq b. \tag{52}$$

b is some positive constant. In this case we can apply Ahlfors' theorem to the mapping $z\to w$ where we now set

$$z=\ln(E/C)-i\pi/2,\quad w(z)=-ig(E). \tag{53}$$

By an argument very similar to that of theorem 2, we obtain

$$\mathrm{Im}g(iE)\geq(2b/\pi)\ln(E/E_0)+\mathrm{const} \tag{54}$$

for all E sufficiently larger than E_0. Using (C8), we therefore obtain:

Theorem 5. If $\mathrm{Re}g(E)$ satisfies (52), then $|g(E)|$ has the lower bound

$$|g(E)|\geq(2b/\pi)\ln(E/E_0)+\mathrm{const} \tag{55}$$

for all E sufficiently larger than E_0.

Similarly, as an analog to theorem 3, we have

Theorem 6. If $\mathrm{Re}g(E)$ satisfies the inequality $0\leq\mathrm{Re}g(E)\leq b'$ for all $E>E_0$, then for sufficiently large E we have

$$|g(E)|\leq(2b'/\pi)\ln(E/E_0)+\mathrm{const}. \tag{56}$$

Finally, we consider the case where $\mathrm{Re}g(E)/\mathrm{Im}g(E)\to 0$ as $E\to+\infty$ but $\mathrm{Re}g(E)$ diverges at the same time. We can now choose the variables as

$$z=\ln(E/C)-i\pi/2,\quad w(z)=[-ig(E)]^{1/\nu},\quad \nu>1. \tag{57}$$

Here $w(z)$ is defined by that branch in which $[\mathrm{Im}g(E)]^{1/\nu}$ is real and positive. For large enough E we have

$$w(z)\simeq[\mathrm{Im}g(E)]^{1/\nu}-\frac{i}{\nu}\frac{\mathrm{Re}g(E)}{[\mathrm{Im}g(E)]^{1-1/\nu}}. \tag{58}$$

Since $w(z)$ is univalent in $\mathrm{Im}E>0$, we can apply Ahlfors' theorem to this case. Suppose we found a ν such that $\nu>1$ and

$$\frac{\mathrm{Re}g(E)}{\nu[\mathrm{Im}g(E)]^{1-1/\nu}}\geq b,\quad E>E_0. \tag{59}$$

Then theorem 5 gives us for $E\gg E_0$

$$[\mathrm{Im}g(E)]^{1/\nu}\geq(2b/\pi)\ln(E/E_0)+\mathrm{const}$$

or

$$\mathrm{Im}g(E)\geq C[\ln(E/E_0)]^\nu. \tag{60}$$

Conversely, if we can find a $\nu>1$ such that

$$\frac{\mathrm{Re}g(E)}{\nu[\mathrm{Im}g(E)]^{1-1/\nu}}\leq b',\quad E>E_0, \tag{61}$$

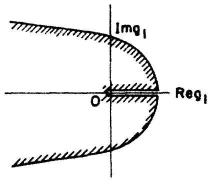

FIG. 7. Schematic drawing of the image of the upper half E plane by the mapping $g_1(E)$.

then, for $E \gg E_0$, we have

$$\operatorname{Im} g(E) \leq C'[\ln(E/E_0)]^{\gamma}. \tag{62}$$

Most results obtained so far in this section are useful when $g(E)$ diverges as E goes to infinity. This would be the case for example when the total cross section tends to a constant as $E \to +\infty$. To obtain some information about $f(E)$ in the case where $g(E)$ tends to a constant as $E \to +\infty$ [i.e., $\sigma_{\text{tot}}(E)$ approaches zero faster than $(\ln E)^{-a}$, $a > 1$], it is more advantageous to look at the function $g_1(E)$ defined by

$$g_1(E) = \int_0^E h(E')dE' = \int_0^E \frac{f(E') - f(0)}{E'} dE'. \tag{63}$$

If $g(\infty)$ is bounded, we need only one subtraction in writing down the dispersion relation for $f(E)$. Nevertheless, this dispersion relation has the same form as (11), and $h(E)$ defined by (12) is still a Herglotz function. It is then easy to show that $g_1(E)$ is regular and univalent in the upper-half E plane. The function $g_1(E)$ is real and monotonically increasing for $0 < E < \mu$, and $\operatorname{Im} g_1(E)$ increases monotonically as E increases from μ to $+\infty$. Thus, if we map the upper-half E plane into the g_1 plane, the image Γ_2 of the positive real E axis will lie in the upper-half g_1 plane and that of the negative real E axis will lie symmetrically below the real g_1 axis (see Fig. 7). We have no specific restriction on the sign of $\operatorname{Re} g_1(E)$. It may take both positive and negative values for $E > \mu$.

With the help of Ahlfors' theorem we can easily obtain the following theorems:

Theorem 7. If $g_1(E)$ satisfies for all $E > E_0$ the inequality

$$\operatorname{Re} g_1(E)/\operatorname{Im} g_1(E) \geq \tan \pi\alpha, \quad -\tfrac{1}{2} < \alpha < \tfrac{1}{2}, \tag{64}$$

we have

$$|g_1(E) \geq C(E/E_0)^{1+2\alpha} \tag{65}$$

for all E sufficiently larger than E_0. This theorem can be proved by the same method as that of theorem 2 except that we have to use (C18) instead of (C8) in the last step.

Theorem 8. If $g_1(E)$ satisfies for all $E > E_0$ the inequality

$$\operatorname{Re} g_1(E)/\operatorname{Im} g_1(E) \leq \tan \pi\alpha', \quad -\tfrac{1}{2} < \alpha' < \tfrac{1}{2}, \tag{66}$$

we have

$$|g_1(E)| \leq C(E/E_0)^{1+2\alpha'} \tag{67}$$

for all E sufficiently larger than E_0.

