is everywhere non-negative. But (4) and (5) imply that

$$\int [f(V_0,\mathbf{V})-g(V_0,\mathbf{V})]P(V_0,\mathbf{V})dV_0d^3V=0$$

from which we conclude that

$$[f(V_0,\mathbf{V})-g(V_0,\mathbf{V})]P(V_0,\mathbf{V})=0$$

which means that

$$f(V_0,\mathbf{V})-g(V_0,\mathbf{V})=0.$$

We can write the conservation laws (2) and (3) in the equivalent form

$$\sum_{n=1}^{N} S_3{}^{(n)}\delta(V_0-V_0{}^{(n)})\delta^{(3)}(\mathbf{V}-\mathbf{V}^{(n)}) = \sum_{n'=1'}^{N'} S_3{}^{(n')}\delta(V_0-V_0{}^{(n')})\delta^{(3)}(\mathbf{V}-\mathbf{V}^{(n')}). \quad (6)$$

Suppose that in the initial state there are no two particles with the same values of $V_0{}^{(n)}$ and $\mathbf{V}^{(n)}$ and with nonzero values of $S_3{}^{(n)}$ and suppose that the same is true for the final state. Then the conservation law (6) implies that the initial and final states contain the same number of particles with nonzero third component of spin, that for these particles the set of values of $V_0{}^{(n)}$ and $\mathbf{V}^{(n)}$ for the initial state is the same as the set of values of $V_0{}^{(n')}$ and $\mathbf{V}^{(n')}$ for the final state, and that each of these values carries the same third component of spin initially and finally. There may be a permutation of these values among particles with different quantum numbers like charge. Particles with zero third component of spin, in particular particles with zero spin, are not restricted by this conservation law. If there is a duplication of values of $V_0{}^{(n)}$ and $\mathbf{V}^{(n)}$ in the initial state or of $V_0{}^{(n')}$ and $\mathbf{V}^{(n')}$ in the final state, there may be a cancellation of terms in (6) so that these particles do not participate in the conservation law.

If $V_0{}^{(n)}$ and $\mathbf{V}^{(n)}$ for a particle in the initial state are equal to $V_0{}^{(n')}$ and $\mathbf{V}^{(n')}$ for a particle in the final state, then these particles have the same velocity. The relativistic velocity of the particle is

$$\mathbf{V}^{(n)}/V_0{}^{(n)}=\mathbf{P}^{(n)}/(\mathbf{P}^{(n)2}+m_n{}^2)^{1/2}$$

and the nonrelativistic velocity is

$$\mathbf{V}^{(n)}/(1-V_0{}^{(n)})=\mathbf{P}^{(n)}/m_n.$$

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Čerenkov Radiation in Inhomogeneous Periodic Media*

K. F. Casey,† C. Yeh, and Z. A. Kaprielian
Electrical Engineering Department, University of Southern California, Los Angeles, California
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The formal exact solution to the problem of the radiation of a charged particle traveling with a constant velocity in a periodically inhomogeneous medium is obtained. As a specific example, the case with a sinusoidally varying dielectric profile is treated in detail. Results of the computation are summarized in two graphs from which information concerning threshold velocity for a particular mode, the emission angles for various radiating modes, and the cutoff frequency for a certain mode can be found. Unlike the case of Čerenkov radiation in a homogeneous medium, there exist radiating modes in this inhomogeneous-dielectric case even when the velocity of the charged particle is below the threshold Čerenkov velocity. A formal expression for the radiation spectrum is also given. Approximate expressions for the radiated fields and for the radiation spectrum are obtained when the variation of the permittivity is small. Results are discussed and interpreted.

I. INTRODUCTION

If the velocity of electrons traveling in a dielectric medium is higher than the phase velocity of light in the medium, radiation is observed. This is the well-known Čerenkov effect.[1] The theoretical analysis of the Čerenkov effect was first obtained by Frank and Tamm,[2] who treated the problem of the radiation from an electron moving uniformly in a homogeneous dielectric medium. Extension of their analysis to anisotropic and dispersive media has been carried out by various authors.[3] The problem of the emission from a particle traversing a piecewise homogeneous dielectric medium

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† Present address: U. S. Air Force Institute of Technology, Wright-Patterson Air Force Base, Dayton, Ohio.

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[2] I. M. Frank and I. Tamm, Dokl. Akad. Nauk. S.S.S.R. **14**, 109 (1937).

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was considered by Fainberg and Khiznyak[4] and by Garybyan.[5] However, very little work has been carried out on the problem of the radiation from an electron moving uniformly in a continuously inhomogeneous periodic dielectric medium. It is expected that if the wavelength of the emitted radiation is much smaller than the period of the inhomogeneity, the Čerenkov radiation would depend to a great extent upon the permittivity in the immediate neighborhood of the particle, which may be assumed nearly constant. The WKB method was used by Ter-Mikaelyan[6] to investigate this problem. On the other hand, if the wavelength of the emitted radiation is much greater than the period of the dielectric variation, the Čerenkov radiation will depend primarily upon the average value of the permittivity of the medium. At wavelength comparable to the period of the medium, the behavior of the radiation is not readily apparent.

It is therefore the purpose of this paper to treat this problem. A formal solution will be obtained for the spectral density of the radiation emitted by a charged-particle traveling uniformly in a dielectric medium which is periodically and continuously inhomogeneous in the direction along the particle path. Detailed analysis is carried out for a specific dielectric variation, i.e.,

$$\epsilon(z)=\epsilon_a[1-\delta\cos(2\pi z/p)],$$

where ϵ_a is the average value of the dielectric constant, δ is the magnitude of the variation and p is the periodicity of the variation. If the magnitude of the dielectric variation is small, approximate analytic results can be found. It is noted that owing to the inhomogeneity of the medium, there exists not only Čerenkov-type radiation by also transition-type radiation as well.

