Time Delay for Wave Packets in Nonrelativistic Scattering Theory with Inelastic Channels Present*

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Time delay is considered in nonrelativistic scattering theory with two-particle inelastic channels present. A comparison method is discussed, which in turn leads to a more natural definition based on the classical definition of time delay for inelastic scattering.

I. INTRODUCTION

TIME delay has been defined using wave packets very narrow in energy[1,2] and even by the use of standing waves.[3] This is perhaps the most useful way of defining it, as we are then left with just the energy derivative of the phase of the S matrix. However, the notion of time delay seems so physical that it should be closely allied to some tangible measurements of time. It would seem then that wave packets which do not have a narrow energy spread but are fairly well located in space should also be considered. For the elastic case, such a treatment can be found in the recent text by Goldberger and Watson.[4] In this paper, then, we look for a "reasonable" definition of time delay for the case when there are nonrelativistic two-particle inelastic channels present.

In Sec. 2, we introduce a "zeroth" wave packet of free particles in the channel of interest and which, by definition, emerges with zero time delay, thereby defining the time delay. Section 2 is, in fact, an attempt to generalize the Goldberger and Watson definition.[4] However, another approach, given in Sec. 3, provides a definition that is perhaps the most direct quantum-mechanical translation of the "classical" definition.[3] The work, as usual, is done in the center-of-mass (c.m.) system and the wave packets are the spherical wave packets. For further simplicity, only spinless particles are considered here. The paper concludes with a short discussion in Sec. 4.

2. BY WAY OF COMPARISON

In this section, we make an attempt to generalize the Goldberger and Watson approach.[4] The attempt is somewhat unsatisfactory since it is left to the approach of Sec. 3 to show exactly how reasonable is the definition to be given in this section. The purpose of the following work is the light it may throw onto the previous work that has been done on time delay and the perspective it gives to Sec. 3.

We shall, for simplicity, consider the case when only two channels are present. The extension to more than two channels is evident. The two free Hamiltonians in the c.m. system are then

$$H_0(i)=M_ic^2+(\mathbf{p}_i^2/2\mu_i),\quad i=1,2, \tag{2.1}$$

where μ_i is the reduced mass in the ith channel and

$$Q=(M_2-M_1)c^2$$

is the threshold energy for channel-1 particles to go into channel-2 particles. The full Hamiltonian is thus

$$H=H_0(1)+H_0(2)+V. \tag{2.2}$$

As usual, we shall assume that there exists some sphere, given by $r=a$, beyond which the potential V vanishes. The scattering operator S now has also matrix elements for cross channels:

$$\begin{aligned}\langle\varphi_{l'm'\epsilon'}|S|\varphi_{lm\epsilon}\rangle&=\delta_{l'l}\delta_{m'm}\delta(\epsilon'-\epsilon)S_l^{11}(\epsilon),\\ \langle\xi_{l'm'\epsilon'}|S|\varphi_{lm\epsilon}\rangle&=\delta_{l'l}\delta_{m'm}\delta(\epsilon'-\epsilon)S_l^{21}(\epsilon),\\ &\text{etc.,}\end{aligned} \tag{2.3}$$

where $|\varphi_{lm\epsilon}\rangle$ is the free spherical wave with total energy ϵ for channel-1 particles and similarly for $|\xi_{lm\epsilon}\rangle$ which belongs to channel 2.

The approach here is to define "zeroth" wave packets for each channel, i.e., free wave packets that are defined to emerge with zero time delay from some large "sphere of observation" centered at the origin with radius $r=R\gg a$. The time delay is then given by comparing the actual wave packet in whichever channel that is of interest with the corresponding zeroth wave packet. The result should be independent of R so long as R is large enough. For the elastic case, the zeroth wave packet is *a priori* the incoming wave packet considered as free. The situation is evidently not so simple for inelastic scattering.

