Sum Rules for the Axial-Vector Coupling-Constant Renormalization in β Decay*

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Starting from the axial-vector current algebra suggested by Gell-Mann and the hypothesis of a partially conserved axial-vector current, we derive a sum rule relating $1-g_A^{-2}$ to off-mass-shell pion-proton total cross sections. Numerical evaluation gives the theoretical prediction $g_A=1.24$, in good agreement with experiment. A similar sum rule for pion-pion scattering can only be satisfied if there is a large low-energy $I=0$, S-wave pion-pion scattering cross section. We suggest tests, in high-energy neutrino reactions, of an algebra suggested by Gell-Mann for the vector and axial-vector current octets.

INTRODUCTION

WITHIN two years after the discovery of parity violation in the weak interactions, the main features of β decay were clarified.[1] It was found that only vector and axial-vector couplings are present. The vector coupling constant was found to be identical with the vector coupling constant in muon decay; the axial-vector coupling constant was found to differ by a factor $g_A\approx 1.2$ from the value expected for a pure $V-A$ interaction. The identity of the vector coupling constants in beta and in muon decay was soon explained by the hypothesis of a conserved vector current (CVC).[2] The value of the axial-vector coupling constant, on the other hand, has remained somewhat of a mystery.[3]

We give, in this paper, a theory of the axial-vector coupling-constant renormalization g_A, based on the axial-vector current algebra suggested by Gell-Mann[4] and on the hypothesis of a partially conserved axial-vector current (PCAC).[5] In Sec. I, we discuss the assumptions made. In Sec. II, we present two derivations of a sum rule relating $1-g_A^{-2}$ to off-mass-shell pion-proton total cross sections. Numerical evaluation of the sum rule, in Sec. III, gives the theoretical prediction $g_A=1.24$. In Sec. IV, we derive a sum rule relating $2g_A^{-2}$ to pion-pion scattering; we find that this sum rule can be satisfied only if there is a large low-energy $I=0$, S-wave pion-pion scattering cross section. In the final section, we propose tests, in high-energy neutrino experiments, of the algebra proposed by Gell-Mann[4] for the vector and the axial-vector current octets. The tests make no assumptions about partial conservation of the currents.

I. ASSUMPTIONS

The sum rules for g_A discussed below are derived from the following assumptions:

(A) The hadronic current responsible for $\Delta S=0$ leptonic decays is

$$J_\lambda=G_V\cos\theta(J_\lambda^{V1}+iJ_\lambda^{V2}+J_\lambda^{A1}+iJ_\lambda^{A2}),\qquad(1)$$

where G_V is the Fermi coupling constant ($G_V\approx 1.02\times 10^{-5}/M_N^2$) and $\cos\theta$ is the Cabibbo angle.[6] Here J_λ^{Va} is the vector current, which we assume to be the same as the isospin current, and J_λ^{Aa} is the axial-vector current. In the Fermi theory, we would have had

$$J_\lambda^{Va}=i:\bar\psi_N\gamma_\lambda\tfrac{1}{2}\tau^a\psi_N:,\qquad(2a)$$

$$J_\lambda^{Aa}=i:\bar\psi_N\gamma_\lambda\gamma_5\tfrac{1}{2}\tau^a\psi_N:.\qquad(2b)$$

Actually, we know that mesonic and other terms must be present. Fortunately, in what follows we will not have to assume any specific expressions for J_λ^V and J_λ^A in terms of particle fields.

Since the vector current is conserved, the vector coupling constant is unrenormalized. The renormalized axial-vector coupling constant g_A is defined by

$$\langle N(q)|J_\lambda|N(q)\rangle=(M_N/q_0)G_V\cos\theta\bar u_N(q)\times(\gamma_\lambda+g_A\gamma_\lambda\gamma_5)\tau^+U_N(q).\qquad(3)$$

(B) The axial-vector current is partially conserved (PCAC),

$$\partial_\lambda J_\lambda^{Aa}=\frac{M_NM_\pi^2g_A}{g_rK^{NN\pi}(0)}\phi_\pi^a.\qquad(4)$$

Here g_r is the rationalized, renormalized pion-nucleon coupling constant ($g_r^2/4\pi\approx 14.6$), $K^{NN\pi}(0)$ is the pionic form factor of the nucleon, normalized so that $K^{NN\pi}(-M_\pi^2)=1$, and ϕ_π^a is the renormalized pion field.

* An abbreviated version of the calculation of g_A has appeared in *Physical Review Letters* [S. L. Adler, Phys. Rev. Letters **14**, 1051 (1965)]. After this calculation was completed, I learned of similar work by Weisberger [W. I. Weisberger, Phys. Rev. Letters **14**, 1047 (1965)].

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[1] M. Goldhaber, *Proceedings of the 1958 Annual International Conference on High Energy Physics* (CERN, Geneva, 1958), p. 233.

[2] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).

[3] Previous papers on the axial-vector coupling constant renormalization include: R. J. Blin-Stoyle, Nuovo Cimento **10**, 132 (1958); S. Okubo, *ibid.* **13**, 292 (1959); J. Bernstein, M. Gell-Mann, and L. Michel, *ibid.* **16**, 560 (1960); A. P. Balachandran, *ibid.* **23**, 428 (1962); H. Banerjee, *ibid.* **23**, 1168 (1962); V. S. Mathur, R. Nath, and R. P. Saxena, *ibid.* **31**, 874 (1964); Y. S. Kim, *ibid.* **36**, 523 (1965); Y. Nambu and G. Jona-Lasinio, Phys. Rev. **124**, 246 (1961); Nguyen-Van-Hieu, Nucl. Phys. **42**, 129 (1963).

[4] M. Gell-Mann, Physics **1**, 63 (1964).

[5] M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960); Y. Nambu, Phys. Rev. Letters **4**, 380 (1960); S. L. Adler, Phys. Rev. **137**, B1022 (1965).

[6] N. Cabibbo, Phys. Rev. Letters **10**, 531 (1963).

According to Eq. (4), the chiralities

$$\chi^{\pm}(t)=-i\int d^3x(J_4^{A1}\pm iJ_4^{A2})$$

satisfy

$$\frac{d}{dt}\chi^{\pm}(t)=\frac{\sqrt{2}M_N M_\pi^2 g_A}{g_r K^{NN\pi}(0)}\int d^3x\, \phi_{\pi^\pm}. \tag{5}$$

(C) The axial-vector current satisfies the equal-time commutation relations

$$[J_4^{Aa}(x),J_4^{Ab}(y)]|_{x_0=y_0}=-\delta(\mathbf{x}-\mathbf{y})\epsilon^{abc}J_4^{Vc}(x). \tag{6}$$

This implies that the chiralities satisfy

$$[\chi^+(t),\chi^-(t)]=2I^3, \tag{7}$$

where I^3 is the third component of the isotopic spin.

The assumptions (A) are the usual ones for the leptonic decays. The vector-axial-vector form of the leptonic weak interactions is, of course, well established.[1] There is also considerable experimental evidence for the hypothesis[2] that the weak vector current J_λ^{Va} is the same as the isospin current.[7]

The hypothesis (B) of a partially conserved axial-vector current (PCAC) was introduced by Gell-Mann and Lévy[5] and by Nambu[5] to explain the successful Goldberger-Treiman relation[8] for charged pion decay. In addition to predicting the Goldberger-Treiman relation, PCAC predicts an experimentally satisfied relation between the pion-nucleon scattering amplitude $A^{\pi N(+)}$ and the pion-nucleon coupling constant g_r.[9]

The commutation relations (C) play an essential role in the calculation. [Note that Eq. (6) is a somewhat stronger assumption than Eq. (7), since even if spatial derivatives of the delta function were present on the right-hand side of Eq. (6), they would integrate to zero in Eq. (7). Only Eq. (7) is actually needed in the derivation below.] The hypothesis that Eq. (6) or Eq. (7) holds exactly is due to Gell-Mann.[4] Gell-Mann and Ne'eman have emphasized[10] that Eq. (7) is the most natural way in which one can make meaningful the idea of universality of strength between the weak couplings of leptons and baryons, without spelling out in detail the construction of J_λ^A from particle fields. Gell-Mann has also pointed out[11] that Eq. (7), by fixing the scale of the axial-vector current relative to the vector current, can, in principle, determine the axial-vector renormalization g_A.

To sum up, Eqs. (1), (3), (5), and (7) are the hypotheses on which our calculation of g_A is based. They are mutually consistent, in the sense that there is a renormalizable field theory (the σ model of Gell-Mann and Lévy[5]), in which they are exactly satisfied.