II. FORMULATION OF THE PROBLEM

It is assumed that an inhomogeneous dielectric medium fills the entire space and possesses a relative permittivity

$$\epsilon(z)/\epsilon_0=\epsilon(z+p)/\epsilon_0, \tag{1}$$

a relative permeability

$$\mu/\mu_0=1, \tag{2}$$

and a conductivity $\sigma=0$, in the (x,y,z) rectangular coordinate system. p is the periodicity of the dielectric variation, ϵ_0 and μ_0 are, respectively, the free-space permeability and $\epsilon(z)$ is an analytic function. A charged particle is moving in the z direction through this medium at a constant speed v. Denoting the charge by q, one has for the current density $\mathbf{J}$ due to the passage of this particle

$$\mathbf{J}=(q/2\pi\rho)\delta(\rho)\delta(t-z/v)\mathbf{e}_z, \tag{3}$$

where $\mathbf{e}_z$ is a unit vector in the z direction, $\rho^2=x^2+y^2$, and δ is the Dirac delta function. Maxwell's equations for the present situation are

$$\nabla\times\mathbf{E}=-\mu\partial\mathbf{H}/\partial t \tag{4}$$

$$\nabla\times\mathbf{H}=\epsilon(z)\frac{\partial\mathbf{E}}{\partial t}+\frac{q}{2\pi\rho}\delta(\rho)\delta\left(t-\frac{z}{v}\right)\mathbf{e}_z, \tag{5}$$

where $\mathbf{E}$ and $\mathbf{H}$ are, respectively, the electric and magnetic fields. Taking the Fourier transformation with respect to time of Eqs. (4) and (5) gives

$$\nabla\times\mathcal{E}=i\omega\mu\mathcal{H} \tag{6}$$

$$\nabla\times\mathcal{H}=-i\omega\epsilon(z)\mathcal{E}+(q/4\pi^2\rho)\delta(\rho)e^{i\omega z/v}\mathbf{e}_z. \tag{7}$$

$\mathcal{E}$ and $\mathcal{H}$ denote the Fourier transforms of $\mathbf{E}$ and $\mathbf{H}$, respectively; they are related to $\mathbf{E}$ and $\mathbf{H}$ by

$$\mathbf{E}=\int_{-\infty}^{\infty}\mathcal{E}e^{-i\omega t}d\omega, \tag{8}$$

$$\mathbf{H}=\int_{-\infty}^{\infty}\mathcal{H}e^{-i\omega t}d\omega. \tag{9}$$

To solve Eqs. (6) and (7) for the region $\rho>0$, let us introduce a scalar function $\psi(\rho,z)$ as follows:

$$\mathcal{H}=\nabla\times[\psi(\rho,z)\mathbf{e}_z], \tag{10}$$

$$\mathcal{E}=\frac{i}{\omega\epsilon(z)}\nabla\times\nabla\times[\psi(\rho,z)\mathbf{e}_z]. \tag{11}$$

$\psi(\rho,z)$ satisfies the following partial differential equation:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial\psi}{\partial\rho}\right]-\frac{1}{\epsilon}\frac{d\epsilon}{dz}\frac{d\psi}{dz}+\frac{\partial^2\psi}{\partial z^2}+\omega^2\mu\rho\psi=0. \tag{12}$$

This method of solving the vector wave equation for an inhomogeneous medium is a modified version of the vector wave-function method of Hansen[7] and Stratton.[8] ψ must also satisfy the following boundary conditions:

(a) the Sommerfeld radiation condition at $\rho=\infty$,
(b) Ampere's law at $\rho=0$, i.e.,

$$\lim_{\rho\to 0}2\pi\rho\frac{\partial\psi}{\partial\rho}=-\frac{q}{2\pi}e^{i\omega z/v}. \tag{13}$$

III. FORMAL SOLUTION FOR ψ

The appropriate solution for ψ can be obtained from Eq. (12) using the method of separation of variables:

$$\psi(\rho,z)=H_0^{(1)}(\gamma\rho)Z(z), \tag{14}$$

[4] Ya. Fainberg and N. Khiznyak, Zh. Eksperim. i Teor. Fiz. 32, 883 (1957) [English transl.: Soviet Phys.—JETP 5, 720 (1957)].
[5] G. Garybyan, Zh. Eksperim. i Teor. Fiz. 35, 1435 (1958) [English transl.: Soviet Phys.—JETP 8, 1003 (1959)].
[6] M. Ter-Mikaelyan, Dokl. Akad. Nauk SSSR 134, 318 (1960) [English transl.: Soviet Phys.—Doklady 5, 1015 (1960)].
[7] W. W. Hansen, Phys. Rev. 47, 139 (1935).
[8] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941).

where $H_0^{(1)}(\gamma\rho)$ is the Hankel function of the first kind of order zero, γ^2 is a separation constant, and $Z(z)$ satisfies the following differential equation:

$$\frac{d^2Z}{dz^2}-\frac{1}{\epsilon}\frac{d\epsilon}{dz}\frac{dZ}{dz}+(\omega^2\mu_0\epsilon-\gamma^2)Z=0. \qquad (15)$$

In order that the radiation condition at infinity can be satisfied, both the real and imaginary parts of γ must be positive.

The solution of Eq. (15) depends, of course, on the dielectric variation, $\epsilon(z)$. Making the substitution

$$\xi=\pi z/p, \qquad (16)$$

$$Z(\xi)=\epsilon^{1/2}(\xi)u(\xi), \qquad (17)$$

one has

$$\frac{d^2u}{d\xi^2}+\left[\frac{1}{2\epsilon}\frac{d^2\epsilon}{d\xi^2}-\frac{3}{4\epsilon^2}\left(\frac{d\epsilon}{d\xi}\right)^2+(\omega^2\mu_0\epsilon-\gamma^2)\frac{p^2}{\pi^2}\right]u=0. \qquad (18)$$