We shall consider specifically an incident wave packet consisting only of channel-1 particles and look at the emerging wave packet in channel 2. The free normalized incoming wave packet is then taken to be

$$|\Phi(t)\rangle\equiv|\Phi_{lm}(t)\rangle=\int d\epsilon\, A(\epsilon)e^{-i\epsilon t/\hbar}|\varphi_{lm\epsilon}\rangle. \tag{2.4}$$

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[1] L. Eisenbud, dissertation, Princeton, 1948 (unpublished).
[2] E. P. Wigner, Phys. Rev. **98**, 145 (1955).
[3] F. T. Smith, Phys. Rev. **118**, 349 (1960).
[4] M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, New York, 1964), pp. 485–490.

Consider now, in channel 2, the free wave packet

$$|\tilde{\chi}_2(t)\rangle = \int d\epsilon\, A(\epsilon) e^{-i\epsilon t/\hbar} |\xi_{lm\epsilon}\rangle,$$

where we assume that $A(\epsilon)$ is zero below threshold. It can then be shown that, for the wave functions,

$$Q=0 \Rightarrow \tilde{\chi}_2(\mathbf{r}_2;t) = (\mu_2/\mu_1)^{3/4}\Phi((\mu_2/\mu_1)^{1/2}\mathbf{r}_2;t),$$

where

$$\tilde{\chi}_2(\mathbf{r}_2;t) = \langle \mathbf{r}_2|\tilde{\chi}_2(t)\rangle, \quad \Phi(\mathbf{r}_1;t) = \langle \mathbf{r}_1|\Phi(t)\rangle,$$

and $\mathbf{r}_i$ is the position variable for the ith channel. That is, for $Q=0$, the wave functions $\tilde{\chi}_2(\mathbf{r}_2;t)$ and $\Phi(\mathbf{r}_1;t)$ are "similar." Thus, in particular, the wave packet $|\tilde{\chi}_2\rangle$ reaches the midpoint of its traverse across any large sphere centered at the origin in the c.m. system at the same time as the free incoming wave packet, and this is a reasonable criterion to demand of a zeroth wave packet. But $Q\neq 0$ in general, and we can only expect to approximate this situation if the wave packets lie in energy well away from threshold. Furthermore, $|S_l^{21}(\epsilon)|$ is dependent on energy, so that the emerging wave packet will have a different energy distribution from that for the incident wave packet. To use $|\tilde{\chi}_2\rangle$ as the zeroth wave packet could then lead to a time delay dependent on R. It is proposed here to use the free normalized wave packet

$$|\chi_2\rangle = N^{-1}\int d\epsilon\, A(\epsilon)|S_l^{21}(\epsilon)|e^{-i\epsilon t/\hbar}|\xi_{lm\epsilon}\rangle \quad (2.5)$$

as the zeroth wave packet, where

$$N = \left\{\int d\epsilon |A(\epsilon)S_l^{21}(\epsilon)|^2\right\}^{1/2}. \quad (2.6)$$

If, in fact, the wave packet is fairly narrow in energy and lies away from threshold, $|S_l^{21}(\epsilon)|$ will be a slowly varying function of energy over the range of $A(\epsilon)$ and $|\chi_2\rangle$ will be close to $|\tilde{\chi}_2\rangle$. In any case, there is a substantial overlap of $|\chi_2\rangle$ with $|\tilde{\chi}_2\rangle$, since

$$|\langle\chi_2(t)|\tilde{\chi}_2(t)\rangle|^2 \geq \tfrac{1}{2}.$$

It must, however, be left to Sec. 3 to indicate exactly how reasonable it is to define $|\chi_2\rangle$ as the zeroth wave packet.

Defining $|\chi_2\rangle$ as the zeroth wave packet, the time delay, according to Goldberger and Watson,[4] is then given by

$$\Delta t_{21}{}^a = \int_{T_0}^{T} dt \int_{X-V_R} d\mathbf{r}_2 \{\chi_2^*(\mathbf{r}_2;t)\chi_2(\mathbf{r}_2;t) - \psi_2^*(\mathbf{r}_2;t)\psi_2(\mathbf{r}_2;t)\}, \quad (2.7)$$

where $|\psi_2\rangle$ is the normalized outgoing wave packet in channel 2, X the total space, V_R the space inside the sphere of observation, and where the times T_0 and T are such that, R having been chosen large enough, the wave packets are totally in V_R at the time T_0 and totally outside of V_R at time T.