II. DERIVATIONS OF THE SUM RULE

We give, in this section, two different derivations of a sum rule expressing g_A in terms of off-mass-shell pion-proton total cross sections. A third derivation has been given by Weisberger.[12]

A. Method of Fubini and Furlan

The simplest derivation uses a method proposed recently by Fubini and Furlan.[13] We take the matrix element of Eq. (7) between single-proton states $\langle p(q)|$ and $|p(q')\rangle$. The right-hand side gives

$$\langle p(q)|2I^3|p(q')\rangle=(2\pi)^3\delta(\mathbf{q}-\mathbf{q}'). \tag{8}$$

In the matrix element of the commutator, we insert a complete set of intermediate states, separating out the one-nucleon term (to which only the neutron contributes):

$$\langle p(q)|[\chi^+(t),\chi^-(t)]|p(q')\rangle=\sum_{\text{spin}}\int\frac{d^3k}{(2\pi)^3}\langle p(q)|\chi^+(t)|n(k)\rangle\langle n(k)|\chi^-(t)|p(q')\rangle + \sum_{j\neq N}\langle p(q)|\chi^+(t)|j\rangle\langle j|\chi^-(t)|p(q')\rangle-(\chi^+\leftrightarrow\chi^-). \tag{9}$$

The one-neutron term is easily evaluated using Eq. (3), giving

$$\sum_{\text{spin}}\int\frac{d^3k}{(2\pi)^3}\langle p(q)|\chi^+(t)|n(k)\rangle\langle n(k)|\chi^-(t)|p(q')\rangle =\int\frac{d^3k}{(2\pi)^3}(2\pi)^3\delta(\mathbf{q}-\mathbf{k})(2\pi)^3\delta(\mathbf{k}-\mathbf{q}')\left(\frac{M_N}{q_0}\frac{M_N}{k_0}\right)g_A^2\bar{u}(q)\gamma_4\gamma_5\left(\frac{k+iM_N}{2iM_N}\right)\gamma_4\gamma_5 u(q') \tag{10}$$
$$=(2\pi)^3\delta(\mathbf{q}-\mathbf{q}')g_A^2(1-M_N^2/q_0^2).$$

[7] C. S. Wu, Rev. Mod. Phys. 36, 618 (1964).
[8] M. L. Goldberger and S. B. Treiman, Phys. Rev. 109, 193 (1958).
[9] S. L. Adler, Ref. 5.
[10] M. Gell-Mann and Y. Ne'eman, Ann. Phys. (N. Y.) 30, 360 (1964).
[11] M. Gell-Mann, Phys. Rev. 125, 1067 (1962).
[12] W. I. Weisberger, Phys. Rev. Letters 14, 1047 (1965).
[13] S. Fubini and G. Furlan, Physics 1, 229 (1965).

In the summation over higher intermediate states we make use of Eq. (5), giving

$$\left[\frac{\sqrt{2}M_N M_\pi{}^2 g_A}{g_r K^{NN\pi}(0)}\right]^2 \sum_{j\neq N} \frac{\langle p(q)|\int d^3x\,\phi_{\pi^+}|j\rangle\langle j|\int d^3x\,\phi_{\pi^-}|p(q')\rangle}{(q_0-q_{j0})^2} - (\pi^+\leftrightarrow\pi^-)\,. \tag{11}$$

From Eqs. (10) and (11), we see that there is a family of sum rules, with q_0 as a parameter. In the limit as q_0 approaches infinity, a sum rule for $1-g_A{}^{-2}$ is obtained. Let us assume that the limiting operation can be taken *inside* the sum over intermediate states in Eq. (11). It is useful to write this sum in the form

$$\sum_{j\neq N} = \int\frac{d^3q_j}{(2\pi)^3}\int_{M_N+M_\pi}^{\infty} dW \sum_{\substack{j\neq N\\ \text{INT}}} \delta(W-M_j)\,, \tag{12}$$

where $\mathbf{q}_j$ is the total momentum and where "INT" denotes the internal variables of the system j. We have denoted by M_j the invariant mass of the system j. The integrations over $\mathbf{x}$ and $\mathbf{q}_j$ can be done explicitly, giving a factor $(2\pi)^3\delta(\mathbf{q}-\mathbf{q}')$ and constraining $\mathbf{q}_j$ to be equal to $\mathbf{q}$. Let us write

$$\langle j|\phi_{\pi^\pm}(0)|p(q)\rangle = ((M_N/q_0)(M_j/q_{j0}))^{1/2}F_j{}^\pm\,, \tag{13}$$

so that $F_j{}^\pm$ is a Lorentz scalar. Then we have for the summation over higher intermediate states,

$$(2\pi)^3\delta(\mathbf{q}-\mathbf{q}')\left[\frac{\sqrt{2}M_N M_\pi{}^2 g_A}{g_r K^{NN\pi}(0)}\right]^2\int_{M_N+M_\pi}^{\infty} dW \sum_{\substack{j\neq N\\ \text{INT}}} \delta(W-M_j)(M_N/q_0)(M_j/q_{j0})(q_0-q_{j0})^{-2}[|F_j{}^-|^2-|F_j{}^+|^2]. \tag{14}$$

Using the equations

$$q_{j0}=(q_0{}^2+M_j{}^2-M_N{}^2)^{1/2}\,, \tag{15a}$$

$$(q_0-q_{j0})^{-2}=(q_0+q_{j0})^2/(M_j{}^2-M_N{}^2)^2\,, \tag{15b}$$

the limit as $q_0\to\infty$ of Eq. (14) becomes

$$\left[\frac{\sqrt{2}M_N g_A}{g_r K^{NN\pi}(0)}\right]^2(2\pi)^3\delta(\mathbf{q}-\mathbf{q}')\int_{M_N+M_\pi}^{\infty} dW\frac{M_N W}{(W^2-M_N{}^2)^2}\lim_{q_0\to\infty}\left\{\frac{[q_0+(q_0{}^2+W^2-M_N{}^2)^{1/2}]^2}{q_0(q_0{}^2+W^2-M_N{}^2)^{1/2}}\right\}$$
$$\times\lim_{q_0\to\infty}[K^-[W,(q-q_j)^2]-K^+[W,(q-q_j)^2]], \tag{16}$$

where we have defined $K^\pm[W,(q-q_j)^2]$ by the equation

$$K^\pm[W,(q-q_j)^2]=\sum_{\substack{j\neq N\\ \text{INT}}}\delta(W-M_j)M_\pi{}^4|F_j{}^\pm|^2\,. \tag{17}$$

Note that $K^\pm$ can only depend on the indicated variables because (i) $K^\pm$ is a Lorentz scalar, and (ii) all internal variables are summed over.[14]

It is now trivial to take the indicated limits. The limit of the quantity in curly brackets is 4, and the limit of the momentum transfer $(q-q_j)^2=-[q_0-(q_0{}^2+W^2-M_N{}^2)^{1/2}]^2$ is 0. Thus we are left with the sum rule

$$1-\frac{1}{g_A{}^2}=\frac{2M_N{}^2}{g_r{}^2K^{NN\pi}(0)^2}\int_{M_N+M_\pi}^{\infty}\frac{4M_N W dW}{(W^2-M_N{}^2)^2}[K^+(W,0)-K^-(W,0)]\,. \tag{18}$$

To complete the derivation, we must express $K^\pm(W,0)$ in terms of pion-proton scattering cross sections. Let $\sigma_0{}^\pm(W)$ denote the total cross section for scattering of a *zero-mass* $\pi^\pm$ on a proton, at center-of-mass energy W. It is easiest to calculate $\sigma_0{}^\pm(W)$ in the center-of-mass frame. If we let k and q be, respectively, the four-momenta of

[14] An average over initial proton spin is understood, but is not indicated explicitly.

the initial pion and proton, then we have[14]

$$\sigma_0^{\pm}(W)\cdot\text{flux}=(2\pi)^4\sum_{j\neq N}\frac{|\langle j|J_{\pi^\pm}(0)|p(q)\rangle|^2}{2k_0}\delta^4(q_j-q-k)$$
$$=(2\pi)^4\int\frac{d^3q_j}{(2\pi)^3}\sum_{\substack{j\neq N\\ \text{INT}}}\frac{|\langle j|J_{\pi^\pm}(0)|p(q)\rangle|^2}{2k_0}\delta^4(q_j-q-k)$$
$$=2\pi\sum_{\substack{j\neq N\\ \text{INT}}}\frac{|\langle j|J_{\pi^\pm}(0)|p(q)\rangle|^2}{2k_0}\delta(q_{j0}-q_0-k_0). \quad (19)$$