If one assumes that $\epsilon(\xi)$ is an even-periodic function of period π, the function in the brackets in Eq. (18) may be represented in terms of a Fourier cosine series. Hence, (18) becomes

$$\frac{d^2u}{d\xi^2}+[\theta_0+2\sum_{n=1}^{\infty}\theta_n\cos 2n\xi]u=0 \qquad (19)$$

which is the canonical form of Hill's equation. The formal solutions of Eq. (19) can be obtained with the help of Floquet's theorem.[9,10] They are

$$u(\xi)=he_\beta^{(1,2)}(\xi)=e^{\pm i\beta\xi}\sum_{n=-\infty}^{\infty}b_n(\beta)e^{\pm 2ni\xi}, \qquad (20)$$

where the superscripts (1) and (2) refer, respectively, to the $+$ and the $-$ signs on the right-hand side of this equation. β are the roots of the following characteristic equation:

$$\sin^2(\tfrac{1}{2}\pi\beta)=\Delta(0)\sin^2(\tfrac{1}{2}\pi\sqrt{\theta_0}) \qquad (21)$$

in which $\Delta(0)$ is an infinite determinant whose elements are

$$\Delta(0)_{mm}=1,$$
$$\Delta(0)_{mn}=\theta_{m-n}/(\theta_0-4m^2) \quad (m\neq n). \qquad (22)$$

The coefficients $b_n(\beta)$ are determined from the recurrence relation

$$-(\beta+2n)^2b_n(\beta)+\sum_{m=-\infty}^{\infty}\theta_m b_{n-m}(\beta)=0. \qquad (23)$$

The most general expression for ψ that satisfies the radiation condition at $\rho=\infty$ is

$$\psi=\epsilon^{1/2}(\xi)\int_{-\infty}^{\infty}F(\beta)he_\beta^{(1)}(\xi)H_0^{(1)}(\gamma\rho)d\beta, \qquad (24)$$

where $F(\beta)$ is an arbitrary function of β. Substituting Eq. (24) into the required boundary condition (13) at $\rho=0$ gives,

$$\epsilon^{1/2}(z)\int_{-\infty}^{\infty}\left(\frac{4}{i}\right)F(\beta)he_\beta^{(1)}(\pi z/p)d\beta=\frac{q}{2\pi}e^{i\omega z/v}. \qquad (25)$$

Division of Eq. (25) by $(4/i)\epsilon^{1/2}(z)e^{i\omega z/v}$ yields

$$\int_{-\infty}^{\infty}F(\beta)he_\beta^{(1)}(\pi z/p)e^{-i\omega z/v}d\beta=\frac{iq}{8\pi}\epsilon^{-1/2}(z). \qquad (26)$$

Since the right-hand side of (26) is periodic with period p, the left-hand side must also be periodic with period p. Hence, substitution of $(z+p)$ for z in (26) yields

$$\int_{-\infty}^{\infty}F(\beta)he_\beta^{(1)}\left(\frac{\pi z}{p}\right)e^{i\beta\pi-i\omega z/v-i\omega p/v}d\beta=\frac{iq}{8\pi}\epsilon^{-1/2}(z). \qquad (27)$$

Subtraction of (26) from (27) yields

$$\int_{-\infty}^{\infty}F(\beta)he_\beta^{(1)}\left(\frac{\pi z}{p}\right)e^{-i\omega z/v}[e^{i\beta\pi-i\omega p/v}-1]d\beta=0. \qquad (28)$$

In order that linear independence of the Hill functions be preserved, the factor in square brackets in (28) must be zero. This requires that

$$\beta=(\omega p/v\pi)+2l, \qquad (29)$$

where l is any integer. Hence, Eq. (24) becomes

$$\psi=\sum_{l=-\infty}^{\infty}\epsilon^{1/2}(z)F_lH_0^{(1)}(\gamma_l^\alpha\rho)he_{\alpha+2l}^{(1)}(\pi z/p), \qquad (30)$$

where $\alpha=\omega p/v\pi$, and γ_l^α denotes the value of γ for which $\beta=\alpha+2l$. F_l can be determined from the relation

$$\frac{iq}{8\pi}\epsilon^{-1/2}(z)e^{i\omega z/v}=\sum_{l=-\infty}^{\infty}F_lhe_{\alpha+2l}^{(1)}\left(\frac{\pi z}{p}\right) \qquad (31)$$

with the help of the orthogonality properties of Hill's functions.[9,10] We have

$$F_l=\frac{\pi}{pC_l}\int_0^p\frac{iq}{8\pi}\epsilon^{-1/2}(z)he_{2l+\alpha}^{(1)*}\left(\frac{\pi z}{p}\right)e^{i\omega z/v}dz, \qquad (32)$$

where C_l is a normalization factor which is defined

$$\int_0^\pi|he_{2l+\alpha}^{(1)}(\xi)|^2d\xi=C_l. \qquad (33)$$

Equation (30) constitutes the formal solution for ψ.

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[9] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953).

[10] N. W. McLachlan, *Theory and Application of Mathieu Functions* (Oxford University Press, Oxford, 1951).

III. THE RADIATION SPECTRUM

We shall now obtain a formal expression for the energy radiated by the particle per unit length and per unit frequency interval. The total energy W radiated by the particle is equal to

$$W=\int_{t=-\infty}^{\infty} P dt, \tag{34}$$

where P denotes the radiated power. By Poynting's theorem,[8] the power radiated into a cylinder of radius ρ coaxial with the track of the particle and of length $2L$ is

$$P=-2\pi\rho\int_{z=-L}^{L} dz\{E_z(\rho,z,t)H_\phi(\rho,z,t)\}. \tag{35}$$

Expressing $E_z(\rho,z,t)$ and $H_\phi(\rho,z,t)$ in terms of their Fourier transforms and substituting (35) into (34) yields

$$W=-2\pi\rho\int_{t=-\infty}^{\infty} dt\int_{z=-L}^{L} dz\int_{\omega=-\infty}^{\infty} d\omega\int_{\omega'=-\infty}^{\infty} d\omega' \times[E_z(\rho,z,\omega)H_\phi(\rho,z,\omega')e^{it(\omega+\omega')}]. \tag{36}$$

Interchanging the order of integration gives

$$W=-4\pi^2\rho\int_{z=-L}^{L} dz\int_{\omega=-\infty}^{\infty} d\omega \times[\mathcal{E}_z(\rho,z,\omega)\mathcal{H}_\phi(\rho,z,-\omega)]. \tag{37}$$