Integrating by parts for the time variable, the expression for this time delay becomes

$$\Delta t_{21}{}^a = -\int_{T_0}^{T} t dt \frac{\partial}{\partial t}\int_{X-V_R} d\mathbf{r}_2\{\chi_2^*\chi_2 - \psi_2^*\psi_2\}.$$

Therefore,

$$\Delta t_{21}{}^a = \int_{T_0}^{T} t dt \int_{S_R} d\mathbf{S}\cdot(\mathbf{j}_\psi - \mathbf{j}_\chi), \quad (2.8)$$

where $\int_{S_R} d\mathbf{S}$ is the surface integral over the sphere of observation and where $\mathbf{j}_\psi$ and $\mathbf{j}_\chi$ are just the probability currents for $|\psi_2\rangle$ and $|\chi_2\rangle$, respectively. For example,

$$\mathbf{j}_\psi = (\hbar/2\mu_2 i)(\psi_2^*\nabla\psi_2 - \psi_2\nabla\psi_2^*).$$

Since $dt\int_{S_R}\mathbf{j}_\psi\cdot d\mathbf{S}$ is just the probability of measuring a time between t and $t+dt$ at the surface S_R for the wave packet $|\psi_2\rangle$, it is evident as to what is the "experiment" being considered here. It consists of time measurements made at the surface of the sphere of observation. The average time for $|\psi_2\rangle$ to emerge from S_R is measured, that for $|\chi_2\rangle$ is calculated from information for the incoming wave packet (and for $|S_l^{21}(\epsilon)|$), and the time delay is then the difference between these two times. In fact, as we shall see in Sec. 3, the direct consideration of such an experiment leads to a more natural definition of time delay.

Now the actual wave packet is

$$|\Psi(t)\rangle = \int d\epsilon\, A(\epsilon)e^{-i\epsilon t/\hbar}|\psi_{lm\epsilon}{}^+\rangle, \quad (2.9)$$

where $|\psi_{lm\epsilon}{}^+\rangle$ is the outgoing eigenstate for the angular quantum numbers (l,m) and energy ϵ. Since, the computation is to be made at large distances, we are only interested in the asymptotic form for $|\psi_{lm\epsilon}{}^+\rangle$:

$${}_\infty\psi_{lm\epsilon} \equiv \begin{pmatrix} {}_\infty\psi_{lm\epsilon}(\mathbf{r}_1) \\ {}_\infty\psi_{lm\epsilon}(\mathbf{r}_2)\end{pmatrix} \equiv \begin{pmatrix}\langle \mathbf{r}_1\to\infty|\psi_{lm\epsilon}{}^+\rangle \\ \langle\mathbf{r}_2\to\infty|\psi_{lm\epsilon}{}^+\rangle\end{pmatrix} = \begin{bmatrix} \left(\dfrac{2\mu_1 k_1}{\pi\hbar^2}\right)^{1/2} \mathcal{Y}_{lm}(\theta_1,\varphi_1)\dfrac{i^l}{(-2ik_1r_1)}[\mathcal{J}_{l\epsilon}(r_1) - S_l^{11}(\epsilon)\mathcal{O}_{l\epsilon}(r_1)] \\ \left(\dfrac{2\mu_2 k_2}{\pi\hbar^2}\right)^{1/2} \mathcal{Y}_{lm}(\theta_2,\varphi_2)\dfrac{i^l}{(-2ik_2r_2)}[-S_l^{21}(\epsilon)\mathcal{O}_{l\epsilon}(r_2)]\end{bmatrix}, \quad (2.10)$$

where

$$\epsilon = M_1c^2 + k_1^2/2\mu_1 = M_2c^2 + k_2^2/2\mu_2,$$

with

$$\mathcal{J}_{l\epsilon}(r) \to i^l e^{-ikr} \quad \text{as} \quad r \to \infty$$