Keeping in mind the fact that the initial pion has zero mass ($k^2=0$), the following center-of-mass-frame equations may be derived:

$$q_0+k_0=W,\quad q_{j0}\equiv M_j; \quad (20a)$$

$$\text{flux}=|\mathbf{k}|/k_0+|\mathbf{k}|/q_0=W/q_0; \quad (20b)$$

$$k_0=(W^2-M_N^2)/(2W); \quad (20c)$$

$$\langle j|J_{\pi^\pm}(0)|p(q)\rangle=M_\pi^2\langle j|\phi_{\pi^\pm}(0)|p(q)\rangle$$
$$=M_\pi^2(M_N/q_0)^{1/2}F_j^{\pm}. \quad (20d)$$

Combining Eqs. (19) and (20) gives

$$\sigma_0^{\pm}(W)=(2\pi M_N/(W^2-M_N^2))\sum_{\substack{j\neq N\\ \text{INT}}}\delta(W-M_j)M_\pi^4|F_j^{\pm}|^2$$
$$=(2\pi M_N/(W^2-M_N^2))K^{\pm}(W,0). \quad (21)$$

Comparing with Eq. (18), we get the simple and exact sum rule

$$1-\frac{1}{g_A^2}=\frac{4M_N^2}{g_r^2K^{NN\pi}(0)^2}\frac{1}{\pi}\int_{M_N+M_\pi}^{\infty}\frac{WdW}{W^2-M_N^2}\times[\sigma_0^+(W)-\sigma_0^-(W)]. \quad (22)$$

While the derivation just given is straight-forward, it suffers from the defect of requiring an additional assumption: We must assume that the limit $q_0\to\infty$ can be taken inside the sum over interemdiate states in Eq. (11). The next derivation which we give clarifies the meaning of this assumption.

B. "PCAC Consistency Condition" Method

In two previous papers[15] (hereinafter called I and II), we showed that the hypothesis of a partially conserved axial-vector current leads to consistency conditions involving strong-interaction scattering amplitudes. The method used is a general one. Suppose that we have local field operators $j_\lambda(x)$ and $d(x)$ which satisfy the equation

$$\partial_\lambda j_\lambda(x)=d(x). \quad (23)$$

Let us take the matrix element of this equation between states $\langle\beta(k_F)|$ and $|\alpha(k_I)\rangle$. We get the equation

$$-i(k_F-k_I)_\lambda\langle\beta(k_F)|j_\lambda(0)|\alpha(k_I)\rangle=\langle\beta(k_F)|d(0)|\alpha(k_I)\rangle. \quad (24)$$

Let us now consider what happens as $(k_F-k_I)\to 0$. In this limit, only those pole terms of $\langle\beta(k_F)|j_\lambda(0)|\alpha(k_I)\rangle$ which behave as $(k_F-k_I)^{-1}$ will contribute to the left-hand side of Eq. (24). It was shown in (II) that these singularities arise only from insertions of the vertex of j_λ on *external* lines of $\langle\beta|\alpha\rangle$. Furthermore, in the limit as $(k_F-k_I)\to 0$, these insertions leave the external particles on mass shell. Thus we get a "consistency condition" expressing

$$\lim_{(k_F-k_I)\to 0}\langle\beta(k_F)|d(0)|\alpha(k_I)\rangle \quad (25)$$

in terms of the physical matrix element $\langle\beta|\alpha\rangle$. Clearly, the same procedure can be applied to the quantities

$$j(t)=\int d^3x\, j_4(\mathbf{x},t)\quad\text{and}\quad d(t)=\int d^3x\, d(\mathbf{x},t),$$

which satisfy the equation

$$dj(t)/dt=id(t). \quad (26)$$

Of course, the resulting formulas will not be manifestly covariant. What was done in (II) was to study in detail the case when $j(t)$ is simply the chirality $\chi^a(t)$. We will now apply the same method to a somewhat more complicated object,

$$j(x_0)=\int dy_0\, e^{-ik_0y_0}\langle N(q)|T[\chi^a(x_0)\chi^b(y_0)]|N(q)\rangle, \quad (27)$$

in order to rederive the sum rule for g_A.

Let us consider the quantity T defined by

$$T=\int dx_0\, e^{il_0x_0}\int dy_0\, e^{-ik_0y_0}\times\langle N(q)|T[\chi^a(x_0)\chi^b(y_0)]|N(q)\rangle$$
$$=\int dx_0\, e^{il_0x_0}j(x_0). \quad (28)$$

[15] S. L. Adler, Phys. Rev. **137**, B1022 (1965), hereinafter called I; S. L. Adler, Phys. Rev. **139**, B1638 (1965), hereinafter called II. See also the related papers: Y. Nambu and D. Lurié, *ibid.* **125**, 1429 (1962); Y. Nambu and E. Shrauner, *ibid.* **128**, 862 (1962).

Let us also define $P^a(x)$ by the equation

$$\partial_\lambda J_\lambda^{Aa}(x)=P^a(x), \tag{29}$$

so that the chirality $\chi^a(x_0)$ satisfies

$$\frac{d}{dx_0}\chi^a(x_0)=\int d^3x\, P^a(x). \tag{30}$$

We will introduce the assumption that $P^a(x)\propto\phi_\pi{}^a(x)$ at a later stage of the calculation.

From time-translation invariance, we know that

$$j(x_0)=e^{-ik_0x_0}\times\text{constant}. \tag{31}$$

Consequently,

$$\begin{aligned}-ik_0j(x_0)=\frac{d}{dx_0}j(x_0)&=\int dy_0\, e^{-ik_0y_0}\langle N(q)|\frac{d}{dx_0}T[\chi^a(x_0)\chi^b(y_0)]|N(q)\rangle\\&=e^{-ik_0x_0}\langle N(q)|[\chi^a(x_0),\chi^b(x_0)]|N(q)\rangle+\int dy_0\int d^3x\, e^{-ik_0y_0}\langle N(q)|T[P^a(x)\chi^b(y_0)]|N(q)\rangle.\end{aligned} \tag{32}$$

Since the second term on the right-hand side of Eq. (32) is proportional to $\exp(-ik_0x_0)$, we can rewrite it as

$$\frac{1}{-k_0^2+M_\pi^2}\int dy_0\int d^3x\, e^{-ik_0y_0}(-\Box_x+M_\pi^2)\langle N(q)|T[P^a(x)\chi^b(y_0)]|N(q)\rangle. \tag{33}$$

We have assumed that we can integrate by parts with respect to the *spatial* variables $\mathbf{x}$; this can be justified by the use of wave packets.[16] Combining Eqs. (28), (32), and (33), and then interchanging the order of the integrations over x_0 and y_0, gives

$$\begin{aligned}-ik_0T=&\int dx_0\, e^{i(l_0-k_0)x_0}\langle N(q)|[\chi^a(x_0),\chi^b(x_0)]|N(q)\rangle\\&+\int dx_0\, e^{il_0x_0}\frac{1}{M_\pi^2-k_0^2}\int dy_0\int d^3x\, e^{-ik_0y_0}(-\Box_x+M_\pi^2)\langle N(q)|T[P^a(x)\chi^b(y_0)]|N(q)\rangle\\=&\,2\pi\delta(l_0-k_0)\langle N(q)|[\chi^a(0),\chi^b(0)]|N(q)\rangle+\frac{1}{M_\pi^2-k_0^2}\int dy_0\, e^{-ik_0y_0}j_1(y_0),\end{aligned} \tag{34}$$

with

$$\begin{aligned}j_1(y_0)&=\int d^4x\, e^{il_0x_0}(-\Box_x+M_\pi^2)\langle N(q)|T[P^a(x)\chi^b(y_0)]|N(q)\rangle\\&=e^{il_0y_0}\times\text{constant}.\end{aligned} \tag{35}$$

Treating $j_1(y_0)$ in the same manner as we treated $j(x_0)$, we get

$$\begin{aligned}il_0j_1(y_0)=&\,M_\pi^2\int d^3x\, e^{il_0y_0}\langle N(q)|[\chi^b(y_0),P^a(\mathbf{x},y_0)]|N(q)\rangle\\&+\frac{1}{M_\pi^2-l_0^2}\int d^4x\int d^3y\, e^{il_0x_0}(-\Box_x+M_\pi^2)(-\Box_y+M_\pi^2)\langle N(q)|T[P^a(x)P^b(y)]|N(q)\rangle.\end{aligned} \tag{36}$$

To sum up, we have derived the identity

$$\begin{aligned}&-ik_0\int dx_0\, e^{il_0x_0}\int dy_0\, e^{-ik_0y_0}\langle N(q)|T[\chi^a(x_0)\chi^b(y_0)]|N(q)\rangle\\&=2\pi\delta(l_0-k_0)\Big[\langle N(q)|[\chi^a(0),\chi^b(0)]|N(q)\rangle+\Big(\frac{M_\pi^2}{M_\pi^2-k_0^2}\Big)\frac{1}{il_0}\int d^3x\langle N(q)|[\chi^b(0),P^a(\mathbf{x},0)]|N(q)\rangle\Big]\\&\quad+\frac{1}{(M_\pi^2-k_0^2)(M_\pi^2-l_0^2)}\frac{1}{il_0}\int d^4x\int d^4y\, e^{il_0x_0-ik_0y_0}(-\Box_x+M_\pi^2)(-\Box_y+M_\pi^2)\langle N(q)|T[P^a(x)P^b(y)]|N(q)\rangle.\end{aligned} \tag{37}$$

[16] We will never integrate by parts with respect to the time variable.