Since $E_z(\rho,z,t)$ and $H_\varphi(\rho,z,t)$ are real, it follows that

$$W=-4\pi^2\rho\int_{z=-L}^{L} dz\int_{\omega=0}^{\infty} d\omega[\mathcal{E}_z(\rho,z,\omega)\mathcal{H}_\phi^*(\rho,z,\omega) +\mathcal{E}_z^*(\rho,z,\omega)\mathcal{H}_\phi(\rho,z,\omega)]. \tag{38}$$

The energy radiated per unit path length and per unit frequency interval $d^2W/dLd\omega$ is given by

$$\frac{d^2W}{dLd\omega}=-4\pi^2\rho\frac{1}{2L}\int_{z=-L}^{L} dz[\mathcal{E}_z(\rho,z,\omega)\mathcal{H}_\phi^*(\rho,z,\omega) +\mathcal{E}_z^*(\rho,z,\omega)\mathcal{H}_\phi(\rho,z,\omega)]. \tag{39}$$

Inserting the relations

$$\mathcal{E}_z(\rho,z,\omega)=\frac{1}{i\omega\epsilon\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right), \tag{40}$$

$$\mathcal{H}_\phi(\rho,z,\omega)=-\partial\psi/\partial\rho \tag{41}$$

into (39), where ψ is given in Eq. (30), and utilizing the large-argument asymptotic expression for the Hankel functions

$$\lim_{\chi\to\infty} H_0^{(1)}(\chi)=\left(\frac{2}{\pi\chi}\right)^{1/2} e^{i\chi-i(\pi/4)}, \tag{42}$$

one has

$$\frac{d^2W}{dLd\omega}=\frac{8\pi}{\omega L}\sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} F_lF_m^*(\gamma_l^\alpha)^{3/2} (\gamma_m^{\alpha*})^{1/2}\exp[i(\gamma_l^\alpha-\gamma_m^{\alpha*})\rho] \times\int_{-L}^{L} he_{\alpha+2l}\left(\frac{\pi z}{p}\right)he_{\alpha+2m}^*\left(\frac{\pi z}{p}\right)dz. \tag{43}$$

Utilizing the orthogonality properties of the Hill functions and carrying out the integral in Eq. (43) gives

$$\frac{d^2W}{dLd\omega}=\frac{16}{\omega}\sum_{l=-\infty}^{\infty}|F_l|^2C_l(\gamma_l^\alpha)^{3/2}(\gamma_l^{\alpha*})^{1/2} \times\exp[i(\gamma_l^\alpha-\gamma_l^{\alpha*})\rho] \tag{44}$$

If γ_l^α is imaginary, the corresponding term in the series becomes zero at large values of ρ, so one has finally

$$\frac{d^2W}{dLd\omega}=\frac{16}{\omega}\sum_l|F_l|^2C_l(\gamma_l^\alpha)^2, \tag{45}$$

where the summation is taken over those values of l for which γ_l^α is real. Equation (45) constitutes the formal solution for the spectral density of the radiation.

V. CLASSIFICATION OF THE RADIATION

Owing to the inhomogeneity of the dielectric medium, the radiation given off by a charged particle moving uniformly through this medium contains not only Čerenkov-type radiation but also transition-type radiation. In order to classify these types of radiation, let us first consider the degenerate case of the radiation in a homogeneous medium. For this degenerate case, the Hill functions, $he_\beta^{(1)}(\xi)$, reduce to the exponential functions, i.e.,

$$he_\beta^{(1)}(\xi)\to e^{i\beta\xi}, \tag{46}$$

where

$$\beta=\sqrt{\theta_0}=(\omega^2\mu_0\epsilon-\gamma^2)(p/\pi)^2. \tag{47}$$

The coefficient F_l given by Eq. (32) reduces to

$$F_l=\frac{1}{p}\left(\frac{iq}{8\pi}\right)\frac{1}{\sqrt{\epsilon}}\int_0^p e^{2li\pi z/p}dz \tag{48}$$

$$\begin{aligned}&=0 &&\text{for}\quad l\neq0\\ &=iq/8\pi\sqrt{\epsilon} &&\text{for}\quad l=0.\end{aligned} \tag{49}$$

Hence, the wave function ψ given by Eq. (30) becomes

$$\psi=\frac{iq}{8\pi}e^{i\omega z/v}H_0^{(1)}(\gamma_0^\alpha\rho), \tag{50}$$

where

$$(\gamma_0^\alpha)^2=\omega^2\mu_0\epsilon-\omega^2/v^2. \tag{51}$$

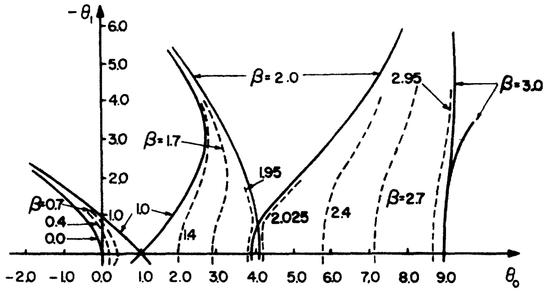

FIG. 1. Stability diagram for Hill's equation with $\delta=0.25$.

And, the energy radiated per unit path length and per unit frequency interval is then given by

$$\frac{d^2W}{dLd\omega}=\frac{q^2}{4\pi\epsilon\omega}(\gamma_0{}^{\alpha})^2 \quad \text{for} \quad \gamma_0{}^{\alpha} \text{ real}$$

$$=0 \qquad \text{for} \quad \gamma_0{}^{\alpha} \text{ imaginary}. \tag{52}$$

Therefore, only the $l=0$ term survives as the inhomogeneous medium degenerates to a homogeneous one. This $l=0$ term corresponds to the Čerenkov radiation term. It is defined that the $l=0$ term in Eq. (30) will be designated as the Čerenkov radiation term and the $l\neq 0$ terms will be designated as the transition radiation terms. Consequently, the $l=0$ term in Eq. (45) will be designated as the radiated energy spectrum due to the Čerenkov effect while the $l\neq 0$ terms in Eq. (45) will be designated as the radiated energy spectrum due to the transition effect.

