region, and thus the result is the same as when only the T-matrix is considered.

Although the present result is only for a spherical wave packet, it is easily extended to any general wave packet if we consider times as averaged over the whole sphere of observation, i.e., including also the particles emerging in the direction of incidence. Then, because of the integration over all angles and the orthogonality of the spherical harmonics, the expression for the time delay, defined in the spirit of this paper, has added just a summation over the spherical quantum numbers.

Finally, it is interesting to note that for Ohmura's result the dependence on the phase of the energy amplitude $A(\epsilon)$ of the wave packet is already important for elastic scatttering, whereas here, as we had noted above, it is only crucial for wave packets "near" and around threshold.

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Conservation Laws Implied by Lorentz Invariance and Conservation of Spin

THOMAS F. JORDAN*
Department of Physics, University of Pittsburgh, Pittsburgh, Pennsylvania
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For a reaction from any number of particles in the initial state to any (possibly different) number of particles in the final state, it is shown that Lorentz invariance and conservation of total spin imply conservation laws involving the velocities of the individual particles. Conservation of spin is defined as in a definition of $SU(6)$ for quarks. The implied conservation laws are too restrictive for the description of any interesting interaction of the spinning particles.

THE relativistic definition of $SU(6)$ for quarks suggested by Mahanthappa and Sudarshan and by Riazuddin and Pandit[1] avoids many of the difficulties encountered in other relativistic versions of $SU(6)$. Those difficulties which can be traced to the failure of the spin matrices of the Dirac equation to commute with the free-particle Hamiltonian—for example, the troubles with unitarity—are obviated by using the Wigner-Foldy[2] canonical particle-spin operators. None of the negative group-theoretic theorems[3] are applicable because, as is shown explicitly below, there is no Lie group with a finite number of parameters which contains both the Lorentz transformations and the transformations generated by the spin. A problem appears, however, when one asks for an invariant interaction. In a previous paper[4] it was shown that for the scattering of two particles, each with positive mass and spin $\frac{1}{2}$, the symmetries generated by the total spin and by the commutator of the total spin with the generator of Lorentz transformations restrict the scattering amplitudes so severely that no interesting interaction can be described. In the present paper conservation laws are constructed from the entire (infinite-dimensional) Lie algebra generated by the total spin and the generators of the Poincaré group for a reaction involving any number of particles with any positive masses and integral or half-integral spins. These conservation laws are too restrictive for the description of any interesting interaction of the spinning particles.

Consider a system of N particles, each with positive mass and integral or half-integral spin. We describe the nth particle ($n=1, 2, \cdots N$) by Hermitian position and momentum operators $\mathbf{Q}^{(n)}$ and $\mathbf{P}^{(n)}$ (which satisfy canonical commutation relations) and Hermitian spin operators $\mathbf{S}^{(n)}$ (which commute with $\mathbf{Q}^{(n)}$ and $\mathbf{P}^{(n)}$ and satisfy angular-momentum commutation relations) in terms of which the generators of the Poincaré group for N noninteracting particles have the form[2]

$$H=\sum_{n=1}^{N} (\mathbf{P}^{(n)2}+m_n^2)^{1/2},$$

$$\mathbf{P}=\sum_{n=1}^{N} \mathbf{P}^{(n)},$$

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* Alfred P. Sloan Research Fellow.

[1] K. T. Mahanthappa and E. C. G. Sudarshan, Phys. Rev. Letters 14, 458 (1965); Riazuddin and L. K. Pandit, *ibid.* 14, 462 (1965). In both of these papers the difficulty of constructing a local four-fermion interaction is noted. F. Gürsey [Phys. Letters 14, 330 (1965)] states that this is the definition of $SU(6)$ intended originally by F. Gürsey and L. A. Radicati, Phys. Rev. Letters 13, 173 (1964).

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$$\mathbf{J}=\sum_{n=1}^{N}(\mathbf{Q}^{(n)}\times\mathbf{P}^{(n)}+\mathbf{S}^{(n)}),$$

$$\mathbf{K}=\sum_{n=1}^{N}[\tfrac{1}{2}(\mathbf{P}^{(n)2}+m_n^2)^{1/2}\mathbf{Q}^{(n)}+\tfrac{1}{2}\mathbf{Q}^{(n)}(\mathbf{P}^{(n)2}+m_n^2)^{1/2}+\mathbf{V}^{(n)}\times\mathbf{S}^{(n)}]$$

with m_n being the mass of particle n and $\mathbf{V}^{(n)}$ being defined by

$$\mathbf{V}^{(n)}=[m_n+(\mathbf{P}^{(n)2}+m_n^2)^{1/2}]^{-1}\mathbf{P}^{(n)}.$$

We define a total spin operator

$$\mathbf{S}=\sum_{n=1}^{N}\mathbf{S}^{(n)}.$$

We construct a basis for the Lie algebra generated by H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$, and $\mathbf{S}$ as follows. We evaluate the commutator