Since we will obtain the sum rule for g_A from the part of Eq. (37) which is *antisymmetric* in a and b, let us drop all terms which are symmetric. Because $[\chi^a(x_0),\chi^b(x_0)]=i\epsilon^{abc}I^c$, and since $dI^c/dx_0=0$, we have $d[\chi^a(x_0),\chi^b(x_0)]/dx_0=0$. In other words,

$$\int d^3x[P^a(\mathbf{x},x_0),\chi^b(x_0)]=\int d^3x[P^b(\mathbf{x},x_0),\chi^a(x_0)], \tag{38}$$

indicating symmetry under interchange of a and b. Thus we can drop the term proportional to

$$\langle N(q)|[\chi^b(0),P^a(\mathbf{x},0)]|N(q)\rangle.$$

Let us now consider the antisymmetric part of Eq. (37) for small k_0. At the end of the calculation, we will let k_0 approach 0. On the left-hand side, only diagrams with χ^a inserted on the external nucleon lines will make a contribution of zeroth order in k_0, as was shown in (II). This can be seen directly by inserting a complete set of intermediate states in the time-ordered product:

$$\begin{aligned}&\int dx_0\int dy_0 e^{il_0x_0-ik_0y_0}\langle N(q)|T[\chi^a(x_0)\chi^b(y_0)]|N(q)\rangle\\
&=\int dx_0\int dy_0 e^{il_0x_0-ik_0y_0}\sum_j [\langle N(q)|\chi^a(x_0)|j\rangle\langle j|\chi^b(y_0)|N(q)\rangle\theta(x_0-y_0)\\
&\qquad\qquad+\langle N(q)|\chi^b(y_0)|j\rangle\langle j|\chi^a(x_0)|N(q)\rangle\theta(y_0-x_0)]\\
&=\sum_j [\langle N(q)|J_4{}^{Aa}(0)|j\rangle\langle j|J_4{}^{Ab}(0)|N(q)\rangle i(k_0-\Delta_j)^{-1}-\langle N(q)|J_4{}^{Ab}(0)|j\rangle\langle j|J_4{}^{Aa}(0)|N(q)\rangle i(k_0+\Delta_j)^{-1}]\\
&\qquad\qquad\times 2\pi\delta(l_0-k_0)(2\pi)^6\delta(\mathbf{0})\delta(\mathbf{q}_j-\mathbf{q}),\end{aligned}\tag{39}$$

where

$$\Delta_j\equiv(q_0^2+M_j^2-M_N^2)^{1/2}-q_0. \tag{40}$$

Clearly, only the one-nucleon intermediate state ($j=N$, $\Delta_j\equiv 0$) gives a singularity behaving as k_0^{-1}. Evaluation of the spin sum, as in Eq. (10), gives, for the left-hand side of Eq. (37),

$$(2\pi)^4\delta(\mathbf{0})\delta(l_0-k_0)g_A^2 i\epsilon^{abc}\langle\tfrac{1}{2}\tau^c\rangle(1-M_N^2/q_0^2)+O(k_0), \tag{41}$$

where $O(k_0)$ indicates terms which vanish as $k_0\to 0$.

Let us now evaluate the terms of the right-hand side of Eq. (37). The commutator of the chiralities is easily evaluated, using Eq. (6), giving

$$2\pi\delta(l_0-k_0)\langle N(q)|[\chi^a(0),\chi^b(0)]|N(q)\rangle=(2\pi)^4\delta(\mathbf{0})\delta(l_0-k_0)i\epsilon^{abc}\langle\tfrac{1}{2}\tau^c\rangle. \tag{42}$$

In the last term of Eq. (37), let us introduce the PCAC hypothesis,

$$P^a(x)=\frac{M_NM_\pi^2g_A}{g_rK^{NN\pi}(0)}\phi_\pi{}^a(x), \tag{43}$$

giving

$$\left(\frac{M_\pi^2}{M_\pi^2-k_0^2}\right)\left(\frac{M_\pi^2}{M_\pi^2-l_0^2}\right)\left[\frac{M_Ng_A}{g_rK^{NN\pi}(0)}\right]^2\frac{1}{il_0}\int d^4x\int d^4y\, e^{il_0x_0-ik_0y_0}$$
$$\times(-\square_x+M_\pi^2)(-\square_y+M_\pi^2)\langle N(q)|T[\phi_\pi{}^a(x)\phi_\pi{}^b(y)]|N(q)\rangle. \tag{44}$$

Apart from factors, this is just a pion-nucleon scattering amplitude. In fact, the off-mass-shell pion-nucleon scattering amplitudes

$$A^{\pi N(-)}(\nu,\nu_B,M_\pi{}^i,M_\pi{}^f)\quad\text{and}\quad B^{\pi N(-)}(\nu,\nu_B,M_\pi{}^i,M_\pi{}^f),$$

where $M_\pi{}^i$ and $M_\pi{}^f$ are, respectively, the masses of the initial and final pion, are defined by[17]

$$\int d^4x \int d^4y\, e^{-il\cdot x}e^{ik\cdot y}(-\Box_x+M_\pi{}^2)(-\Box_y+M_\pi{}^2)\langle N(q_2)|T[\phi_\pi{}^a(x)\phi_\pi{}^b(y)]|N(q_1)\rangle$$

$$\equiv -i(2\pi)^4\delta(q_1+k-q_2-l)((M_N/q_{10})(M_N/q_{20}))^{1/2}$$

$$\times \bar{u}_N(q_2)\{[A^{\pi N(-)}(\nu,\nu_B,M_\pi{}^i,M_\pi{}^f)-ikB^{\pi N(-)}(\nu,\nu_B,M_\pi{}^i,M_\pi{}^f)]\tfrac{1}{2}[\tau^a,\tau^b]+\text{isospin symmetric}\}u_N(q_1)\,, \tag{45a}$$

$$\nu_B=k\cdot l/(2M_N),\quad \nu=-k\cdot(q_1+q_2)/(2M_N)\,. \tag{45b}$$

The term B can be separated into pole terms,[17] and a nonpole part which we label $\bar{B}$:

$$B^{\pi N(-)}=(g_r{}^2/2M_N)K^{NN\pi}[-(M_\pi{}^i)^2]K^{NN\pi}[-(M_\pi{}^f)^2]((\nu_B-\nu)^{-1}+(\nu_B+\nu)^{-1})+\bar{B}^{\pi N(-)}\,. \tag{46}$$

The integral in Eq. (44) is identical with Eq. (45), with

$$l=(0,il_0)=k=(0,ik_0),\quad M_\pi{}^i=M_\pi{}^f=k_0\,;\qquad \nu_B=-k_0{}^2/(2M_N),\quad \nu=q_0k_0/M_N\,. \tag{47}$$

Combining Eqs. (44), (45), (46), and (47), we find that Eq. (44) becomes

$$(2\pi)^4\delta(0)\delta(l_0-k_0)i\epsilon^{abc}\langle\tfrac{1}{2}\tau^c\rangle\left\{-g_A{}^2M_N{}^2/q_0{}^2-\frac{2M_N{}^2}{g_r{}^2K^{NN\pi}(0)^2}g_A{}^2\frac{1}{\nu}[A^{\pi N(-)}(\nu,0,0,0)+\nu\bar{B}^{\pi N(-)}(\nu,0,0,0)]\right\}+O(k_0{}^2)\,, \tag{48}$$

with $\nu=q_0k_0/M_N$. The term proportional to $-g_A{}^2M_N{}^2/q_0{}^2$ arises from the Born term in Eq. (46) when the substitutions of Eq. (47) are made, and just cancels the similar term in Eq. (41). Thus, in the limit as $k_0\to 0$, we obtain from Eq. (37) the Lorentz-invariant identity

$$1-\frac{1}{g_A{}^2}=\frac{-2M_N{}^2}{g_r{}^2K^{NN\pi}(0)^2}G(0)\,, \tag{49}$$

where

$$G(\nu)=\nu^{-1}[A^{\pi N(-)}(\nu,0,0,0)+\nu\bar{B}^{\pi N(-)}(\nu,0,0,0)]$$
$$=\nu^{-1}[A^{\pi N(-)}(\nu,0,0,0)+\nu B^{\pi N(-)}(\nu,0,0,0)]\,. \tag{50}$$

We are able to drop the bar on B because the Born term $(\nu_B-\nu)^{-1}+(\nu_B+\nu)^{-1}$ vanishes identically at $\nu_B=0$.