and

$$\mathcal{O}_{l\epsilon}(r) = \mathcal{J}_{l\epsilon}^*(r).$$

Since R has been chosen so large that only outgoing wave packets contribute to Eq. (2.8), we may rewrite it as

$$\Delta t_{21}{}^a = \int_{-\infty}^{\infty} t dt \int_{S_R} d\mathbf{S} \cdot (\mathbf{j}_\psi - \mathbf{j}_\chi)_{\text{out}}, \tag{2.11}$$

where the subscript "out" indicates that the incoming part, i.e., any term involving $\mathcal{J}_{l\epsilon}$, is to be left out of the computation of $\Delta t_{21}{}^a$.

We can absorb the factor t as an energy derivative of the time dependence of the currents to yield

$$\int_{-\infty}^{\infty} t dt \int_{S_R} d\mathbf{S} \cdot (\mathbf{j}_\psi)_{\text{out}} = \int_{-\infty}^{\infty} dt \int_{S_R} d\mathbf{S} \cdot \frac{\hbar}{2\mu_2 i N^2} \int d\epsilon \int d\epsilon' A(\epsilon) S_l^{21}(\epsilon) A^*(\epsilon') S_l^{21*}(\epsilon') \times (\xi_{lm\epsilon'}{}^* \nabla \xi_{lm\epsilon} - \xi_{lm\epsilon} \nabla \xi_{lm\epsilon'}{}^*)_{\text{out}} i\hbar \frac{\partial}{\partial \epsilon} e^{-i(\epsilon - \epsilon')t/\hbar},$$

and similarly for $|\chi_2\rangle$ except for absolute value signs on the S-matrix factors. Now integrating by parts for the energy variable ϵ, then integrating over time to give a $\delta(\epsilon - \epsilon')$ term, and noting that $A(\infty) = 0$ and that the current vanishes for zero kinetic energy, we obtain for the time delay

$$\Delta t_{21}{}^a = \int_{S_R} d\mathbf{S} \cdot \frac{\hbar^2}{2\mu_2 N^2} \int d\epsilon 2\pi\hbar |A(\epsilon) S_l^{21}(\epsilon)|^2 \times \Big[-\exp(-2i\delta_l^{21}) \xi_{lm\epsilon}{}^* \nabla \frac{\partial}{\partial \epsilon} \{\exp(2i\delta_l^{21}) \xi_{lm\epsilon}\} + \frac{\partial}{\partial \epsilon} \{\exp(2i\delta_l^{21}) \xi_{lm\epsilon}\} \nabla \xi_{lm\epsilon}{}^* + \xi_{lm\epsilon}{}^* \nabla \frac{\partial}{\partial \epsilon} \xi_{lm\epsilon} - \Big\{\frac{\partial}{\partial \epsilon} \xi_{lm\epsilon}\Big\} \nabla \xi_{lm\epsilon}{}^* \Big]_{\text{out}},$$

where we have written

$$S_l^{21}(\epsilon) = |S_l^{21}(\epsilon)| e^{2i\delta_l^{21}(\epsilon)}, \quad \delta_l^{21} \equiv \delta_l^{21}(\epsilon). \tag{2.12}$$

It might here be noted that for the elastic case, the expression in the square brackets was also obtained by Smith[3] except for the oscillatory terms which do not contribute here as they come from the cross terms between the incoming and outgoing parts. It can be seen now that in the wave-packet calculation these oscillatory terms would in any case vanish as we take $R \to \infty$ according to the Riemann-Lebesgue lemma. In fact, in the one-dimensional case, the contribution here due to such oscillatory terms would essentially be the scalar product between the incoming free Heisenberg wave-packet state and the corresponding outgoing state with one of them translated through a distance of $2R$.