For α, $\alpha' > 0$, these theorems have essentially the same content as theorems 2 and 3. However, for α, $\alpha' < 0$ they give new information. Note also that

$$\alpha' > -\tfrac{1}{2}\left(1 + \frac{1}{\ln(E/\mu)}\right), \quad E \gg E_0, \tag{68}$$

according to (C16).

V. FROISSART BOUND AND THE ASYMPTOTIC PROPERTIES OF $g(E)$ AND $f(E)$

In order to discuss physical implications of the results obtained in Sec. IV, we shall assume now that the scattering amplitude $f(E)$ satisfies the Froissart bound

$$|f(E)| \leq C|E|(\ln|E|)^2 \tag{69}$$

for all energy E greater than some fixed E_1. Then $g(E)$ is subject to the condition

$$|g(E)| \leq C(\ln|E|)^3. \tag{70}$$

First we note that, if (70) is valid, the theorems of Sec. IV put severe restrictions on the possible asymptotic behavior of the ratio $\operatorname{Re} g(E)/\operatorname{Im} g(E)$. For example, one immediately sees from theorem 2 and its corollary (44) that, if $\operatorname{Re} g(E)/\operatorname{Im} g(E) \geq C(\ln E)^{-a}$, $0 < a < 1$, for all $E > E_0$, then $g(E)$ grows more rapidly than the right-hand side of (70). Thus such an asymptotic behavior for $\operatorname{Re} g(E)/\operatorname{Im} g(E)$ must be excluded if (70) is valid.

This result can be improved further by noting that, if

$$\frac{\operatorname{Re} g(E)}{\operatorname{Im} g(E)} > \frac{3\pi}{2} \frac{\ln(\ln E)}{\ln E}$$

holds for $E > E_0$, we are already in contradiction with the Froissart bound. This is easily shown using (42). In fact even better results might be obtained if we can estimate the integral of (28) more accurately than we did in Sec. IV. Instead of pursuing this line further, however, we shall give here a result obtained by utilizing the univalence of $g(E)$ in a somewhat different manner:

Theorem 9. If $g(E)$ satisfies the inequality

$$\operatorname{Re} g(E)/\operatorname{Im} g(E) \geq C/\ln E \tag{71}$$

for all $E > E_0$, then we have

$$|g(E)| > C'(\ln E)^{C/C_1}, \tag{72}$$

where C_1 is a sufficiently large fixed constant.

Proof. According to lemma 3 of Appendix C, $|\operatorname{Re} g(E)|$ is bounded from above by $C_1(E)|Eg'(E)|$, where $C_1(E)$ is finite for all E except possibly for those corresponding to very high and narrow peaks of $\operatorname{Im} f(E)/E^2$. For any fixed positive constant C_1, consider the set $R(C_1)$ of all points of the real E axis satisfying $C_1(E) < C_1$. Obviously $R(C_1)$ consists of a finite or infinite number of open intervals of the real E axis, grows monotonically as C_1 increases, and covers the whole real E axis in the limit $C_1=\infty$ except possibly for a set of measure zero. Thus, if we choose a sufficiently large C_1, $R(C_1)$ covers most of the real E axis. On such a set $R(C_1)$, we obtain from (71), (C8), and (C19) the inequality

$$\left|\frac{g'(iE)}{g(iE)}\right|\geq\frac{C}{C_1}\frac{1}{E(C+\ln E)}. \tag{73}$$

If we integrate both sides of (73) over the intersection of the interval (E_0,E) and the set $R(C_1)$, the result will not be substantially different from that of integration over the interval (E_0,E) insofar as C_1 is sufficiently large. Thus we obtain

$$|g(iE)|>C''(\ln E)^{C/C_1}.$$

Making use of (36) we arrive at (72). Q.E.D.

Obviously this theorem is rather weak if we have to choose a large C_1. Of course, if $\mathrm{Im}f(E)/E^2$ is a slowly varying function of E almost everywhere, we can choose a small C_1 as is seen from the considerations in Appendix C. In any case, theorem 9 makes it clear that the condition (71) is already inconsistent with (70) if the constant C is larger than $3C_1$. To be consistent with the Froissart bound, there must therefore be at least an infinite sequence of points $\{E_i\}$, $E_i\rightarrow+\infty$ as $i\rightarrow\infty$, or an infinite sequence of intervals on the positive real E axis for which

$$\mathrm{Re}g(E_i)/\mathrm{Im}g(E_i)\leq C/\ln E_i \tag{74}$$

holds for an appropriately chosen finite constant C. This is as far as we can go without making any specific assumption on possible oscillations of $\mathrm{Re}g(E)$ as $E\rightarrow+\infty$.

As is seen from theorem 3 and others, the smaller is the upper bound on $\mathrm{Re}g(E)/\mathrm{Im}g(E)$, the slower is the growth of $|g(E)|$ as $E\rightarrow\infty$. Thus, for instance, if (74) holds for all $E>E_0$, $g(E)$ is bounded by $(\ln E)^{C'}$ according to (51), where $C'=2\sqrt{2}C$. If $g(E)$ satisfies an even stronger condition such as $\mathrm{Re}g(E)/\mathrm{Im}g(E)\leq C(\ln E)^{-a}$, $a>1$, for all $E>E_0$, then $|g(E)|$ is bounded by a constant in the limit $E\rightarrow+\infty$, according to theorem 4. This would correspond to a total cross section that vanishes faster than $1/\ln E$ as $E\rightarrow+\infty$.[17] Thus, if the total cross section should approach a finite constant value at very large energy as is strongly indicated by the experimental data, $\mathrm{Re}g(E)/\mathrm{Im}g(E)$ must tend to zero as $E\rightarrow+\infty$ not much faster or much slower than $C/\ln E$.

