Instabilities and Growing Waves

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(Received 4 December 1963; revised manuscript received 14 June 1965)

The character, stable or unstable, of a medium can be deduced from the behavior of an ideal model of a semi-infinite medium which is subjected to an excitation only at the boundary. A new analytic method is used to solve this particular problem. The results obtained show a connection between the character of the medium and certain properties of the dispersion equation, and agree with those derived from other methods by Sturrock, Jackson, Bers, and Briggs.

I. INTRODUCTION

A. The Physical Aspect of the Problem

A MEDIUM subjected to an excitation at its boundary or at any other place, or to an initial perturbation, can behave in different ways: If, in an infinite medium, an initial perturbation dies out, the medium is said to be *stable*. If it grows while propagating the medium is said to exhibit a *convective instability*. If it grows in time at every point, the medium is said to exhibit a *nonconvective instability*.

In a semi-infinite medium, an excitation at the boundary may generate either a nongrowing wave, or a wave growing in space, or may start a nonconvective instability which invades the whole medium and prevents the establishment of a permanent regime. This behavior is strongly connected with the behavior of an initial perturbation, and reflects the intrinsic character—stable or unstable—of the medium.

B. The Formulation of the Problem

Let us consider a medium subjected to a perturbation of a known type; we wish to predict whether there will be an instability, whether it will be convective or nonconvective, and also the character of the waves of the eventual permanent regime, i.e., whether they will be growing or decaying in space.

The main feature is the dispersion relation of the medium, as has been clearly understood by Sturrock. Nevertheless, the direct use of the dispersion relation leads to a well-known difficulty in the simplest case of a loss free medium: When solved for real values of ω (the angular frequency), it yields complex-conjugate values of the wave numbers k and k^*, which represent growing as well as decaying waves. In the absence of any criterion, a similar difficulty would appear when k is taken to be real.

There are two ways of approach: One can, as did Sturrock,[1] examine the kinematic properties of a wave packet satisfying the dispersion relation. One can also start with the differential equations of the problem and try to establish a connection between the form of the solution and certain properties of the dispersion relation. The classical Fourier-Laplace transform used by Jackson[2] is an illustration of such an approach.

In spite of the various mathematical precisions which have been added to it, in particular by Polovin,[3] Sturrock's elegant theory remains rather intuitive and it seems very difficult to recast it into a rigorous and quite general form. Moreover, this theory is difficult to extend to the case where the coefficients of the dispersion relation are complex quantities and where it is not possible to plot the dispersion curve $\omega(k)$ in real coordinates.

The Fourier-Laplace transform is more rigorous. If the Fourier inversion is made first, as Jackson did, it is possible to predict the existence of an instability, but it is difficult to obtain information on its convective or nonconvective nature. If one makes Laplace inversion first, as Bers and Briggs did recently,[4] and then deforms the contours of integration properly, one may obtain complete information on the behavior of a system submitted to a well-localized external source. But this procedure makes use of the causality principle, whereas in the following method the causality principle is automatically fulfilled thanks to a purely analytic condition, namely, a holomorphy condition, which eliminates erroneous solutions.

The basic physical idea of this paper is to deduce the character of the medium from its behavior when it is semi-infinite and when the only excitation is an oscillation at its boundary. We shall first investigate the behavior of this ideal model, either for $x\geq 0$ or for $x\leq 0$, using a purely analytic method. Then, using physical considerations, we shall deduce the character of the infinite medium when submitted to any kind of excitation.

II. BEHAVIOR OF A SEMI-INFINITE MEDIUM

A. Limitations of This Study

The medium considered in this section is one-dimensional, semi-infinite, and the only disturbance is an excitation at its boundary. Thus, the conclusions of this section may not be used directly for an actual plasma, which is subjected to other excitations, such as initial dis-

[1] P. A. Sturrock, Phys. Rev. **112**, 1488 (1958).

[2] J. D. Jackson, J. Nucl. Energy: Pt. C **1**, 171 (1960).

[3] R. V. Polovin, Zh. Tekhn. Fiz. **31**, 1220 (1961) [English transl.: Soviet Phys.—Tech. Phys. **6**, 889 (1962)].

[4] A. Bers and R. J. Briggs, MIT Research Laboratory of Electronics Report No. 71, 1963, p. 122 (unpublished).

turbance, thermal noise, etc., and is generally bounded in space, so that a convective instability is generally reflected and leads to a nonconvective one.

B. The Merits of the Double Laplace Transform

As Sturrock has pointed out, the growth or decay of a wave in space is essentially dependent on the stability or instability of the transient regime preceding it. As shown by the calculation and as is almost evident intuitively, a growing wave follows a convective growing transient, whereas a decaying wave is preceded by a decaying transient regime, while a nonconvective instability does not constitute a transient regime and is not followed by a permanent regime.

We have used the double Laplace transform in order to obtain simultaneously the permanent regime and the transient, so as to avoid the difficulties involved in the study of an isolated wave.

As we shall see, a double difficulty is usually encountered when a double Laplace transform is performed: First, there appear superfluous initial and boundary values, which cannot be distinguished *a priori* from the necessary ones; then the final transform obtained—which should be holomorphic—does not necessarily have this property. The solution of this double difficulty is the main feature of our method: By imposing the holomorphy condition, we eliminate certain classes of singularities and this automatically removes the indicated ambiguities and eliminates the superfluous initial and boundary values.