Substituting in the asymptotic form for $\xi_{lm\epsilon}$, we get that

$$\Delta t_{21}{}^a = N^{-2} \int d\epsilon |A(\epsilon) S_l^{21}(\epsilon)|^2 2\hbar (\partial \delta_l^{21}(\epsilon)/\partial \epsilon). \tag{2.13}$$

3. A QUANTUM-THEORETIC FORMULATION OF THE "CLASSICAL" DEFINITION

Taking the "experimental" situation mentioned in Sec. 2 as the basis of our discussion here, we then have the average times for the incoming and outgoing wave packets at the sphere of observation. Furthermore, we can, in principle, extrapolate these wave packets as if free and find then the average times the incoming state would emerge from and the outgoing state enter this same sphere. Using these average times in the "classical" definition, see Eq. (1) of Ref. 3, the time delay is then given by

$$\Delta t_{21} = \tfrac{1}{2}(\bar{T}_\psi{}^{\text{in}} + \bar{T}_\psi{}^{\text{out}}) - \tfrac{1}{2}(\bar{T}_\Phi{}^{\text{in}} + \bar{T}_\Phi{}^{\text{out}}), \tag{3.1}$$

where $\bar{T}_\psi{}^{\text{in}}$ and $\bar{T}_\psi{}^{\text{out}}$ are the average in and out times, respectively, for the state $|\psi_2\rangle$ at the sphere of observation and similarly for $\bar{T}_\Phi{}^{\text{in}}$ and $\bar{T}_\Phi{}^{\text{out}}$ for the state $|\Phi\rangle$.

Performing the same type of computation as was done in Sec. 2, it can then be shown that

$$\bar{T}_\Phi{}^{\text{out}} = \int_{T_0}^{T} t dt \int_{S_R} d\mathbf{S} \cdot \mathbf{j}_\Phi = \int_{-\infty}^{\infty} t dt \int_{S_R} d\mathbf{S} \cdot (\mathbf{j}_\Phi)_{\text{out}} = \bar{T}_\Phi + \tau_\Phi,$$

with

$$\bar{T}_\Phi = \text{Re} \int_{S_R} d\mathbf{S} \cdot \left(-\frac{\hbar^2}{2\mu_1}\right) \int d\epsilon\, 2\pi\hbar |A(\epsilon)|^2 \times \left(\varphi_{lm\epsilon}{}^* \nabla \frac{\partial \varphi_{lm\epsilon}}{\partial \epsilon} - \frac{\partial \varphi_{lm\epsilon}}{\partial \epsilon} \nabla \varphi_{lm\epsilon}{}^*\right)_{\text{out}}$$

and

$$\tau_\Phi=\int_{S_R} d\mathbf{S}\cdot\left(-\frac{\hbar^2}{2\mu_1}\right)\int d\epsilon\, 2\pi\hbar|A(\epsilon)|^2 i \times\frac{\partial a(\epsilon)}{\partial\epsilon}(\varphi_{lm\epsilon}{}^*\nabla\varphi_{lm\epsilon}-\varphi_{lm\epsilon}\nabla\varphi_{lm\epsilon}{}^*)_{\text{out}},$$

where we have written

$$A(\epsilon)=|A(\epsilon)|e^{ia(\epsilon)}. \tag{3.2}$$

Similarly,

$$\bar{T}_\Phi{}^{\text{in}}=\int_{-T}^{T_0} tdt\int_{S_R}(-d\mathbf{S}\cdot\mathbf{j}_\Phi)=-\int_{-\infty}^{\infty} tdt\int_{S_R} d\mathbf{S}\cdot(\mathbf{j}_\Phi)_{\text{in}},$$

where the subscript "in" denotes that the out part, i.e., any term involving $\mathcal{O}_{l\epsilon}$, is to be left out of the computation of the integral. Then, putting in the $\mathcal{J}_{l\epsilon}$ factors explicitly and noting that $\mathcal{J}_{l\epsilon}(r)=\mathcal{O}_{l\epsilon}{}^*(r)$, we obtain

$$\bar{T}_\Phi{}^{\text{in}}=-\bar{T}_\Phi+\tau_\Phi.$$

Therefore,

$$\tfrac{1}{2}(\bar{T}_\Phi{}^{\text{in}}+\bar{T}_\Phi{}^{\text{out}})=\tau_\Phi.$$