[17] When $g(E)$ tends to a finite limit as $E\rightarrow+\infty$, one can gain more detailed information on the scattering amplitude by studying $g_1(E)$ defined by (63) rather than $g(E)$ itself. According to theorem 8, if $\mathrm{Re}g_1(E)/\mathrm{Im}g_1(E)\leq\tan\pi\alpha'$, which for negative α' means $|\mathrm{Re}g_1(E)|/\mathrm{Im}g_1(E)\geq|\tan\pi\alpha'|$, then $|g_1(E)|$ is bounded by $C|E|^{1-2|\alpha'|}$. This result is equivalent to the one obtained by theorem 1 of Ref. 2. Theorem 2 of Ref. 2 can be reproduced in the same manner.

The theorems of the previous section lead us to other interesting results if we make the physical assumption that $\mathrm{Re}f(E)$ has a definite sign beyond some energy E_1. For example, if $\mathrm{Re}f(E)\leq0$ for all real $E>E_1$, $\mathrm{Re}g(E)$ is monotonically decreasing for $E>E_1$. Hence

$$\mathrm{Re}g(E)\leq\mathrm{Re}g(E_1),\quad E>E_1.$$

According to theorem 6 we thus have the upper bound

$$|g(E)|<(2/\pi)\mathrm{Re}g(E_1)\ln E+\text{const} \tag{75}$$

for all $E\gg E_1$. This means that the total cross section must be bounded by some constant for almost all E in the sense that $\int_{E_1}^{E}[\sigma(E')/E']dE'\leq C\ln E$ as $E\rightarrow+\infty$. This result is valid irrespective of whether $\mathrm{Re}f(E)$ oscillates or not as far as it is negative for $E>E_1$. Obviously we do not have to assume the Froissart bound in this consideration.

On the other hand, if $\mathrm{Re}f(E)\geq0$ for $E>E_1$, then $\mathrm{Re}g(E)$ is monotonically increasing and thus

$$\mathrm{Re}g(E)>\text{const},\quad E>E_1.$$

Following theorem 5 we have

$$|g(E)|>(2/\pi)\mathrm{Re}g(E_1)\ln E+\text{const}. \tag{76}$$

Thus, in this case, the total cross section cannot go to zero smoothly as $E\rightarrow+\infty$. Conversely, if the total cross section diverges in such a way that $\mathrm{Im}g(E)>C(\ln E)^{\nu}$, $\nu>1$, it is impossible to find a finite constant C' such that $\mathrm{Re}g(E)<C'$ holds for all large E (see theorem 6). This means that $\mathrm{Re}g(E)$ must tend to infinity. In such a case $\mathrm{Re}f(E)$ cannot stay negative for all large E.

These results show that there is a strong correlation between the sign of $\mathrm{Re}f(E)$ and the boundedness of the total cross section at high energies. [Note, however, that $f(E)$ differs from the actual forward scattering amplitude by nucleon pole terms which tend to a constant as $E\rightarrow+\infty$.]

VI. POSSIBLE EXPERIMENTAL TEST OF ANALYTICITY AND CROSSING

One of the main purposes of the present series of investigations is to find out devices by means of which we can test experimentally theoretical predictions of local field theory. In Refs. 2 and 3 we proposed several inequalities which may be used for this purpose. Besides some quantities that are not sensitive to high-energy data, these inequalities contain only those quantities that can be determined from the experimental data of forward scattering amplitude at finite energies. In this section we shall give two more inequalities of this kind which are byproducts of Ahlfors' theorem.

We recall that the function $g(E)$ for real $E>\mu$ is fully determined by a measurement of $\mathrm{Re}f(E)$ and $\mathrm{Im}f(E)$ over physically accessible energy range, if the constants $g(\mu)$ and $f(0)$ are found by some means. These constants can be estimated using the dispersion relations. For $E<\mu$ the dispersion relation is known to work well and not sensitive to the actual value of the total cross section at very high energies.

We shall assume in the following that $\mathrm{Re}\,g(E)$ and $\mathrm{Im}\,g(E)$ have been measured for all energies in the range $E_1\leq E\leq E_2$, where $E_2\gg E_1$, and that the function $g(E)$ for real E, plotted in the w plane, looks like the curve in Fig. 8(b), where we put

$$z=\ln(E/\mu)-i\pi/2\,,\quad w(z)=-ig(E)\,. \tag{77}$$

In drawing this curve we have assumed that $\mathrm{Re}\,g(E)$ is nonincreasing [i.e., $\mathrm{Re}\,f(E)\leq 0$] for large E, although it is not absolutely necessary for our purpose. The data available at present is not qualitatively different from this figure.[18]

Let us now consider the mapping by $w(z)$ of the domain D in the z plane [see Fig. 8(a)] onto the strip of width $2\,\mathrm{Re}\,g(E_2)$ in the w plane shown in Fig. 8(b). The domain D is bounded by two vertical line segments with $\mathrm{Re}z=\ln(E_1/\mu)$ and $\mathrm{Re}z=\ln(E_2/\mu)$ and two Jordan curves which are maps of the boundary lines $\mathrm{Im}w(z)=\pm\mathrm{Re}\,g(E_2)$ of the strip by the inverse transformation $z=z(w)$. The images of the lines $\mathrm{Re}z=\ln(E_1/\mu)$ and $\mathrm{Re}z=\ln(E_2/\mu)$ are represented by the curves in the w plane connecting $-ig(E_1)$ with $-ig(-E_1)$ and $-ig(E_2)$ with $-ig(-E_2)$. We may now apply Ahlfors' inequality (28) to this mapping by putting $a=2\,\mathrm{Re}\,g(E_2)$, $x_1=\ln(E_1/\mu)$, $x_2=\ln(E_2/\mu)$, and $\theta(x)<\pi$. Thus we obtain

$$u_1(x_2)-u_2(x_1)\geq 2\,\mathrm{Re}\,g(E_2)[(1/\pi)\ln(E_2/E_1)-4]\,, \tag{78}$$