Equation (49), which follows solely from the assumptions of Sec. I, is our final result. From the crossing and analyticity properties of $A^{\pi N(-)}$ and $B^{\pi N(-)}$, we know that $G(\nu)$ is an even function of ν and is analytic in the ν plane, apart from cuts running from $\pm[M_\pi+M_\pi{}^2/(2M_N)]$ to $\pm\infty$. Let us assume that $G(\nu)$ satisfies an unsubtracted dispersion relation in the variable ν. Then we may write

$$G(0)=\frac{2}{\pi}\int_{M_\pi+M_\pi{}^2/(2M_N)}^{\infty}\frac{d\nu}{\nu}\,\mathrm{Im}G(\nu)\,. \tag{51}$$

It is easily verified that

$$\mathrm{Im}G(\nu)=\tfrac{1}{2}(\sigma_0{}^- - \sigma_0{}^+)\,. \tag{52}$$

Changing the integration variable from ν to the center-of-mass energy W $[\nu=(W^2-M_N{}^2)/(2M_N)]$, and combining Eqs. (49), (51), and (52) leads to the sum rule of Eq. (22). *Thus, the assumption that the limit $q_0\to\infty$ may be taken inside the sum over intermediate states in the method of Fubini and Furlan is equivalent to the assumption that $G(\nu)$ obeys an unsubtracted dispersion relation.*

There is evidence that an unsubtracted dispersion relation for $G(\nu)$ is valid. First of all, provided that the Pomeranchuk theorem is valid, the integral in Eq. (22) is convergent. Secondly, Amblard *et al.* and Höhler *et al.* have shown[18] that the forward charge-exchange scattering amplitude

$$A^{\pi N(-)}(\nu,-M_\pi{}^2/(2M_N),M_\pi,M_\pi)$$
$$+\nu B^{\pi N(-)}(\nu,-M_\pi{}^2/(2M_N),M_\pi,M_\pi)$$

satisfies an unsubtracted dispersion relation. It would be surprising if this result were changed by the extrapolation of the external pion mass from M_π to 0. Clearly, if a subtraction were required, the sum rule for g_A would be useless.

By writing a dispersion relation for the last term in Eq. (37), *without* assuming the PCAC hypothesis, one gets a sum rule relating $1-g_A{}^2$ to cross sections measurable in high-energy neutrino experiments. This sum rule is discussed further in Sec. V.

III. NUMERICAL EVALUATION

Because Eq. (22) involves off-mass-shell pion-proton scattering cross sections, a little work is necessary to compare it with experiment. Let us split the right-hand side of Eq. (22) into the sum of three terms:

$$1-g_A{}^{-2}=(4M_N{}^2/g_r{}^2)(R_1+R_2+R_3)\,, \tag{53}$$

[17] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).

[18] B. Amblard *et al.*, Phys. Letters **10**, 138 (1964); G. Höhler, G. Ebel, and J. Giesecke, Z. Physik **180**, 430 (1964).

with

$$R_1=-\frac{1}{\pi}\int_{M_\pi}^{\infty}\frac{d\nu}{\nu}\,\mathrm{Im}G\left(\nu,-\frac{M_\pi^2}{2M_N},M_\pi,M_\pi\right)$$
$$=\frac{1}{2\pi}\int_{M_\pi}^{\infty}\frac{d\nu}{\nu^2}(\nu^2-M_\pi^2)^{1/2}[\sigma^+(\nu)-\sigma^-(\nu)], \quad (54a)$$

$$R_2=\frac{1}{\pi}\int_{M_\pi}^{\infty}\frac{d\nu}{\nu}\,\mathrm{Im}G\left(\nu,-\frac{M_\pi^2}{2M_N},M_\pi,M_\pi\right)$$
$$-\frac{1}{\pi}\int_{M_\pi+M_\pi^2/(2M_N)}^{\infty}\frac{d\nu}{\nu}\,\mathrm{Im}G(\nu,0,M_\pi,M_\pi), \quad (54b)$$

$$R_3=\frac{1}{\pi}\int_{M_\pi+M_\pi^2/(2M_N)}^{\infty}\frac{d\nu}{\nu}\,\mathrm{Im}\left[G(\nu,0,M_\pi,M_\pi)-\frac{G(\nu,0,0,0)}{K^{NN\pi}(0)^2}\right], \quad (54c)$$

$$G(\nu,\nu_B,M_\pi^i,M_\pi^f)\equiv\nu^{-1}[A^{\pi N(-)}(\nu,\nu_B,M_\pi^i,M_\pi^f)+\nu B^{\pi N(-)}(\nu,\nu_B,M_\pi^i,M_\pi^f)]. \quad (54d)$$

There is a definite reason for splitting things up this way. Numerically, we find that $|R_1|>|R_2|>|R_3|$. The dominant term, R_1, involves only the physical pion-proton scattering cross sections $\sigma^\pm$, and thus can be reliably determined. The terms R_2 and R_3 are corrections, which take into account the fact that the sum rule involves the forward charge-exchange scattering amplitude, with both external pions of *zero mass.* The term R_2 can be calculated in terms of pion-nucleon scattering phase shifts. Since it is dominated by the (3,3) resonance, it can be fairly reliably calculated. The term R_3 is less well known, because a model is needed to calculate the off-mass-shell partial wave amplitudes.

We get the following numerical results[19]

$$(4M_N^2/g_r^2)R_1=0.254\,,$$
$$(4M_N^2/g_r^2)R_2=0.155\,, \quad (55)$$
$$(4M_N^2/g_r^2)R_3=-0.061\,,$$

giving

$$g_A^{\text{theory}}=1.24. \quad (56)$$

A reasonable error estimate, based upon the variations among the several calculations of R_2 and R_3 discussed below, is ±0.03. The best experimental value is[20]

$$g_A^{\text{expt}}=1.18\pm0.02. \quad (57)$$

Thus, the sum rule agrees with experiment to within 5%.

[19] For the pion-nucleon coupling constant, we used the value $f^2=g_r^2M_\pi^2/(16\pi M_N^2)=0.081\pm0.002$ quoted by W. S. Woolcock, *Proceedings of the Aix-en-Provence International Conference on Elementary Particles* (Centre d'Etudes Nucléaires de Saclay, Seine et Oise, 1961), Vol. I, p. 459.

[20] C. S. Wu (private communication).

It is interesting that the region around the 600- and 900-MeV pion-nucleon resonances makes an important contribution to the sum rule. If only the contribution of the (3,3) resonance is retained, we get the result $g_A=1.44$. In other words, the (3,3) resonance does not exhaust the sum rule.

The remainder of this section deals with the details of the numerical evaluation

A. Calculation of R_1

As stated above, R_1 is calculated directly from the physical pion-proton total cross sections $\sigma^\pm$. Values of $\sigma^\pm$ from 0 to 110 MeV were taken from the smoothed fit of Klepikov *et al.*[21] From 110 to 4950 MeV, we used the tabulation of Amblard *et al.*[22] Above 4950 MeV, we used the asymptotic formula $\sigma^--\sigma^+=7.73$ mb $\times[k/(\mathrm{BeV}/c)]^{-0.7}$ given by von Dardel *et al.*[23] This formula gives a good fit to the experimental data up to 20 BeV/c. Use of this formula beyond 20 BeV/c represents an extrapolation from the present experimental data, and gives

$$\frac{4M_N^2}{g_r^2}\frac{1}{2\pi}\int_{20\text{ BeV}}^{\infty}\frac{d\nu}{\nu^2}(\nu^2-M_\pi^2)^{1/2}(\sigma^+-\sigma^-)\approx-0.011. \quad (58)$$

Thus, unless the $[k/(\mathrm{BeV}/c)]^{-0.7}$ asymptotic behavior is very much in error, the region above 20 BeV/c contributes only a few percent of $1-g_A^{-2}$.