The angles of emission of the radiation can be readily obtained from Eq. (30) and the expression for the Hill functions, Eq. (20). The radiation is composed of an infinite number of cylindrical waves emitted at angles

$$\theta_{ln}=\arctan\left[\frac{\gamma_l{}^{\alpha}}{(\beta_l+2n)\pi/p}\right] \tag{53}$$

with respect to the positive z-axis, where n and l are any integers and β_l is given by Eq. (29). It will be noted that for any value of l for which $\gamma_l{}^{\alpha}$ is real, radiation is emitted at an infinite number of angles ranging from zero, when n is very large and positive, to π, when n is very large and negative. One would

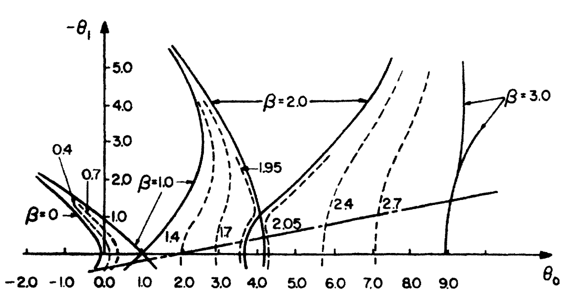

FIG. 2. Stability diagram for Hill's equation with $\delta=0.4$. The $\gamma_l{}^{\alpha}=0$ curve is given by the dot-dashed line.

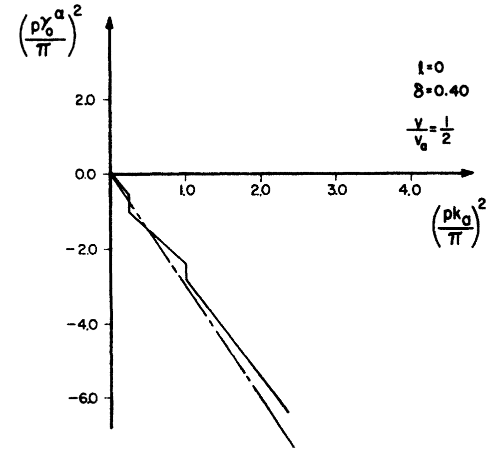

FIG. 3. Transverse separation constant $(p\gamma_0/\pi)^2$ as a function of frequency $(pk_a/\pi)^2$ for Čerenkov mode ($l=0$) well below the threshold velocity. The dot-dashed line indicates the curve for the homogeneous case. No radiation is emitted.

expect that the angle at which the dominant Čerenkov radiation will be emitted is the angle θ_{00}:

$$\theta_{00}=\arctan\left(\frac{\gamma_0{}^{\alpha}}{\beta_0\pi/p}\right),$$

$$=\arctan\left(\frac{v\gamma_0{}^{\alpha}}{\omega}\right), \tag{54}$$

since this is the angle at which the Čerenkov radiation is emitted in a homogeneous medium.

VI. AN EXAMPLE: THE DIELECTRIC PROFILE $\epsilon(z)=\epsilon_a[1-\delta\cos(2\pi z/p)]$

In order to obtain quantitative results from the formal solution, it is necessary to assume a specific dielectric profile. Let us consider the following dielectric variation:

$$\epsilon(z)=\epsilon_a[1-\delta\cos(2\pi z/p)], \tag{55}$$

where ϵ_a is the average value of the permittivity and δ gives the relative amplitude of the variation. Furthermore, $0\leq\delta<1$. Substituting Eq. (55) into Eq. (18) gives the coefficients θ_0 and θ_n in Eq. (19). They are

$$\theta_0=\left(\frac{p}{\pi}\right)^2(\omega^2\mu_0\epsilon_a-\gamma_l{}^{\alpha 2})-\left(\frac{1}{(1-\delta^2)^{1/2}}-1\right), \tag{56}$$

$$\theta_1=-\frac{\delta}{2}\left(\frac{p}{\pi}\right)^2\omega^2\mu_0\epsilon_a+\frac{4C_1{}^3-2C_1}{C_1{}^2-1}, \tag{57}$$

$$\theta_n=\frac{(3n+1)C_1{}^{n+2}-(3n-1)C_1{}^n}{C_1{}^2-1}, \quad (n\geq 2), \tag{58}$$

in which

$$C_1=(1/\delta)-(1/\delta)(1-\delta^2)^{1/2}. \tag{59}$$

For a given δ, depending upon the values of θ_0 and θ_1, Eq. (19) would yield solutions that are stable or unstable.[11] In order to satisfy Sommerfeld's radiation condition only the stable solutions are allowed. For a given value of δ, a diagram giving the stable and unstable regions can be constructed. Two such stability diagrams for $\delta=0.25$ and $\delta=0.40$ are given in Figs. 1 and 2. These diagrams provide information concerning the threshold velocity for each mode (i.e., for each l) and the angles of emission for various radiating components.

To illustrate how one may obtain this information from this diagram, let us consider the following procedures. Combining Eqs. (56) and (57) and eliminating $\omega^2\mu_0\epsilon_a p^2/\pi^2(k_a^2p^2/\pi^2)$ gives

$$-\theta_1=\frac{\delta}{2}\theta_0+\left(\frac{p\gamma_l^{\alpha}}{\pi}\right)^2\frac{\delta}{2}+\left[\frac{\delta}{2}\left(\frac{1}{(1-\delta^2)^{1/2}}-1\right)-\frac{4C_1^3-2C_1}{C_1^2-1}\right], \qquad (60)$$

where C_1 is given by Eq. (59). For **a** fixed value of δ expression (60) gives a family of straight lines corresponding to various values of $(p\gamma_l^{\alpha}/\pi)^2$. The line for which $(p\gamma_l^{\alpha}/\pi)^2=0$ is drawn in Fig. 2. Above this line, $(p\gamma_l^2/\pi)^2$ is positive corresponding to radiation, while below this line, it is negative corresponding to radially evanescent fields and therefore no radiation is emitted. For a given velocity of the charged particle (v/v_a), for a given frequency (k_ap/π) and for a given mode (l), one may compute β from Eq. (29). $-\theta_1$ can also be computed knowing (k_ap/π). The intersection between the lines $\beta=$constant and $-\theta_1=$constant provides the point from which one may obtain the value for θ_0.