where u_1, u_2 are defined in the same manner as in (28). By definition

$$u_1(x_2)\leq \mathrm{Im}\,g(E_2)\,. \tag{79}$$

We are thus left with the job of estimating $u_2(x_1)$. In the case considered here, namely that of Fig. 8(b), it will be advantageous to take E_1 as small as possible. For instance, we may choose E_1 which corresponds to the intersection of the line $\mathrm{Im}w(z)=\mathrm{Re}\,g(E_2)$ with the low-energy data curve. If E_1 is small enough, we obtain $u_2(x_1)=|g(iE_1)|$ which can be easily estimated. From (78) and (79) we therefore obtain

$$\mathrm{Im}\,g(E_2)\geq 2\,\mathrm{Re}\,g(E_2)[(1/\pi)\ln(E_2/E_1)-4]+u_2(x_1)\,, \tag{80}$$

where $u_2(x_1)$ may be ignored for large E_2. This inequality would be most useful if the data are such that $\mathrm{Re}\,f(E)$ is negative but approaches zero at high energies. Then beyond certain energy $\mathrm{Re}\,g(E)$ does not decrease rapidly.

The inequality (80) can be somewhat improved if we use instead of (28) the more accurate formula

$$u_1(x_2)-u_2(x_1)>a\left\{\int_{x_1}^{x_2}\frac{dx}{\theta(x)}-\frac{4}{\pi}\ln 2+\frac{1}{\pi}\ln\left[1-8\exp\left(-\pi\int_{x_1}^{x_2}\frac{dx}{\theta(x)}\right)\right]\right\} \tag{81}$$

due to Teichmüller.[19] In the case where $E_2/E_1\gg 1$ this formula gives us the inequality

$$\mathrm{Im}\,g(E_2)>2\,\mathrm{Re}\,g(E_2)\left[\frac{1}{\pi}\ln\frac{E_2-8E_1}{E_1}-\frac{4}{\pi}\ln 2\right]\,, \tag{82}$$

where $u_2(x_1)$ is ignored.

We also note that, if $\mathrm{Re}\,g(E)$ is not monotonic, we have only to replace in the above argument the quantity $\mathrm{Re}\,g(E_2)$ by the smallest value of $\mathrm{Re}\,g(E)$ in the interval $E_1\leq E\leq E_2$.

We can obtain another useful inequality by switching the definition of z and w in (77). Namely we put

$$z=-ig(E)\,,\quad w(z)=\ln(E/\mu)-i\pi/2\,. \tag{83}$$

The domain D is now of the form indicated in Fig. 9(a). The function $w(z)$ maps D into a strip S of width π in the w plane, shown in Fig. 9(b). In applying (28) we note that $x_2=\mathrm{Im}\,g(E_2)$, $x_1=\mathrm{Im}\,g(E_1)$, and $a=\pi$. Thus we obtain

$$u_1(x_2)-u_2(x_1)\geq\pi\left[\int_{\mathrm{Im}g(E_1)}^{\mathrm{Im}g(E_2)}\frac{dx}{\theta(x)}-4\right]. \tag{84}$$

If we choose $E_1(\geq\mu)$ close enough to μ, we find that $u_2(x_1)=\ln(E_1/\mu)$. We also have

$$u_1(x_2)\leq\ln(E_2/\mu)\,. \tag{85}$$

Thus, choosing $E_1=\mu$, we obtain from (84) and (85) the

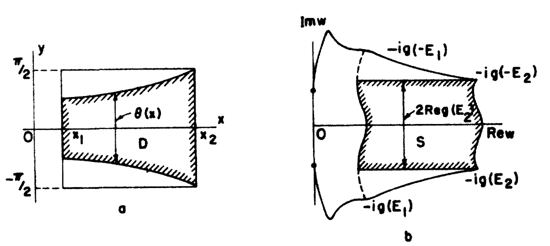

FIG. 8. The curve in (b) is a schematic drawing of the experimental data of $w(z)=-i\,g(E)$. The domain D in (a) is mapped onto the strip S in (b) by $w(z)$.

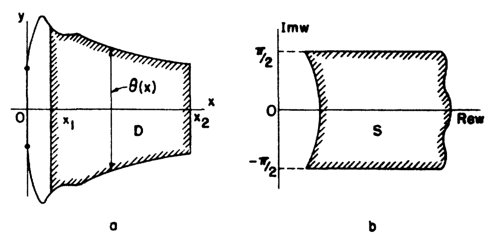

FIG. 9. The curve in (a) is the same as the curve in Fig. 8(b). The domain D in (a) is mapped onto the strip S in (b) by $w(z)$.

[18] Preliminary calculation of $\mathrm{Re}\,g(E)$ and $\mathrm{Im}\,g(E)$ based on the available experimental data has been carried out by E. Paschos.

[19] See Ref. 15, p. 100.

inequality

$$\ln(E_2/\mu)\geq\pi\left[\int_0^{\mathrm{Im}g(E_2)}\frac{dx}{\theta(x)}-4\right]. \qquad (86)$$

Here $\theta(x)$ is given by the data and is equal to 2 $\mathrm{Re}g(E_x)$. Again Teichmüller's inequality (81) improves (86) by allowing us to replace 4 inside the bracket of (86) by $(4/\pi)\ln 2$.

It is clear that (86) can be violated if the total cross section at high energies behaves roughly like a constant and if at the same time $\mathrm{Re}f(E)$ remains negative and large. For in such a case $\mathrm{Im}g(E_2)$ will grow as $\ln E_2$ and $\theta(x)$ will be a decreasing function of x.