B. Calculation of R_2

It is convenient to express R_2 as a single integral over center-of-mass energy W, the integrand of which is the difference of terms referring to $\nu_B=0$ and to $\nu_B=-M_\pi^2/(2M_N)$. The center-of-mass scattering angle ϕ is given by

$$y\equiv\cos\phi=1+M_\pi^2/|\mathbf{k}|^2 \quad \text{at} \quad \nu_B=0\,,$$
$$y\equiv\cos\phi=1 \quad \text{at} \quad \nu_B=-M_\pi^2/(2M_N)\,, \quad (59)$$

where $|\mathbf{k}|$ is the center-of-mass frame pion momentum. Thus we get

$$R_2=-16\int_{M_N+M_\pi}^{\infty}dW\Delta(W)\,,$$

$$\Delta(W)=\frac{W^2}{(W^2-M_N^2)^2}\left[f_1\left(W,1+\frac{M_\pi^2}{|\mathbf{k}|^2}\right)\frac{(W+M_N)^2}{(W+M_N)^2-M_\pi^2}+f_2\left(W,1+\frac{M_\pi^2}{|\mathbf{k}|^2}\right)\frac{(W-M_N)^2}{(W-M_N)^2-M_\pi^2}\right]$$
$$-\frac{W^2}{(W^2-M_N^2-M_\pi^2)^2}[f_1(W,1)+f_2(W,1)], \quad (60)$$

[21] N. P. Klepikov *et al.*, Dubna report D-584, 1960 (unpublished).

[22] B. Amblard *et al.*, Ref. 18 and private communication.

[23] G. von Dardel *et al.*, Phys. Rev. Letters **8**, 173 (1962).

with $f_1(W,y)$ and $f_2(W,y)$ the usual center-of-mass pion-nucleon scattering amplitudes. Since f_1 and f_2 are analytic functions of y in an ellipse with foci ± 1 and with semimajor axis $1+2M_\pi^2/|\mathbf{k}|^2$,[24] we can legitimately use partial-wave expansions in calculating f_1 and f_2 in both terms of Eq. (60). The integral is rapidly convergent, since the two terms in $\Delta(W)$ tend to cancel at high energies.

The number $(4M_N^2/g_r^2)R_2=0.155$ quoted in Eq. (55) was obtained by using Roper's $l_m=3$ phase-shift fit,[25] truncating the integral at $W=11.20M_\pi$. (Beyond this energy no phase-shift fit is available.) The integral is dominated by the (3,3) resonance; extending the integral *only* over the (3,3) resonance gave $(4M_N^2/g_r^2)R_2=0.166$. A third calculation, using simple Breit-Wigner forms for the (3,3) and the 600- and 900-MeV resonances, and neglecting all other partial waves, gave $(4M_N^2/g_r^2)R_2=0.156$ when the integral was truncated at $11.20M_\pi$, and $(4M_N^2/g_r^2)R_2=0.145$ when the integral was extended to an upper limit of $W\approx 17M_\pi$. The good agreement of these numbers indicates that R_2 is insensitive to "controversial" features of Roper's phases, such as whether the P_{11} wave resonates.

C. Calculation of R_3

The term R_3, which describes corrections arising from taking the external pion off the mass shell, cannot be calculated directly from experimental data. In order to estimate this term, we must assume a model for the off-mass-shell partial wave amplitude $f_{lJI}(W,M_\pi^i,M_\pi^f)$. (Here l= orbital angular momentum, J= total angular momentum, and I= isospin.)

Actually, in order to evaluate R_3, we only need to know the imaginary part of $f_{lJI}(W,M_\pi^i,M_\pi^f)$. Below the inelastic threshold at $W=M_N+2M_\pi$, generalized unitarity tells us that

$$\mathrm{Im} f_{lJI}(W,M_\pi^i,M_\pi^f) = |\mathbf{k}| f_{lJI}(W,M_\pi^i,M_\pi) f_{lJI}(W,M_\pi^f,M_\pi)^* . \quad (61)$$

The intermediate state pion is, of course, on the mass shell. Since only the region around the (3,3) resonance is appreciably affected by taking the external pions off the mass shell, it suffices to study $f_{lJI}(W,M_\pi^i,M_\pi)$ and then to use the elastic unitarity relation of Eq. (61) to get $\mathrm{Im} f_{lJI}(W,M_\pi^i,M_\pi^f)$.

In constructing a model, we use the following information about f_{lJI}:

(i) *Threshold behavior.* From kinematic considerations, we know that near the threshold at $W=M_N+M_\pi$, $f_{lJI}(W,M_\pi^i,M_\pi^f)$ will be equal to $(|\mathbf{k}^i||\mathbf{k}^f|)^l$ times slowly varying factors, with

$$|\mathbf{k}^{i,f}|=[(k_0^{i,f})^2-M_\pi^2]^{1/2}, \quad k_0^{i,f}=[W^2-M_N^2+(M_\pi^{i,f})^2]/(2W). \quad (62)$$

Here $|\mathbf{k}^i|$ and $|\mathbf{k}^f|$ are the center-of-mass momenta of the initial and final pions; when $M_\pi^i=0(M_\pi)$, we denote $|\mathbf{k}^i|$ by $|\mathbf{k}^0|$ ($|\mathbf{k}|$).

(ii) *Unitarity.* Setting either M_π^i or M_π^f equal to M_π in Eq. (61), we see that $f_{lJI}(W,M_\pi^i,M_\pi)$ has the same phase δ_{lJI} as the true pion-nucleon partial wave amplitude $f_{lJI}(W,M_\pi,M_\pi)$.

(iii) *Left-hand singularities.* Changing the external pion mass changes the left-hand singularities in the partial wave amplitude $f_{lJI}(W,M_\pi^i,M_\pi^f)$. The left-hand singularities closest to the physical region come from the partial wave projection $f_{lJI}^B(W,M_\pi^i,M_\pi^f)$ of the Born approximation (the pole term) in Eq. (46). Reference to Eq. (46) shows that $f_{lJI}^B(W,M_\pi^i,M_\pi^f)$ contains a factor $K^{NN\pi}[-(M_\pi^i)^2]K^{NN\pi}[-(M_\pi^f)^2]$ arising from the change in strength of the coupling of the external pions to nucleons when the external pion mass is changed from the physical value.

A simple model, which takes into account the considerations (i)–(iii), is to take

$$f_{lJI}(W,M_\pi^i,M_\pi) \approx \frac{f_{lJI}^B(W,M_\pi^i,M_\pi)}{f_{lJI}^B(W,M_\pi,M_\pi)} f_{lJI}(W,M_\pi,M_\pi). \quad (63)$$

Equation (63) gives $f_{lJI}(W,M_\pi^i,M_\pi)$ the same phase as $f_{lJI}(W,M_\pi,M_\pi)$. Multiplying the physical f_{lJI} by the ratio of the Born approximations gives the off-mass-shell f_{lJI} the correct threshold behavior and, approximately, the correct nearby left-hand singularities. A second model is to take

$$f_{lJI}(W,M_\pi^i,M_\pi) \approx (|\mathbf{k}^i|/|\mathbf{k}|)^l K^{NN\pi}[-(M_\pi^i)^2] f_{lJI}(W,M_\pi,M_\pi). \quad (64)$$

Here we have put in only a threshold correction factor and a constant factor $K^{NN\pi}[-(M_\pi^i)^2]$ to account for the change in strength of the nearby left-hand singularities. According to Eq. (61), the first model gives

$$\mathrm{Im} f_{lJI}(W,0,0) = \left[\frac{f_{lJI}^B(W,0,0)}{f_{lJI}^B(W,M_\pi,M_\pi)}\right]^2 \mathrm{Im} f_{lJI}(W,M_\pi,M_\pi), \quad (65)$$

while the second model gives

$$\mathrm{Im} f_{lJI}(W,0,0) = (|\mathbf{k}^0|/|\mathbf{k}|)^{2l} K^{NN\pi}(0)^2 \times \mathrm{Im} f_{lJI}(W,M_\pi,M_\pi). \quad (66)$$

Although Eq. (61) is valid only below the inelastic threshold, we will use Eq. (65) and Eq. (66) above the inelastic threshold as well as below.

Numerical evaluation of Eq. (54c) gives $(4M_N^2/g_r^2)R_3$

[24] This statement assumes the validity of the Mandelstam representation.

[25] L. D. Roper, Phys. Rev. Letters 12, 340 (1964) and private communication.

$=-0.061$ when the model of Eq. (65) is used, and $(4M_N^2/g_r^2)R_3=-0.051$ when we assume Eq. (66). In both cases, Roper's phase-shift fit was used, and the integral was truncated at $W=11.20M_\pi$. Using Eq. (65) integrated only over the (3,3) resonance gave $(4M_N^2/g_r^2)R_3=-0.066$. Evaluating the integral with only Breit-Wigner terms for the low-lying resonances gave similar results. Thus, the quoted value of R_3, while dependent on the model used for going off mass shell, is insensitive to "controversial" features of the phase shifts.

D. Remarks

The terms R_2 and R_3, which come largely from the (3,3) resonance region, give a combined contribution of 0.094, as compared with the contribution of 0.254 coming from R_1. It may at first seem surprising that the effect of R_2 and R_3 is so big, but it is easy to understand this. From Eq. (66), we can see that the main effect of R_2 and R_3 is to multiply $\sigma_{3,3}$, the (3,3) resonance contribution to the integrand of R_1, by a factor

$$|\mathbf{k}^0|^2/|\mathbf{k}|^2. \tag{67}$$

At the peak of the (3,3) resonance, this factor is 1.27. Since the (3,3) contribution to R_1 is 0.43, we expect R_1 to be increased by an amount of order

$$0.27\times0.43\approx0.12, \tag{68}$$

in rough agreement with the sum of R_2 and R_3.