$$i(K_kS_j-S_jK_k)=\sum_{n=1}^{N}V_j^{(n)}S_k^{(n)}$$

for values of j, $k=1, 2, 3$ such that $j\neq k$. By taking the commutator of this with S_m for values of $m=1, 2, 3$ such that $m\neq j$ and $m\neq k$, we get operators like the right-hand side above for values of j, $k=1, 2, 3$ such that $j=k$. Taking repeated commutators of these operators with each other yields all operators of the form

$$\sum_{n=1}^{N}V_{j_1}^{(n)}V_{j_2}^{(n)}\cdots V_{j_p}^{(n)}S_k^{(n)}$$

with any positive integral number p of factors of components of $\mathbf{V}^{(n)}$ and all values of $j_1, j_2, \cdots j_p, k=1, 2, 3$. To include commutators again with $\mathbf{K}$, we must admit also operators of the above form with any number of factors $V_j^{(n)}$ replaced by

$$V_0^{(n)}=[m_n+(\mathbf{P}^{(n)2}+m_n^2)^{1/2}]^{-1}(\mathbf{P}^{(n)2}+m_n^2)^{1/2}.$$

Thus we get all operators of the form

$$\sum_{n=1}^{N}V_{\mu_1}^{(n)}V_{\mu_2}^{(n)}\cdots V_{\mu_p}^{(n)}S_k^{(n)} \quad (1)$$

for positive integral p with all values of $\mu_1, \mu_2, \cdots \mu_p =0, 1, 2, 3$ and $k=1, 2, 3$. These operators, together with H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$, and $\mathbf{S}$, form a basis for the Lie algebra generated by H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$, and $\mathbf{S}$; the commutator (divided by i) of any two of these operators is a (real) linear combination of these operators. The Lie algebra is infinite-dimensional. Theorems proved for a finite-dimensional Lie group containing the Poincaré and spin groups[3] are not applicable here. The Lie algebra contains the spin-dependent terms of $\mathbf{J}$ and $\mathbf{K}$ as separate elements and contains the generators of the Poincaré group with the spin-dependent terms removed.

Consider a reaction described by a scattering operator T. Lorentz invariance implies that T commutes with the generator $\mathbf{K}$ of Lorentz transformations. We postulate that T commutes with the total spin operator $\mathbf{S}$. It follows that T commutes with all operators (1) obtained from $\mathbf{K}$ and $\mathbf{S}$ by taking commutators. Consider the amplitude for the reaction to proceed from an initial state of N particles which is an eigenstate of $\mathbf{P}^{(n)}$ and $S_3^{(n)}$ for each $n=1, 2, \cdots N$ to a final state of N' particles which is an eigenstate of $\mathbf{P}^{(n')}$ and $S_3^{(n')}$ for each $n'=1', 2', \cdots N'$. This matrix element of T is zero unless the conservation laws

$$\sum_{n=1}^{N}S_3^{(n)}=\sum_{n'=1'}^{N'}S_3^{(n')} \quad (2)$$

and

$$\sum_{n=1}^{N}V_{\mu_1}^{(n)}V_{\mu_2}^{(n)}\cdots V_{\mu_p}^{(n)}S_3^{(n)}=\sum_{n'=1'}^{N'}V_{\mu_1}^{(n')}V_{\mu_2}^{(n')}\cdots V_{\mu_p}^{(n')}S_3^{(n')} \quad (3)$$

are satisfied for each positive integral number p of factors of $V_\mu^{(n)}$ and all values of $\mu_1, \mu_2, \cdots \mu_p=0, 1, 2, 3$. To see the effect of these conservation laws we define

$$f(V_0,\mathbf{V})=\sum_{n=1}^{N}S_3^{(n)}\delta(V_0-V_0^{(n)})\delta^{(3)}(\mathbf{V}-\mathbf{V}^{(n)})$$

$$g(V_0,\mathbf{V})=\sum_{n'=1'}^{N'}S_3^{(n')}\delta(V_0-V_0^{(n')})\delta^{(3)}(\mathbf{V}-\mathbf{V}^{(n')})$$

for which (2) and (3) imply that

$$\int f(V_0,\mathbf{V})dV_0d^3V=\int g(V_0,\mathbf{V})dV_0d^3V \quad (4)$$

and

$$\int f(V_0,\mathbf{V})V_{\mu_1}V_{\mu_2}\cdots V_{\mu_p}dV_0d^3V=\int g(V_0,\mathbf{V})V_{\mu_1}V_{\mu_2}\cdots V_{\mu_p}dV_0d^3V \quad (5)$$

for each positive integral p and all values of $\mu_1, \mu_2, \cdots \mu_p =0, 1, 2, 3$, from which we can show that