As was mentioned in the Introduction, it is not easy to decide at present which one of the inequalities (80), (86), or others proposed earlier[3,20] is the most useful since it will depend on the detailed features of the data over the energy range $E_1\leq E\leq E_2$. Here we shall simply point out that (80) and (86) are complementary in the sense that while (80) gives an upper bound for $\mathrm{Re}g(E)$ [assuming that $\mathrm{Im}g(E)$ is known], (86) gives essentially a lower bound for $\mathrm{Re}g(E)$.

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APPENDIX A: REMARKS ON MEIMAN'S THEOREMS

In Ref. 2 we studied the high-energy behavior of the forward scattering amplitude making use of the theorems derived by Meiman.[21] In the proofs of these theorems given by him and also in the Appendix of Ref. 2, a strong, though not unreasonable, assumption was made concerning the absence of violent oscillations in the amplitude at large energies. (See, for example, footnote 23 of Ref. 2.) As one can see from theorem 1 of the present paper, however, such an assumption is actually not necessary to prove that only two subtractions are needed in the forward dispersion relation. Other results of this paper also show that major conclusions of Ref. 2 can be derived without using this assumption.

To see how much of these results can in fact be obtained within the framework of Ref. 2, it is sufficient to replace Meiman's theorem by theorem A given below. This theorem is weaker than Meiman's theorem but holds even if the scattering amplitude has violent oscillations. We shall first discuss theorem A and then state more precisely what we mean by "violent oscillation."

We consider a function $\phi(E)$ which is regular and bounded by a polynomial of E in the region $\mathrm{Im}E>0$ of the complex E plane. On the real axis it is assumed to be continuous and have the symmetry property

$$\phi(-E+i0)=\phi^*(E+i0). \qquad (A1)$$

Furthermore we assume

$$\lim_{E\to+\infty}\phi(E+i0)=0. \qquad (A2)$$

Thus $\phi(E)$ maps $\mathrm{Im}E>0$ into a domain of the ϕ plane, a neighborhood of $E=\infty$ being mapped into a neighborhood of $\phi=0$, which is perhaps many-sheeted. The upper edges of the semireal axes $(0,-\infty)$ and $(0,+\infty)$ are mapped onto the curves Γ_1 and Γ_2 symmetrically located with respect to the real ϕ axis. We shall assume that there exists some real large E_0 such that, for $E\geq E_0$, Γ_1 and Γ_2 have no common point except the point $\phi=0$. We further assume that for all $E\geq E_0$

$$|\mathrm{Im}\phi(E)/\mathrm{Re}\phi(E)|\geq\tan\pi\alpha,\quad 0<\alpha\leq\tfrac{1}{2}. \qquad (A3)$$

Then, without any further restrictions on Γ_1 and Γ_2, we obtain:

Theorem A. If the function $\phi(E)$ has the properties described above, we can find a sequence of real intervals $I_1, I_2, \cdots, I_n, \cdots$ such that

$$|\phi(E_n)|\leq C(E_0/E_n)^\alpha \qquad (A4)$$

holds for $E_n\in I_n$, $n=1, 2, \cdots$, where C is independent of E_n and $E_n\to\infty$ as $n\to\infty$.

The proof of this theorem follows closely that of theorem I in the Appendix of Ref. 2. It shows that, given any real interval (E_0,E), there is an E', $E_0\leq E'\leq E$ such that the inequality

$$|\phi(E')|\leq C'(E_0/E')^\alpha \qquad (A5)$$

holds. Obviously we can choose as E' the point at which $|\phi(x)|$ takes its least value in the interval $E_0\leq x\leq E$. E' thus defined is a (discontinuous) function of E which increases indefinitely to ∞ as $E\to\infty$ because of the property (A2) of the function $\phi(E)$. Since $\phi(E)$ is continuous in E for real E, the inequality (A5) can be satisfied in a small neighborhood of E' if we choose a somewhat larger C'. This completes the proof of theorem A.

We note that the power of E_0/E_n in (A4) is improved by a factor of 2 compared with Ref. 2. This is obtained by using a better estimate of the harmonic measure due to Hersch[7] according to which the formula (A6) of Ref. 2 can be replaced by the stronger inequality

$$\int_{s_1}^{s_2}\frac{ds}{\rho(s)}\geq-\ln\tan\left[\frac{\pi}{4}(1-m_w)\right]. \qquad (A6)$$

In order to make (A4) hold for all $E>E_0$ as in Meiman's theorem, it is necessary to impose some restrictions on the boundary curves Γ_1 and Γ_2, or, in other words, to exclude certain types of violent oscil-

[20] A. Martin, Phys. Letters **15**, 76 (1965).

[21] N. N. Meiman, Zh. Eksperim. i Teor. Fiz. **43**, 2277 (1962) [English transl.: Soviet Phys.—JETP **16**, 1609 (1963)].

lations of $\phi(E)$ for large E. For instance we may require that, for a sufficiently large constant C', $\phi(E)$ satisfies the "smoothness" condition

$$\max_{E_n\leq E\leq E_{n+1}}|\phi(E)/\phi(E_{n+1})|\leq C' \tag{A7}$$

for any $n\geq n_0>0$, where $E_1, E_2, \cdots, E_n, \cdots$ is the sequence for which (A4) holds. Then we obtain from (A4) and (A7) the relation

$$|\phi(E)|\leq CC'(E_0/E_{n+1})^\alpha\leq CC'(E_0/E)^\alpha. \tag{A8}$$

This shows that Meiman's theorem is valid even if the boundary curves Γ_1 and Γ_2 oscillate as far as the oscillation is mild in the sense of (A7).