IV. PION-PION SCATTERING SUM RULE

In Sec. II, we took the matrix element of Eq. (7) between proton states and derived a sum rule relating g_A to pion-proton scattering. Now let us take the matrix element of Eq. (7) between π^+ states. The same manipulations used in the proton case lead to the sum rule

$$\frac{2}{g_A^2}=\frac{4M_N^2}{g_r^2K^{NN\pi}(0)^2}\frac{1}{\pi}\int_{2M_\pi}^{\infty}\frac{WdW}{W^2-M_\pi^2}\times[\sigma_{0\pi}^-(W)-\sigma_{0\pi}^+(W)], \tag{69}$$

where $\sigma_{0\pi}^\pm(W)$ is the total cross section for scattering of a zero mass $\pi^\pm$ on a physical π^+, at center-of-mass energy W. Equation (69) involves g_A^{-2}, rather than $g_A^{-2}-1$, because the one-pion intermediate state contribution vanishes on account of parity. The factor 2 on the left-hand side of Eq. (69) comes from the fact that $\langle\pi^+(q)|2I^3|\pi^+(q')\rangle=2\cdot(2\pi)^3\delta(\mathbf{q}-\mathbf{q}')$.

While, of course, no direct pion-pion scattering data is available, there is enough information on pion-pion resonances to compare Eq. (69) with experiment. First of all, $\sigma_{0\pi}^+(W)$ comes only from $I=2$ scattering. While there are resonances in the low energy $I=0$ and $I=1$ pion-pion scattering, the $I=2$ scattering seems to be small. Thus the right-hand side of Eq. (69) is positive, agreeing in sign with the left-hand side.

Now let us make a quantitative analysis. According to Eq. (57), the left-hand side of Eq. (69) is

$$2/g_A^2=1.43. \tag{70}$$

Let us express the right-hand side of Eq. (69) in terms of the variable $s=W^2$, giving

$$\frac{4M_N^2}{g_r^2K^{NN\pi}(0)^2}\frac{1}{2\pi}\int_{4M_\pi^2}^{\infty}\frac{ds}{s-M_\pi^2}[\sigma_{0\pi}^-(s)-\sigma_{0\pi}^+(s)]. \tag{71}$$

As in the proton case, we take account of the fact that the external pion in Eq. (71) is of zero mass by writing

$$\begin{aligned}\sigma_{0\pi}^{l,I}(s)&=K^{NN\pi}(0)^2(|\mathbf{k}^0|/|\mathbf{k}|)^{2l}\sigma_\pi^{l,I}(s)\\&=K^{NN\pi}(0)^2[(s-M_\pi^2)^2/s(s-4M_\pi^2)]^l\sigma_\pi^{l,I}(s),\end{aligned} \tag{72}$$

where l=orbital angular momentum, I=isospin, and $\sigma_\pi^{l,I}(s)$ is the on-mass-shell partial wave cross section. Thus Eq. (71) becomes

$$\frac{4M_N^2}{g_r^2}\frac{1}{2\pi}\int_{4M_\pi^2}^{\infty}\frac{ds}{s-M_\pi^2}\times\left\{\sum_{\substack{l=0\\l\text{ even}}}^{\infty}\left[\frac{(s-M_\pi^2)^2}{s(s-4M_\pi^2)}\right]^l\tfrac{2}{3}[\sigma_\pi^{l,0}(s)-\sigma_\pi^{l,2}(s)]+\sum_{\substack{l=1\\l\text{ odd}}}^{\infty}\left[\frac{(s-M_\pi^2)^2}{s(s-4M_\pi^2)}\right]^l\sigma_\pi^{l,1}(s)\right\}. \tag{73}$$

Let us first evaluate the contributions of the two well-established $\pi\pi$ resonances, the $l=I=1$ ρ and the $l=2, I=0$ f^0. We parametrize $\sigma_\pi^{1,1}$ and $\sigma_\pi^{2,0}$ in the form[26]

$$\begin{aligned}\sigma_\pi^{1,1}(s)&=\frac{12\pi\gamma_\rho^2\nu^2/(\nu+M_\pi^2)}{(s_\rho-s)^2+\gamma_\rho^2\nu^3/(\nu+M_\pi^2)},\\ \sigma_\pi^{2,0}(s)&=\frac{20\pi\gamma_f^2\nu^4/(\nu+M_\pi^2)}{(s_f-s)^2+\gamma_f^2\nu^5/(\nu+M_\pi^2)},\quad \nu=\tfrac{1}{4}s-M_\pi^2.\end{aligned} \tag{74}$$

The reduced widths γ_ρ^2 and γ_f^2 are related to the experimental full widths at half-maximum Γ_ρ and Γ_f by

$$\gamma_\rho^2=\frac{\nu_\rho+M_\pi^2}{\nu_\rho^3}s_\rho\Gamma_\rho^2,\quad \gamma_f^2=\frac{\nu_f+M_\pi^2}{\nu_f^5}s_f\Gamma_f^2,\quad \nu_{\rho,f}=\tfrac{1}{4}s_{\rho,f}-M_\pi^2. \tag{75}$$

Using the experimental values[27] $s_\rho=29.7M_\pi^2$, $\Gamma_\rho=0.755M_\pi$, $s_f=80.0M_\pi^2$, $\Gamma_f=0.716M_\pi$, we get, for the ρ and f^0 contributions to Eq. (73),

$$\begin{aligned}\rho\text{ contribution}&=0.42,\\ f^0\text{ contribution}&=0.11.\end{aligned} \tag{76}$$

As a check, we also calculated the ρ and f^0 contribu-

[26] L. A. P. Balázs, Phys. Rev. **129**, 872 (1963).
[27] A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **36**, 977 (1964).

tions in the narrow resonance approximation. This gave 0.35 for the ρ and 0.09 for the f^0 contribution, indicating that resonance shape corrections will not substantially change the numbers of Eq. (76).

The contribution of 0.53 from the ρ and the f^0 is only 37% of the total of 1.43 required by the sum rule. Since the f^0 contribution is so small, and since there seem to be no resonances with $l \geq 3$ in the low-energy region,[27] it should be reasonable to neglect the contribution of terms with $l \geq 3$ in Eq. (73). Rearranging Eq. (69), we get

$$\frac{4M_N^2}{g_r^2}\frac{1}{2\pi}\int_{4M_\pi^2}^{\infty}\frac{ds}{s-M_\pi^2}\tfrac{2}{3}\sigma_\pi^{0,0}(s)$$
$$=\frac{4M_N^2}{g_r^2}\frac{1}{2\pi}\int_{4M_\pi^2}^{\infty}\frac{ds}{s-M_\pi^2}\times\frac{2}{3}\left\{\sigma_\pi^{0,2}(s)+\left[\frac{(s-M_\pi^2)^2}{s(s-4M_\pi^2)}\right]^2\sigma_\pi^{2,2}(s)\right\}$$
$$+1.43-0.42-0.11\geq 0.9. \quad (77)$$

Thus, the pion-pion sum rule can be satisfied only if there is a large low-energy $I=0$, S-wave pion-pion scattering cross section.

In order to get an idea of how big the $I=0$, S-wave scattering cross section would have to be in order to satisfy Eq. (77), we evaluated the left-hand side of Eq. (77) using a simple scattering-length parametrization of the $I=0$, S-wave phase shift,[28]

$$(\nu/(\nu+M_\pi^2))^{1/2}\cot\delta^{0,0}=1/a_0+H(\nu),$$
$$H(\nu)=(2/\pi)(\nu/(\nu+M_\pi^2))^{1/2}\times\ln[(\nu/M_\pi^2)^{1/2}+(\nu/M_\pi^2+1)^{1/2}], \quad (78)$$

which gives

$$\sigma_\pi^{0,0}=\frac{4\pi a_0^2}{a_0^2\nu+(\nu+M_\pi^2)[1+a_0H(\nu)]^2}. \quad (79)$$

We find that Eq. (77) can be satisfied only if $a_0>1.3$ or if $a_0<-0.85$. It is interesting that an $I=0$, S-wave scattering length of the order of a pion Compton wavelength is also suggested by studies of low-energy pion-nucleon scattering[29] and of K_{l4} decays.[30] Needless to say, there is nothing unique about the parametrization of Eq. (78).