Now the wave function of the outgoing state $|\psi_2\rangle$ is given asymptotically as

$$\psi_2(\mathbf{r}_2;t)=\frac{1}{N}\int d\epsilon\, A(\epsilon)S_l^{21}(\epsilon)e^{-i\epsilon t/\hbar}(\xi_{lm\epsilon})_{\text{out}},\quad r_2 \text{ large.}$$

Thus, the outgoing state considered as free is just

$$|\psi_2(t)\rangle=\frac{1}{N}\int d\epsilon\, A(\epsilon)S_l^{21}(\epsilon)e^{-i\epsilon t/\hbar}|\xi_{lm\epsilon}\rangle. \tag{3.3}$$

By applying the same type of computation as before, we get that

$$\tfrac{1}{2}(\bar{T}_\psi{}^{\text{in}}+\bar{T}_\psi{}^{\text{out}})=\tau_\psi,$$

where

$$\tau_\psi=\int_{S_R} d\mathbf{S}\cdot\left(-\frac{\hbar^2}{2\mu_2}\right)\int d\epsilon\, 2\pi\hbar|A(\epsilon)S_l^{21}(\epsilon)|^2 i \times\frac{\partial}{\partial\epsilon}\{a(\epsilon)+2\delta_l^{21}(\epsilon)\}(\xi_{lm\epsilon}{}^*\nabla\xi_{lm\epsilon}-\xi_{lm\epsilon}\nabla\xi_{lm\epsilon}{}^*)_{\text{out}}.$$

τ_Φ and τ_ψ are evidently the times at which the respective free wave packets reach the midpoint (in time) of their traverse across the sphere of observation.

Since R is large, we can now substitute in the asymptotic forms for the spherical wave functions to obtain for the time delay

$$\Delta t_{21}=\tau_\Phi-\tau_\psi =N^{-2}\left[\int d\epsilon|A(\epsilon)S_l^{21}(\epsilon)|^2 2\hbar\frac{\partial\delta_l^{21}(\epsilon)}{\partial\epsilon} +\int d\epsilon|A(\epsilon)|^2\hbar\frac{\partial a(\epsilon)}{\partial\epsilon}\{|S_l^{21}(\epsilon)|^2-N^2\}\right], \tag{3.4}$$

with

$$N^2=\int d\epsilon|A(\epsilon)S_l^{21}(\epsilon)|^2.$$

4. DISCUSSION

The result for the elastic channel and the extension to more two-particle channels and to spin are evident. In fact, this present work rests mainly on being able to write the asymptotic expression for the stationary spherical scattering state $|\psi_{lm\epsilon}{}^+\rangle$ in a form similar to that of Eq. (2.10). The result then is valid for any local or nonlocal potential with a finite range. More strongly, since we assume that we can take the sphere of observation to infinity, it should be valid for any potential for which scattering theory is applicable. In practice, the particles going in and coming out are detected as free particles, so that the range is "naturally" finite or else is finite owing to some form of shielding.

In the elastic case, since $|S_l^{11}(\epsilon)|$ is then unity, we get back the Goldberger and Watson result for a general wave packet.[4] For a wave packet with a narrow energy spread, such that $|S_l^{21}(\epsilon)|$ and $\partial\delta_l^{21}(\epsilon)/\partial\epsilon$ can be considered as constants as far as Eq. (3.4) is concerned, the result of Eisenbud is obtained.[1] The interest of the present result is that it is for inelastic scattering with a general wave packet incident. This result, Eq. (3.4), depends, however, on how the wave packet is made up. Though, away from threshold we can assume that $|S_l^{21}(\epsilon)|^2$, which expresses the probability of finding the inelastic channel, is so slowly varying over the energy range of the wave packet that we can consider it as constant, in which case the second integral vanishes. Such an assumption for a wave packet about the threshold energy would, of course, be ridiculous. For this case, even if such time measurements could be made, we need, in order to be able to say something concerning the phase of the S matrix, to know not only the energy distribution of the wave packet as well as $|S_l^{21}(\epsilon)|$ but also the phase of the energy amplitude $A(\epsilon)$ for the wave packet. This would seem to belie the usefulness, at least hypothetically, of time delay in giving information concerning the phase of the S matrix about the threshold energy. On the other hand, it is amusing to conjecture that, if we did know the S matrix, then perhaps we might through time delay be able to say something about the phase of the energy amplitude of a wave packet.