If the oscillation is not so mild, we may no longer assume (A7). However, we may still obtain a useful upper bound on $\phi(E)$ if we can replace C' in (A7) by some known function of E. For example, if

$$\max_{E_n\leq E\leq E_{n+1}}|\phi(E)/\phi(E_{n+1})|\leq C'(\ln E_n)^p,\ p>0, \tag{A9}$$

holds for all $n\geq n_0$, we obtain

$$|\phi(E)|\leq CC'(E_0/E)^\alpha(\ln E)^p. \tag{A10}$$

A condition like (A9) may be a reasonable one to make in the case where the Froissart bound is assumed to hold as in Sec. II of Ref. 2. However, for Sec. III of Ref. 2 where only the Greenberg-Low bound was assumed, (A9) may have to be replaced by an even weaker one. Of course, theorem 1 of the present paper does not make use of any extra assumptions like (A7) and (A9).

It is worthwhile to emphasize again that some of the results of Sec. II of Ref. 2 follow from theorem A and the inequality (A4) alone. This is because they are proved essentially by producing contradictions, and having (A4) on a sequence of points is enough to produce such contradictions.

APPENDIX B: REMARK ON THE GREENBERG-LOW BOUND

We should like to give here an alternative proof of theorem 1 on the improvement of the Greenberg-Low bound making use of the techniques of univalent functions. We assume that $f(E)$ satisfies conditions (i)–(v) of Sec. II. We also assume for the moment that the dispersion relation for $f(E)$ does actually require three subtractions and that it diverges with two subtractions. We thus write

$$f(E)-f(0)-\tfrac{1}{2}f''(0)E^2=\frac{2E^4}{\pi}\int_\mu^\infty \frac{dE'\,\mathrm{Im}f(E')}{E'^3(E'^2-E^2)}. \tag{B1}$$

It is easily seen from (B1) that the function $H(E)$ defined by

$$H(E)\equiv[f(E)-f(0)-\tfrac{1}{2}f''(0)E^2]/E^3 \tag{B2}$$

is a Herglotz function.

We now consider the function

$$G_1(E)=\int_0^E H(E')dE'. \tag{B3}$$

This function has a property very similar to that of $g_1(E)$ defined by (63). It is univalent in the upper-half E plane and theorems 7 and 8 apply to it just as well as to $g_1(E)$. Under our assumption about the necessity for three subtractions $\int^E \mathrm{Im}f(E')E'^{-3}dE'$ diverges as $E\to\infty$. If we now make the assumption made in theorem 1, namely, that for real $E>E_0$

$$|\mathrm{Im}f(E)/\mathrm{Re}f(E)|\geq\tan\pi\alpha,\quad 0<\alpha\leq\tfrac{1}{2}, \tag{B4}$$

then we get a contradiction as in theorem 1. In order to see how this contradiction comes about, we first note that if $\int^E \mathrm{Im}f(E')E'^{-3}dE'$ diverges as $E\to+\infty$, (B4) leads us to

$$\tan\pi(\alpha-\tfrac{1}{2})\leq\mathrm{Re}G_1(E)/\mathrm{Im}G_1(E)\leq\tan\pi(\tfrac{1}{2}-\alpha) \tag{B5}$$

for $E\gg E_0$. We can now apply theorem 7 and obtain for large enough E

$$|G_1(E)|\geq C(E/E_0)^{1+2(\alpha-\frac{1}{2})}=C(E/E_0)^{2\alpha}. \tag{B6}$$

This will contradict the Greenberg-Low bound if $\alpha>0$, since the latter requires that

$$|G_1(E)|\leq C(\ln E)^3. \tag{B7}$$

Thus $\int^E \mathrm{Im}f(E')E'^{-3}dE'$ cannot diverge as $E\to+\infty$ if $\alpha>0$. We then find from (B4) that

$$\int^E|\mathrm{Re}f(E')|E'^{-3}dE'$$

cannot diverge too. This proves theorem 1.

By an argument similar to that of theorem 9, it may be possible to show further that the Greenberg-Low bound can be satisfied only if

$$\mathrm{Im}G_1(E)/(-\mathrm{Re}G_1(E))\leq C/\ln E \tag{B8}$$

for almost all real E greater than some E_1, where C is a certain positive number.

APPENDIX C: LEMMAS ON $g(E)$ AND $g_1(E)$

We shall first prove several lemmas on $g(E)$ making use of the formulas

$$\mathrm{Re}g(E)=\frac{1}{\pi}\int_\mu^\infty dE'\frac{\mathrm{Im}f(E')}{E'^2}\ln\left|\frac{E'+E}{E'-E}\right|, \tag{C1}$$

$$\mathrm{Im}g(E)=\int_\mu^E dE'\frac{\mathrm{Im}f(E')}{E'^2}, \tag{C2}$$

$$\mathrm{Im}g(iE)=\frac{2}{\pi}\int_\mu^\infty dE'\frac{\mathrm{Im}f(E')}{E'^2}\tan^{-1}\left(\frac{E}{E'}\right), \tag{C3}$$

for real positive E, which can be derived easily from the formulas (11) and (13).

Lemma 1. For any given real positive λ, there exists a real positive E_λ such that

$$\mathrm{Im}g(E_\lambda)=\mathrm{Im}g(i\lambda)=(1/i)g(i\lambda) \tag{C4}$$

holds.

Proof. Consider the integral

$$\mathrm{Im}g(i\lambda)=\int_\mu^\infty dE'\frac{\mathrm{Im}f(E')}{E'^2}\frac{2}{\pi}\tan^{-1}\left(\frac{\lambda}{E'}\right). \tag{C5}$$

Noting that $\mathrm{Im}f(E')$ is positive and $(2/\pi)\tan^{-1}(\lambda/E')$ is positive and less than one for all E' in the interval $(\mu,+\infty)$, we obtain

$$\int_\mu^\infty dE'\frac{\mathrm{Im}f(E')}{E'^2}\frac{2}{\pi}\tan^{-1}\left(\frac{\lambda}{E'}\right)<\int_\mu^\infty dE'\frac{\mathrm{Im}f(E')}{E'^2}. \tag{C6}$$

Since the function

$$\int_\mu^E dE'\frac{\mathrm{Im}f(E')}{E'^2}$$

is continuous, positive, and monotonically increasing for $\mu<E<+\infty$, we can find a positive real E_λ such that $\mu<E_\lambda<+\infty$ and

$$\int_\mu^\infty dE'\frac{\mathrm{Im}f(E')}{E'^2}\frac{2}{\pi}\tan^{-1}\left(\frac{\lambda}{E'}\right)=\int_\mu^{E_\lambda} dE'\frac{\mathrm{Im}f(E')}{E'^2}. \tag{C7}$$

We obtain (C4) from (C5) and (C7). Q. E. D.