$$f(V_0,\mathbf{V})=g(V_0,\mathbf{V})$$

as follows. Since $f(V_0,\mathbf{V})-g(V_0,\mathbf{V})$ is nonzero at only a finite number of points $(V_0,\mathbf{V})$, we can find a polynomial $P(V_0,\mathbf{V})$ which is positive at all points where $f(V_0,\mathbf{V})-g(V_0,\mathbf{V})$ is positive and negative at all points where $f(V_0,\mathbf{V})-g(V_0,\mathbf{V})$ is negative, so that

$$[f(V_0,\mathbf{V})-g(V_0,\mathbf{V})]P(V_0,\mathbf{V})$$

is everywhere non-negative. But (4) and (5) imply that

$$\int [f(V_0,\mathbf{V})-g(V_0,\mathbf{V})]P(V_0,\mathbf{V})dV_0d^3V=0$$

from which we conclude that

$$[f(V_0,\mathbf{V})-g(V_0,\mathbf{V})]P(V_0,\mathbf{V})=0$$

which means that

$$f(V_0,\mathbf{V})-g(V_0,\mathbf{V})=0.$$

We can write the conservation laws (2) and (3) in the equivalent form

$$\sum_{n=1}^{N} S_3{}^{(n)}\delta(V_0-V_0{}^{(n)})\delta^{(3)}(\mathbf{V}-\mathbf{V}^{(n)}) = \sum_{n'=1'}^{N'} S_3{}^{(n')}\delta(V_0-V_0{}^{(n')})\delta^{(3)}(\mathbf{V}-\mathbf{V}^{(n')}). \quad (6)$$

Suppose that in the initial state there are no two particles with the same values of $V_0{}^{(n)}$ and $\mathbf{V}^{(n)}$ and with nonzero values of $S_3{}^{(n)}$ and suppose that the same is true for the final state. Then the conservation law (6) implies that the initial and final states contain the same number of particles with nonzero third component of spin, that for these particles the set of values of $V_0{}^{(n)}$ and $\mathbf{V}^{(n)}$ for the initial state is the same as the set of values of $V_0{}^{(n')}$ and $\mathbf{V}^{(n')}$ for the final state, and that each of these values carries the same third component of spin initially and finally. There may be a permutation of these values among particles with different quantum numbers like charge. Particles with zero third component of spin, in particular particles with zero spin, are not restricted by this conservation law. If there is a duplication of values of $V_0{}^{(n)}$ and $\mathbf{V}^{(n)}$ in the initial state or of $V_0{}^{(n')}$ and $\mathbf{V}^{(n')}$ in the final state, there may be a cancellation of terms in (6) so that these particles do not participate in the conservation law.

If $V_0{}^{(n)}$ and $\mathbf{V}^{(n)}$ for a particle in the initial state are equal to $V_0{}^{(n')}$ and $\mathbf{V}^{(n')}$ for a particle in the final state, then these particles have the same velocity. The relativistic velocity of the particle is

$$\mathbf{V}^{(n)}/V_0{}^{(n)}=\mathbf{P}^{(n)}/(\mathbf{P}^{(n)2}+m_n{}^2)^{1/2}$$

and the nonrelativistic velocity is

$$\mathbf{V}^{(n)}/(1-V_0{}^{(n)})=\mathbf{P}^{(n)}/m_n.$$

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Čerenkov Radiation in Inhomogeneous Periodic Media*

K. F. CASEY,† C. YEH, AND Z. A. KAPRIELIAN

Electrical Engineering Department, University of Southern California, Los Angeles, California

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The formal exact solution to the problem of the radiation of a charged particle traveling with a constant velocity in a periodically inhomogeneous medium is obtained. As a specific example, the case with a sinusoidally varying dielectric profile is treated in detail. Results of the computation are summarized in two graphs from which information concerning threshold velocity for a particular mode, the emission angles for various radiating modes, and the cutoff frequency for a certain mode can be found. Unlike the case of Čerenkov radiation in a homogeneous medium, there exist radiating modes in this inhomogeneous-dielectric case even when the velocity of the charged particle is below the threshold Čerenkov velocity. A formal expression for the radiation spectrum is also given. Approximate expressions for the radiated fields and for the radiation spectrum are obtained when the variation of the permittivity is small. Results are discussed and interpreted.

I. INTRODUCTION

If the velocity of electrons traveling in a dielectric medium is higher than the phase velocity of light in the medium, radiation is observed. This is the well-known Čerenkov effect.[1] The theoretical analysis of the Čerenkov effect was first obtained by Frank and Tamm,[2] who treated the problem of the radiation from an electron moving uniformly in a homogeneous dielectric medium. Extension of their analysis to anisotropic and dispersive media has been carried out by various authors.[3] The problem of the emission from a particle traversing a piecewise homogeneous dielectric medium

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† Present address: U. S. Air Force Institute of Technology, Wright-Patterson Air Force Base, Dayton, Ohio.

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