Note added in proof. After this manuscript had been submitted for publication, the author became aware of similar work done by T. Ohmura [Progr. Theoret. Phys. (Kyoto) (suppl.) **29**, 108 (1964)], to which the present work can be thought of as complementary. Ohmura considered the problem of a burst of particles and defined a time delay for particles emerging at some angle not in the incident wave packet, so that it is the T-matrix which then determines the time delay. The S-matrix could also have been used. However, when the computation is made over all partial waves, the contribution due to the incident wave packet vanishes as the incident wave packet has value zero in this region, and thus the result is the same as when only the T-matrix is considered.

Although the present result is only for a spherical wave packet, it is easily extended to any general wave packet if we consider times as averaged over the whole sphere of observation, i.e., including also the particles emerging in the direction of incidence. Then, because of the integration over all angles and the orthogonality of the spherical harmonics, the expression for the time delay, defined in the spirit of this paper, has added just a summation over the spherical quantum numbers.

Finally, it is interesting to note that for Ohmura's result the dependence on the phase of the energy amplitude $A(\epsilon)$ of the wave packet is already important for elastic scatttering, whereas here, as we had noted above, it is only crucial for wave packets "near" and around threshold.

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Conservation Laws Implied by Lorentz Invariance and Conservation of Spin

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For a reaction from any number of particles in the initial state to any (possibly different) number of particles in the final state, it is shown that Lorentz invariance and conservation of total spin imply conservation laws involving the velocities of the individual particles. Conservation of spin is defined as in a definition of $SU(6)$ for quarks. The implied conservation laws are too restrictive for the description of any interesting interaction of the spinning particles.

THE relativistic definition of $SU(6)$ for quarks suggested by Mahanthappa and Sudarshan and by Riazuddin and Pandit[1] avoids many of the difficulties encountered in other relativistic versions of $SU(6)$. Those difficulties which can be traced to the failure of the spin matrices of the Dirac equation to commute with the free-particle Hamiltonian—for example, the troubles with unitarity—are obviated by using the Wigner-Foldy[2] canonical particle-spin operators. None of the negative group-theoretic theorems[3] are applicable because, as is shown explicitly below, there is no Lie group with a finite number of parameters which contains both the Lorentz transformations and the transformations generated by the spin. A problem appears, however, when one asks for an invariant interaction. In a previous paper[4] it was shown that for the scattering of two particles, each with positive mass and spin $\frac{1}{2}$, the symmetries generated by the total spin and by the commutator of the total spin with the generator of Lorentz transformations restrict the scattering amplitudes so severely that no interesting interaction can be described. In the present paper conservation laws are constructed from the entire (infinite-dimensional) Lie algebra generated by the total spin and the generators of the Poincaré group for a reaction involving any number of particles with any positive masses and integral or half-integral spins. These conservation laws are too restrictive for the description of any interesting interaction of the spinning particles.

Consider a system of N particles, each with positive mass and integral or half-integral spin. We describe the nth particle ($n=1, 2, \cdots N$) by Hermitian position and momentum operators $\mathbf{Q}^{(n)}$ and $\mathbf{P}^{(n)}$ (which satisfy canonical commutation relations) and Hermitian spin operators $\mathbf{S}^{(n)}$ (which commute with $\mathbf{Q}^{(n)}$ and $\mathbf{P}^{(n)}$ and satisfy angular-momentum commutation relations) in terms of which the generators of the Poincaré group for N noninteracting particles have the form[2]

$$H=\sum_{n=1}^{N} (\mathbf{P}^{(n)2}+m_n^2)^{1/2},$$

$$\mathbf{P}=\sum_{n=1}^{N} \mathbf{P}^{(n)},$$

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