It follows from Lemma 1 that, if $\mathrm{Im}g(i\lambda)\to+\infty$ as $\lambda\to+\infty$, $\mathrm{Im}g(E_\lambda)$ must necessarily go to infinity, too.

Lemma 2. For any real positive E, we have the inequality

$$|\mathrm{Im}g(iE)-\mathrm{Im}g(E)|<\mathrm{Re}g(E). \tag{C8}$$

Proof. Our proof is based on the inequality

$$\ln\frac{x+y}{x-y}>2\tan^{-1}\left(\frac{y}{x}\right)\quad\text{for}\quad x>y>0, \tag{C9}$$

which can be easily obtained by expanding both sides in power series in y/x. Now, from (C1), (C2), and (C3) we obtain

$$\begin{aligned}&\mathrm{Re}g(E)+\mathrm{Im}g(iE)-\mathrm{Im}g(E)\\&>\frac{1}{\pi}\int_\mu^E dE'\frac{\mathrm{Im}f(E')}{E'^2}\left\{\ln\frac{E+E'}{E-E'}+2\tan^{-1}\left(\frac{E}{E'}\right)-\pi\right\}\\&=\frac{1}{\pi}\int_\mu^E dE'\frac{\mathrm{Im}f(E')}{E'^2}\left\{\ln\frac{E+E'}{E-E'}-2\tan^{-1}\left(\frac{E'}{E}\right)\right\},\end{aligned} \tag{C10}$$

where the last term is positive according to (C9). Similarly, we have

$$\begin{aligned}&\mathrm{Re}g(E)+\mathrm{Im}g(E)-\mathrm{Im}g(iE)\\&>\int_\mu^E dE'\frac{\mathrm{Im}f(E')}{E'^2}\left\{1-\frac{2}{\pi}\tan^{-1}\left(\frac{E}{E'}\right)\right\}\\&+\frac{1}{\pi}\int_E^\infty dE'\frac{\mathrm{Im}f(E')}{E'^2}\left\{\ln\frac{E'+E}{E'-E}-2\tan^{-1}\left(\frac{E}{E'}\right)\right\},\end{aligned} \tag{C11}$$

where the right-hand side is again positive because of (C9). Q. E. D.

For the function $g_1(E)$ defined by (16), we have for positive real E

$$\mathrm{Re}g_1(E)=\frac{1}{\pi}\int_\mu^\infty dE'\frac{\mathrm{Im}f(E')}{E'}\ln\frac{E'^2}{|E'^2-E^2|}, \tag{C12}$$

$$\mathrm{Im}g_1(E)=\int_\mu^E dE'\frac{\mathrm{Im}f(E')}{E'}, \tag{C13}$$

$$g_1(iE)=-\frac{1}{\pi}\int_\mu^\infty dE'\frac{\mathrm{Im}f(E')}{E'}\ln\frac{E^2+E'^2}{E'^2}. \tag{C14}$$

Obviously, $\mathrm{Im}g_1(E)$ is positive and monotonically increasing, and $g_1(iE)$ is real, negative, and monotonically decreasing as E increases. From the inequality

$$\begin{aligned}\mathrm{Re}g_1(E)&>\frac{1}{\pi}\int_\mu^E dE'\frac{\mathrm{Im}f(E')}{E'}\left\{\ln\frac{E^2}{E^2-E'^2}+\ln\frac{E'^2}{E^2}\right\}\\&>\frac{2}{\pi}\int_\mu^E dE'\frac{\mathrm{Im}f(E')}{E'}\ln\frac{E'}{E},\end{aligned} \tag{C15}$$

we obtain

$$\mathrm{Re}g_1(E)>-\frac{2}{\pi}\ln\left(\frac{E}{\mu}\right)\mathrm{Im}g_1(E). \tag{C16}$$

Noting that

$$\mathrm{Re}g_1(E)+g_1(iE)=\frac{1}{\pi}\int_\mu^\infty dE'\frac{\mathrm{Im}f(E')}{E'}\ln\frac{E'^4}{|E'^4-E^4|}, \tag{C17}$$

we can also derive the inequality

$$\mathrm{Re}g_1(E)+\frac{1}{\pi}\ln\left(\frac{E}{\mu}\right)\mathrm{Im}g_1(E)>|g_1(iE)| \tag{C18}$$

by a method similar to (C15).

Lemma 3. For any real positive E, $\mathrm{Re}g(E)$ is bounded from above as follows:

$$\mathrm{Re}g(E)\le C_1(E)E|g'(iE)|, \tag{C19}$$

where $C_1(E)$ is finite for all E except possibly for those corresponding to very high and narrow peaks of $\mathrm{Im}f(E)/E^2$.