V. TESTS OF THE CURRENT ALGEBRA IN HIGH-ENERGY NEUTRINO REACTIONS

The sum rules discussed in the preceding three sections are derived from two principal hypotheses: the axial-vector current commutation relations of Eq. (7) and the partially conserved axial-vector current hypothesis of Eq. (5). In this section, we discuss a sum rule which follows from the axial-vector current algebra alone, regardless of whether PCAC is true. We will also derive sum rules which follow from a proposed algebra of the strangeness-changing currents.

Let us begin by reviewing the theory of leptonic weak interactions of the hadrons. According to Gell-Mann[4] and to Cabibbo,[6] the hadronic weak current is[31]

$$J_\lambda^h=(\mathfrak{F}_{1\lambda}+i\mathfrak{F}_{2\lambda}+\mathfrak{F}_{1\lambda}^5+i\mathfrak{F}_{2\lambda}^5)G_V\cos\theta+(\mathfrak{F}_{4\lambda}+i\mathfrak{F}_{5\lambda}+\mathfrak{F}_{4\lambda}^5+i\mathfrak{F}_{5\lambda}^5)G_V\sin\theta. \quad (80)$$

Here G_V is the Fermi coupling constant and θ is the Cabibbo angle. The vector currents $\mathfrak{F}_{j\lambda}$ and the axial currents $\mathfrak{F}_{j\lambda}^5$ ($j=1,\cdots,8$) each form an SU_3 octet. The SU_3 generalization of the conserved-vector-current (CVC) hypothesis is to assume that the vector currents $\mathfrak{F}_{j\lambda}$ are just the unitary spin currents, with

$$\mathfrak{F}_{a\lambda}=I_\lambda^a,\quad a=1,2,3; \quad (81)$$
$$\mathfrak{F}_{8\lambda}=\tfrac{1}{2}\sqrt{3}Y_\lambda,$$

where I_λ^a is the isotopic spin current and Y_λ is the hypercharge current. In our new notation, the currents defined in Sec. I are

$$J_\lambda^{Va}=\mathfrak{F}_{a\lambda},\quad J_\lambda^{Aa}=\mathfrak{F}_{a\lambda}^5,\quad a=1,2,3. \quad (82)$$

Let us define vector and axial-vector "charges" F_j and F_j^5 according to

$$F_j=-i\int d^3x\,\mathfrak{F}_{j4};\quad F_j^5=-i\int d^3x\,\mathfrak{F}_{j4}^5. \quad (83)$$

Gell-Mann[4] has postulated that even in the presence of the SU_3 symmetry-breaking interaction, the following commutation relations hold exactly:

$$[F_i,F_j]=if_{ijk}F_k,$$
$$[F_i,F_j^5]=if_{ijk}F_k^5, \quad (84)$$
$$[F_i^5,F_j^5]=if_{ijk}F_k.$$

The chirality commutation relation of Eq. (7) is, of course, just a special case of Eq. (84):

$$[F_1^5+iF_2^5,F_1^5-iF_2^5]=2F_3. \quad (85)$$

From Eq. (84), we also get the following commutation relation for the "charge" associated with the strangeness changing part of J_λ^h:

$$[F_4+iF_5+F_4^5+iF_5^5,F_4-iF_5+F_4^5-iF_5^5]=2\sqrt{3}F_8+2F_3+2\sqrt{3}F_8^5+2F_3^5. \quad (86)$$

Assuming that we can integrate by parts with respect to the spatial variables $\mathbf{x}$, we can express the time derivatives of the "charges" in terms of the divergences of

[28] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).
[29] J. Hamilton, *Strong Interactions and High Energy Physics—Scottish Universities' Summer School, 1963*, edited by R. G. Moorhouse (Plenum Press, New York, 1964).
[30] C. Kacser, P. Singer, and T. Truong, Phys. Rev. **137**, B1605 (1965).
[31] In this section, we use the notation of Ref. 4 for the currents.

the corresponding currents:

$$\frac{d}{dt}F_j=\int d^3x\,\partial_\lambda\mathfrak{F}_{j\lambda},$$
$$\frac{d}{dt}F_j^5=\int d^3x\,\partial_\lambda\mathfrak{F}_{j\lambda}^5. \tag{87}$$

Let us now derive sum rules which provide tests of the commutation relations of Eq. (85) and Eq. (86), considering first the strangeness-conserving case, Eq. (85). We proceed exactly as in Sec. II, taking the matrix element of Eq. (85) between proton states. The only difference is that we do *not* assume that the divergence $\partial_\lambda\mathfrak{F}_{a\lambda}^5$ is proportional to the pion field. We thus get the sum rule

$$1=g_A^2+\int_{M_N+M_\pi}^{\infty}\frac{4M_NWdW}{(W^2-M_N^2)^2}[N_p^-(W)-N_p^+(W)], \tag{88}$$

with

$$N_p^\pm(W)=\sum_{\substack{j\neq N\\ \mathrm{INT}}}\delta(W-M_j)|\mathfrak{F}_j^\pm|^2|_{(q-q_j)^2=0},$$

$$\langle j|\partial_\lambda\mathfrak{F}_{1\lambda}^5\pm i\partial_\lambda\mathfrak{F}_{2\lambda}^5|p(q)\rangle =((M_N/q_0)(M_j/q_{j0}))^{1/2}\mathfrak{F}_j^\pm. \tag{89}$$

In other words, $\mathfrak{F}_j^\pm$ is the matrix element of the divergence of the axial-vector current; the sum rule of Eq. (88) involves this matrix element only at zero four-momentum transfer $(q-q_j)^2$.

The matrix element needed to evaluate the right-hand side of Eq. (88) can be directly measured in high-energy neutrino reactions. Consider the inelastic reaction

$$\nu_l+N\to l+j, \tag{90}$$

with ν_l a neutrino, l a lepton, N a nucleon, and j a system of strongly interacting particles with $M_j\neq M_N$. In a previous paper,[32] we showed that when the lepton emerges parallel to the incident neutrino direction, and when the lepton mass is neglected, the matrix element for Eq. (90) depends only on the *divergences* of the hadronic current. Clearly, under these hypotheses the momentum transfer $(q-q_j)^2$ is zero, so we are measuring just the matrix element needed in Eq. (88). (In the $\Delta S=0$ case, the divergence of the vector current vanishes.) Summing over final states j of strangeness zero $(S=0)$ leads to the relations, for forward lepton,

$$\frac{d^2\sigma[\nu+p\to l^-+(S=0)]}{d\Omega_l dE_l}=G_V^2\cos^2\theta f(W)N_p^+(W),$$
$$\frac{d^2\sigma[\bar\nu+p\to l^++(S=0)]}{d\Omega_l dE_l}=G_V^2\cos^2\theta f(W)N_p^-(W), \tag{91}$$

with

$$f(W)=\frac{1}{2\pi^2}\left[\frac{M_N^2+2M_NE-W^2}{W^2-M_N^2}\right]^2. \tag{92}$$

Here E is the incident-neutrino energy, E_l is the final-lepton energy, and Ω_l is the lepton solid angle (all in the laboratory frame, where the initial proton is at rest). In terms of W and E, E_l is given by

$$E_l=(M_N^2+2M_NE-W^2)/(2M_N). \tag{93}$$

We can apply the same method to the commutator of the strangeness-changing currents[33] [Eq. (86)], giving the two sum rules

$$4=\int\frac{4M_NWdW}{(W^2-M_N^2)^2}[S_p^-(W)-S_p^+(W)], \tag{94a}$$

$$2=\int\frac{4M_NWdW}{(W^2-M_N^2)^2}[S_n^-(W)-S_n^+(W)]. \tag{94b}$$

Equation (94a) has discrete contributions at $W=M_\Lambda$, $W=M_\Sigma$ and a continuum from $W=M_\pi+M_\Lambda$ to ∞. Equation (94b) has a discrete contribution at $W=M_\Sigma$ and a continuum from $W=M_\pi+M_\Lambda$ to ∞. The functions $S_{p,n}^\pm$ are measurable in strangeness-changing high-energy neutrino reactions, since for forward lepton,

$$\frac{d^2\sigma[\nu+(p,n)\to l^-+(S=+1)]}{d\Omega_l dE_l}=G_V^2\sin^2\theta f(W)S_{(p,n)}^+(W), \tag{95}$$
$$\frac{d^2\sigma[\bar\nu+(p,n)\to l^++(S=-1)]}{d\Omega_l dE_l}=G_V^2\sin^2\theta f(W)S_{(p,n)}^-(W).$$

Thus, Eqs. (88), (91), (94), and (95) can be used to directly test the algebra proposed by Gell-Mann for the vector and the axial-vector currents.

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[32] S. L. Adler, Phys. Rev. **135**, B963 (1964).

[33] The nucleon matrix element of the axial-vector terms on the right-hand side of Eq. (86) vanishes when we average over nucleon spin.