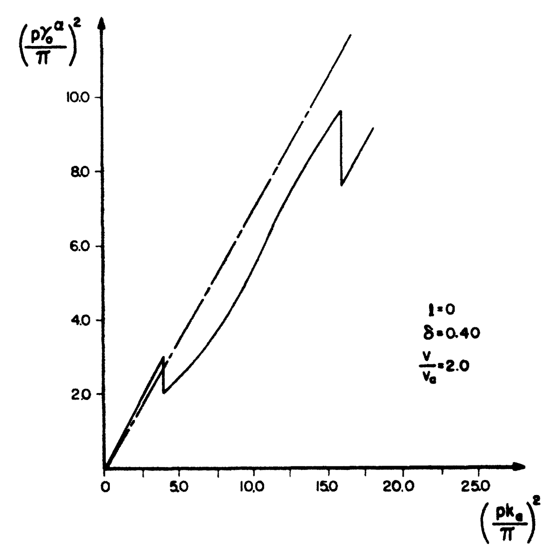

FIG. 4. Transverse separation constant $(p\gamma_0/\pi)^2$ as a function of frequency $(pk_a/\pi)^2$ for Čerenkov mode $(l=0)$ above the threshold velocity. Dot-dashed line indicates the curve for the homogeneous case. Radiation is emitted at all frequencies.

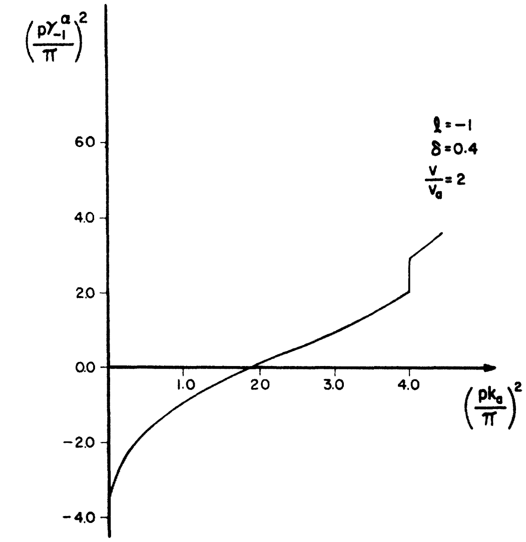

FIG. 5. Transverse separation constant $(p\gamma_{-1}/\pi)^2$ as a function of frequency $(pk_a/\pi)^2$ for $l=-1$ transition radiation mode above the Čerenkov threshold velocity. Radiation is emitted when $(pk_a/\pi)^2$ is greater than 1.9.

Hence the value of $(p\gamma_l^{\alpha}/\pi)^2$ can be computed from Eq. (56). The sign of $(p\gamma_l^{\alpha}/\pi)^2$ provides the information whether this particular mode radiates or not. As a specific example, let $\delta=0.4$, $pk_a/\pi=2.0$, $v/v_a=1.414$, $l=0$; we then have $\beta=1.414$, and $-\theta_1=0.4$. The intersection of $\beta=1.414$ and $-\theta_1=0.4$ yields the value for θ_0 which is 2.05. Using Eq. (56) we have $(p\gamma_l^{\alpha}/\pi)^2=2.87$, which is positive, hence radiation does occur for this particular case. Knowing $(p\gamma_l^{\alpha}/\pi)^2$, it is a simple matter to compute the emission angles from Eq. (53). They are for $n=0$, $\theta_{00}=50.2°$; $n=1$, $\theta_{01}=26.1°$; $n=2$, $\theta_{02}=17.4°$, etc.

It is noted from Eq. (29) that no matter how small the velocity of the charged particle is, β can always be adjusted using l to give modes which would radiate.

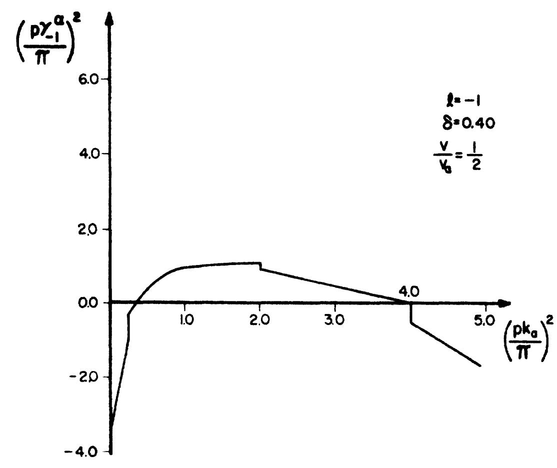

FIG. 6. Transverse separation constant $(p\gamma_{-1}/\pi)^2$ as a function of frequency $(pk_a/\pi)^2$ for $l=-1$ transition radiation mode below the Čerenkov threshold. Radiation is emitted only in the frequency range $0.35<(pk_a/\pi)^2<4.0$.

[11] Stable solutions refer to solutions which possess real values of β while the unstable solutions refer to solutions which possess complex values of β. [See Eq. (20).]

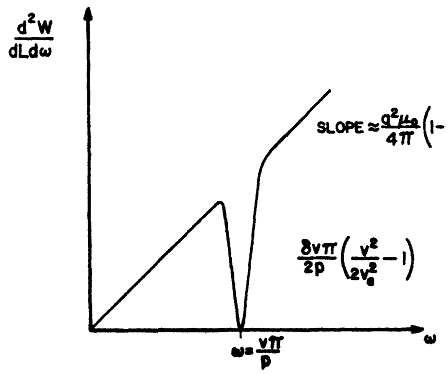

FIG. 7. Spectral density of Čerenkov radiation ($l=0$ mode) above the Čerenkov threshold velocity.