Proof. For $0\leq\theta\leq\pi$, $\mathrm{Re}g(Ee^{i\theta})$ can be expressed as

$$\mathrm{Re}g(Ee^{i\theta})=\frac{1}{\pi}\int_{\mu}^{\infty}dE'\frac{\mathrm{Im}f(E')}{E'^2}\ln\left|\frac{E'+Ee^{i\theta}}{E'-Ee^{i\theta}}\right|, \quad \text{(C20)}$$

as is seen from (15). Using the expansion

$$\ln\left|\frac{E'+Ee^{i\theta}}{E'-Ee^{i\theta}}\right|=\sum_{n=0}^{\infty}\frac{1}{2n+1}\left(\frac{2EE'\cos\theta}{E^2+E'^2}\right)^{2n+1}, \quad \text{(C21)}$$

which is convergent for $0<\theta<\pi$, we may put (C20) in the form

$$\mathrm{Re}g(Ee^{i\theta})=\sum_{n=0}^{\infty}\frac{1}{2n+1}a_{2n+1}(E)(\cos\theta)^{2n+1}, \quad \text{(C22)}$$

where

$$a_m(E)=\frac{1}{\pi}\int_{\mu}^{\infty}dE'\frac{\mathrm{Im}f(E')}{E'^2}\left(\frac{2EE'}{E^2+E'^2}\right)^m. \quad \text{(C23)}$$

Since $\mathrm{Im}f(E')$ is positive, $a_m(E)$ are all positive and decrease monotonically as m increases. Thus the series (C22) converges for all values of $\cos\theta$ in the range $-1<\cos\theta<1$. Furthermore, even at $\cos\theta=\pm1$, the series may converge if $a_m(E)$ decreases sufficiently rapidly as m increases. To find out how rapidly $a_m(E)$ decreases, let us examine the difference

$$a_{2n+1}(E)-a_{2n+3}(E)=\frac{1}{\pi}\int_{\mu}^{\infty}dE'\frac{\mathrm{Im}f(E')}{E'^2}\times\left[\left(\frac{2EE'}{E^2+E'^2}\right)^{2n+1}-\left(\frac{2EE'}{E^2+E'^2}\right)^{2n+3}\right]. \quad \text{(C24)}$$

Making use of the obvious inequality

$$\begin{aligned} x^m-x^{m+2}&\geq k^2x^m \quad \text{for} \quad k^2\leq1-x^2\leq1,\\ &\geq 0 \quad \text{for} \quad 0\leq1-x^2\leq k^2, \end{aligned} \quad \text{(C25)}$$

where k is a positive constant less than 1, we see that the integral in (C24) is bounded from below by

$$\frac{k^2}{\pi}\int_{\mu}^{\infty}dE'\frac{\mathrm{Im}f(E')}{E'^2}\left(\frac{2EE'}{E^2+E'^2}\right)^{2n+1}-\frac{k^2}{\pi}\int_{E_-}^{E_+}dE'\frac{\mathrm{Im}f(E')}{E'^2}\left(\frac{2EE'}{E^2+E'^2}\right)^{2n+1}, \quad \text{(C26)}$$

where

$$E_\pm=E\left(\frac{1\pm k}{1\mp k}\right)^{1/2}. \quad \text{(C27)}$$

Thus, we obtain a positive lower bound for $a_{2n+1}(E)-a_{2n+3}(E)$ if we can make the second integral definitely smaller than the first one by a proper choice of k. For this purpose we note that, if $k^2=2/(2n+3)$, the factor $[2EE'/(E^2+E'^2)]^{2n+1}$ in the second integral of (C26) varies gradually from 1 at $E=E'$ to

$$\left(\frac{2EE_\pm}{E^2+E_\pm^2}\right)^{2n+1}=\left(1+\frac{2}{2n+1}\right)^{-(2n+1)/2}\simeq e^{-1} \quad \text{(C28)}$$

at $E'=E_\pm$. More generally, for $k^2=1/(qn)$ where q is a positive constant, e^{-1} in (C28) is replaced by $e^{-1/q}$. From (C27) we also obtain

$$\frac{E_+-E_-}{2E}\simeq\frac{1}{qn}. \quad \text{(C29)}$$

Suppose that we have chosen E that does not correspond to the maxima of $\mathrm{Im}f(E)/E^2$. Then, for fixed $q(>1)$, we can always find a finite positive integer $n_0=n_0(E)$ such that the average value of $\mathrm{Im}f(E)/E^2$ in the interval $(E-E/qn_0, E+E/qn_0)$ is of the same order of magnitude as the average value of $\mathrm{Im}f(E)/E^2$ in the interval $(E-E/n_0, E+E/n_0)$. For such a choice of n_0, we find that the second integral of (C26) for $n>n_0$ is smaller than the first one by a factor of order $1/q$. Even when E is at the maximum of $\mathrm{Im}f(E)/E^2$, we can find a finite n_0 insofar as the width of this peak is not infinitesimally small. In all these cases we can obtain from (C24) and (C26) the inequality

$$a_{2n+3}(E)<a_{2n+1}(E)\left(1-\frac{C}{n}\right) \quad \text{(C30)}$$

for $n\geq n_0$, where C is a positive constant independent of $n(0<C<q^{-1}<1)$. This leads us to

$$\frac{a_{2n+1}(E)}{a_{2n_0+1}(E)}\leq\frac{C'}{n^C}, \quad n=n_0, \quad n_0+1, \quad \cdots, \quad \text{(C31)}$$

where C' is another positive constant. From (C22), (C31), and the fact that $a_{2n+1}(E)$ decreases monotonically for increasing n, we obtain

$$\lim_{\theta\to0}|\mathrm{Re}g(Ee^{i\theta})|<C_1(E)a_1(E), \quad \text{(C32)}$$

where

$$C_1(E)=\sum_{n=0}^{n_0-1}\frac{1}{2n+1}+\sum_{n=n_0}^{\infty}\frac{1}{2n+1}\frac{C'}{n^C}. \quad \text{(C33)}$$

Obviously $C_1(E)$ is finite if n_0 is finite. According to the above argument this means that $C_1(E)$ is finite for all E except possibly for those corresponding to very high peaks of $\mathrm{Im}f(E)/E^2$ with infinitesimally narrow widths. Noting that $a_1(E)=E|g'(iE)|$, we obtain (C19). Q. E. D.