This is because l may take on positive as well as negative integer values. This conclusion is in contrast with the case of Čerenkov radiation in a homogeneous medium in which radiation will take place only if a certain threshold velocity is reached. This observation may be understood by the fact that transition radiation occurs for any velocities.

Figures 3 through 6 display the functional variation of $(p\gamma_l{}^\alpha/\pi)^2$ with respect to $(pk_a/\pi)^2$ for various values of l and the ratio v/v_a when $\delta=0.40$. The fact that $(p\gamma_l{}^\alpha/\pi)^2$ can be positive even if $v<v_a$ is apparent in Fig. 6.

VII. APPROXIMATE SOLUTIONS FOR SMALL δ

If the amplitude of the dielectric variation given by Eq. (55) is very small, i.e., if $\delta \ll 1$, it is possible to derive approximate analytic expressions for the scalar wave function ψ as well as the energy radiated per unit path length and per unit frequency interval $d^2W/dLd\omega$. Retaining terms correct to the first order in δ, we have

$$\begin{aligned}
&\theta_0 \approx (p/\pi)^2(\omega^2\mu\epsilon_a-\gamma_l{}^{\alpha 2}),\\
&\theta_1 \approx \delta[1-(p^2/2\pi^2)\omega^2\mu\epsilon_a],\\
&\theta_n \approx 0 \quad (n\geq 2),\\
&b_{-1}(\beta)/b_0(\beta) \approx -\theta_1/[\theta_0-(\beta-2)^2],\\
&b_1(\beta)/b_0(\beta) \approx -\theta_1/[\theta_0-(\beta+2)^2],\\
&b_n(\beta)/b_0(\beta) \approx 0 \quad (|n|\geq 2),\\
&C_l \approx \pi|b_0(\alpha+2l)|^2,\\
&\beta \approx \sqrt{\theta_0}.
\end{aligned} \tag{61}$$

The Hill's function then reduces to

$$he_\beta{}^{(1)}(\pi z/p) \approx e^{i\omega z/v}e^{i2l\pi z/p}[b_{-1}(\beta)e^{-2\pi iz/p}+b_0(\beta) +b_1(\beta)e^{2\pi iz/p}]. \tag{62}$$

Substituting Eq. (62) into Eq. (32) and carrying out the integration gives

$$\begin{aligned}
&F_{-1} \approx \frac{iq}{8\pi\epsilon_a}\left\{\frac{b_1(\alpha-2)}{[b_0(\alpha-2)]^2}+\frac{\delta}{4b_0(\alpha-2)}\right\},\\
&F_0 \approx \frac{iq}{8\pi\epsilon_a}\left\{\frac{1}{b_0(\alpha)}\right\},\\
&F_1 \approx \frac{iq}{8\pi\epsilon_a}\left\{\frac{b_{-1}(\alpha+2)}{[b_0(\alpha+4)]^2}+\frac{\delta}{4b_0(\alpha+2)}\right\},\\
&F_{\pm l} \approx 0, \quad l>1.
\end{aligned} \tag{63}$$

Inserting Eq. (63) into Eq. (30), one obtains the approximate expression for the scalar wave function:

$$\begin{aligned}
\psi \approx \frac{iq}{8\pi}e^{i\omega z/v}\Bigg(&H_0{}^{(1)}\left\{\left[k_a{}^2-\left(\frac{\omega}{v}-\frac{2\pi}{p}\right)^2\right]^{1/2}\rho\right\}e^{-2i\pi z/p}\frac{\delta}{4}\left[1+\frac{1-\frac{1}{2}(pk_a/\pi)^2}{(\omega p/v\pi)-1}\right]\\
&+H_0{}^{(1)}\left[\left(k_a{}^2-\frac{\omega^2}{v^2}\right)^{1/2}\rho\right]\left[1-e^{-2\pi iz/p}\frac{\delta}{4}\left(1+\frac{1-\frac{1}{2}(pk_a/\pi)^2}{(\omega p/v\pi)-1}\right)-e^{2\pi iz/p}\frac{\delta}{4}\left(1-\frac{1-\frac{1}{2}(pk_a/\pi)^2}{(\omega p/v\pi)+1}\right)\right]\\
&+H_0{}^{(1)}\left\{\left[k_a{}^2-\left(\frac{\omega}{v}+\frac{2\pi}{p}\right)^2\right]^{1/2}\rho\right\}e^{2i\pi z/p}\frac{\delta}{4}\left[1-\frac{1-\frac{1}{2}(pk_a/\pi)^2}{(\omega p/v\pi)+1}\right]\Bigg).
\end{aligned} \tag{64}$$

It is noted that as δ approaches zero, ψ reduces to the expression for the homogeneous case. As p approaches zero, ψ also reduces to the same expression for the homogeneous case. This is because as p approaches zero, even if δ is finite, the medium becomes macroscopically homogeneous with an average permittivity ϵ_a. Equation (64) also shows that to the first order in δ, the threshold velocity for the Čerenkov component ($l=0$) is $1/(\mu_0\epsilon_a)^{1/2}$ which is identical to the threshold velocity for the homogeneous case. The threshold conditions for the $l=\pm 1$ components of the field are, respectively,

$$k_a{}^2-[(\omega/v)\pm(2\pi/p)]^2=0. \tag{65}$$

It is also possible from the threshold conditions, Eq. (65), to obtain the frequency ranges for the $l=\pm 1$ component radiation. For $v>1/(\mu\epsilon_a)^{1/2}$, we have

$$\omega \geq 2\pi v/p[v(\mu\epsilon_a)^{1/2}+1] \tag{66}$$

for the $l=-1$ component and

$$\omega \geq 2\pi v/p[v(\mu\epsilon_a)^{1/2}-1] \tag{67}$$

for the $l=+1$ component. When $v<1/(\mu\epsilon_a)^{1/2}$, we have

$$2\pi v/p[v(\mu\epsilon_a)^{1/2}+1] \leq \omega \leq 2\pi v/p[1-v(\mu\epsilon_a)^{1/2}] \tag{68}$$

for the $l=-1$ component. Although radiation occurs below the usual Čerenkov threshold velocity, the spectral density of the radiation for the $l=-1$ component is quite small. (It is of the order of δ^2.) This fact will be shown below.

An approximate expression for the spectral density can be obtained with the help of Eqs. (61) through (64). Substituting these equations into expression (45) and carrying out the algebraic manipulation gives

$$\frac{d^2W}{dLd\omega}\approx\frac{q^2}{4\pi\epsilon_a\omega}\times\left\{(\gamma_{-1}^{\alpha})^2\frac{[f^2-2\delta f(\alpha-1)+\delta^2(\alpha-1)^2](\alpha-3)^2}{f^2[(\alpha-1)^2+(\alpha-3)^2]+16(\alpha-1)^2(\alpha-3)^2}\right.$$
$$+(\gamma_0^{\alpha})^2\frac{\left[16(\alpha^2-1)\left(1+\frac{3\delta^2}{8}\right)+\delta f\right](\alpha^2-1)}{f^2[(\alpha-1)^2+(\alpha+1)^2]+16(\alpha^2-1)^2}$$
$$\left.+(\gamma_1^{\alpha})^2\frac{[f^2+2\delta f(\alpha+1)+\delta^2(\alpha+1)^2](\alpha+3)^2}{f^2[(\alpha+1)^2+(\alpha+3)^2]+16(\alpha+1)^2(\alpha+3)^2}\right\} \quad (69)$$

in which γ_{-1}^{α}, γ_0^{α}, and γ_1^{α} must be real,

$$f=\frac{\delta}{2}\left(\frac{pk_a}{\pi}\right)^2-\delta$$

and

$$\omega p/v\pi=\alpha.$$

If any of these γ_l^{α} in Eq. (69) is imaginary, the term containing this γ_l^{α} should be set to zero.

The behavior of the $l=0$ (Čerenkov) term in Eq. (69) is very similar to that of the spectral density of Čerenkov radiation in a homogeneous medium, i.e.,

$$\left.\frac{d^2W}{dLd\omega}\right|_{l=0\text{ term}}\approx\frac{q^2\mu\omega}{4\pi}\left(1-\frac{v_a^2}{v^2}\right) \quad (70)$$

with $v_a=1/(\mu\epsilon_a)^{1/2}$, except in the vicinity of $\omega p/v\pi=1$, When $\omega p/v\pi=1$, this $l=0$ term is zero. A sketch of the spectral density for the $l=0$ term as a function of frequency is given in Fig. 7.

The $l=+1$ (transition) term in Eq. (69) is of the order of δ^2 over the whole frequency spectrum. The $l=-1$ (transition) term is also of the order of δ^2 over the whole frequency spectrum except around $\omega p/v\pi=1$. At this point, the $l=-1$ term is of the order of unity and has the value

$$\left.\frac{d^2W}{dLd\omega}\right|_{l=-1\text{ term at }\omega p/v\pi=1}\approx\frac{q^2\mu\omega_0}{4\pi}\left(1-\frac{v_a^2}{v^2}\right).$$

In other words, a peak occurs for the $l=-1$ term at a point where a null occurs for the $l=0$ term. A sketch of the spectral density for the $l=-1$ term as a function of frequency is also given in Fig. 7.

It can be shown from the approximate expression for the scalar wave function, Eq. (64), that at frequencies other than $\omega p/v\pi=1$, the phase for the $l=0$ Čerenkov radiation is stationary at the angle

$$\theta|_{l=0}\approx\arctan\left(\frac{v^2}{v_a^2}-1\right)^{1/2} \quad (71)$$

which is simply the Čerenkov radiation angle in a homogeneous medium. On the other hand, around the frequency $\omega p/v\pi=1$ the phase for the $l=-1$ transition radiation is stationary at the angle

$$\theta|_{l=-1}\approx\arctan\left[-\left(\frac{v^2}{v_a^2}-1\right)^{1/2}\right] \quad (72)$$

which is $(\pi-\theta_c)$, where θ_c is the Čerenkov angle. Thus the peak $l=-1$ transition radiation is emitted in the backward direction while the $l=0$ Čerenkov radiation is emitted in the forward direction.

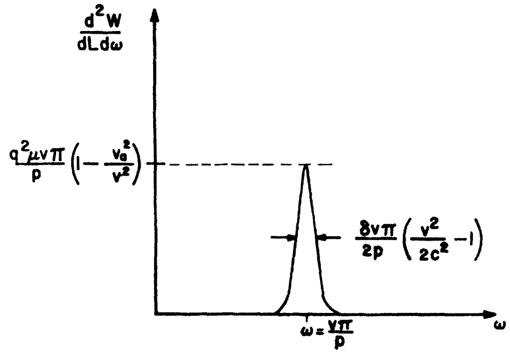

FIG. 8. Spectral density of $l=-1$ transition radiation mode above the Čerenkov threshold velocity.

One way of interpreting the above result qualitatively may be given in terms of the Bragg reflection condition. Above the Čerenkov threshold velocity, the charged particle emits radiation at an angle θ_c with respect to the direction of travel of the particle. The radiated wave then undergoes multiple reflections due to the striations of the medium. Bragg reflections occur when the following condition is satisfied:

$$n\lambda=2p\cos\theta_c, \quad (73)$$

where n are positive integers,

$$\lambda=2\pi v_a/\omega \quad (74)$$

and

$$\cos\theta_c=v_a/v. \quad (75)$$

Hence, the Bragg condition becomes

$$\omega p/v\pi=n. \quad (76)$$

When $n=1$, we have the condition described earlier. Therefore according to this argument one would expect a null in the Čerenkov radiation spectrum and a corresponding peak in the transition radiation spectrum when $\omega p/v\pi=1$. The Bragg-reflection analysis indicates that one might expect other peaks and nulls in the various spectra when $\omega p/v\pi$ takes on integer value other than unity.