Schematically our calculation is as follows: The linear system of equations of the multi-fluid type is first transformed in x and t (space and time coordinates); one then obtains by reduction the transform of one of the functions $F(p,q)=A(p,q)/H(p,q)$; then the behavior of the singularities of the function $q(p)$ and the position of the cuts in the p plane yield the required information on the form of the solution $F(x,t)$ obtained by a double inversion.

C. Outline of the Method

(1) The following basic hypotheses are made: The medium is semi-infinite and one-dimensional. The functions defining the perturbation, such as the electric field $E(x,t)$, are continuous, at least piecewise; they are of exponential order in x and t, for an observer subjected to a uniform translation; this means that there are two fixed numbers p_0 and q_0 such that

$$|E(x,t)|\leq M\exp(q_0x+p_0t).$$

(2) The transforms are defined by the following relation:

$$E(p,q)=\int_0^\infty\int_0^\infty E(x,t)\exp(-qx-pt)\,dx\,dt,$$

where $E(x,t)$ is defined for $x\geq 0$ and $t\geq 0$. The integral defining $E(p,q)$ converges for the values $p=p_0$ and $q=q_0$, as defined above. It follows that it is *a fortiori* convergent in the associated half-planes:

$$\mathrm{Re}(p)>p_0,\quad \mathrm{Re}(q)>q_0.$$

It can also be shown that $E(p,q)$ *is analytic* in this region.

If $E(p,q)$ were not holomorphic in the associated half-planes $\mathrm{Re}(p)>p_0$, $\mathrm{Re}(q)>q_0$, this function would not be a Laplace transform of a function $E(x,t)$ piecewise continuous and of exponential order. This statement will play an essential role in what follows.

(3) By applying this transformation to the system of partial differential equations, one gets the transformed system in p and q, in which appear explicitly the initial conditions at $t=0$ and the boundary conditions at $x=0$. Among these there are, in general, superfluous values, introduced by the transformations; we will have to eliminate them.

(4) One can deduce from this system one of the transforms, e.g., $E(p,q)$, which is obtained in the following form:

$$E(p,q)=A(p,q)/H(p,q).$$

The function $H(p,q)$, when set equal to zero, gives the dispersion relation of the medium, expressed with the variables $p=-i\omega$ and $q=ik$. The function $A(p,q)$ contains all the initial and boundary values including the superfluous ones.

A and H are algebraic if the "multifluid" model is adopted. In order that $E(p,q)$ should be a Laplace transform, there should exist two associated half-planes, such that $\mathrm{Re}(p)>p_0$ and $\mathrm{Re}(q)>q_0$, in which the function $E(p,q)$ has no singularities.

(5) In fact, the zeros $q(p)$ of $H(p,q)$ may behave differently in the q plane when one increases $\mathrm{Re}(p)$. One or more of them may be such that when $\mathrm{Re}(p)$ is increasing, the real part of q, $\mathrm{Re}(q)$, increases indefinitely. Such poles of H, of course, do not remain on the left of the two limiting straight lines $\mathrm{Re}(p)=p_0$ and $\mathrm{Re}(q)=q_0$ in the associated p and q complex planes; thus they would violate the holomorphy condition imposed on $E(p,q)$ and we have to eliminate them.

To do so we must set the additional condition that these particular roots of the function $H(p,q)$ should also be roots of the function $A(p,q)$. This condition corresponds to a certain number of equations (of the functional type in general) which permit the elimination of the superfluous initial or boundary values.

Finally, $A(p,q)$ becomes, after division by the common factors, a new function $B(p,q)$ and we are then left only with singularities such that

$$\mathrm{Re}(p)>p_0 \text{ entails } \mathrm{Re}(q)<q_0.$$

In general, the required discrimination between the various roots of $H(p,q)$ leads to the provision of "cuts" in one of the associated planes, e.g., in the p plane, in order to make the various determinations $q(p)$ uniform.

(6) Now we have only to make the double Mellin-Fourier inversion, by a successive integration along the Bromwich contours of the q and p planes, such that all the singularities and cuts are on the left-hand side.

The following example, particularly simple, illustrates the method.

D. Illustrative Example

The problem of propagation in a Maxwellian plasma: macroscopic approximation.

We shall suppose all initial values at $t=0$ to be zero. We shall consider the following boundary values, at $x=0$:

the fluctuation of the electron density $n(0,t)$, which we shall take to be of the form $\exp(-i\omega t)$ for $t\geq 0$; and

the boundary value of the electric field $E(0,t)$, without knowing *a priori* if this latter value is superfluous. We assume $v(0,t)=0$ (specular reflection). All quantities are defined for $x\geq$ and $t\geq 0$.

(*a*) *The system of macroscopic equations.* The system of macroscopic equations, for electron oscillations, is as follows:

$$\partial n/\partial t+n_0(\partial v/\partial x)=0,$$

$$\partial E/\partial x+4\pi en=0,$$

$$\frac{e}{m}E+\frac{a^2}{n_0}\frac{\partial n}{\partial x}+\frac{\partial v}{\partial t}=0,$$

where n_0 is the average electron density, n is the fluctuating part of it and a is the thermal velocity, v being the particle's velocity. A double Laplace transform of this system yields

$$pn(p,q)+n_0qv(p,q)=0,$$

$$qE(p,q)-d(p)+4\pi en(p,q)=0,$$

$$(e/m)E(p,q)+(a^2/n_0)[qn(p,q)-c(p)]+pv(p,q)=0.$$

In this new system we have

$$c(p)=\int_0^\infty n(0,t)\exp(-pt)\,dt,$$

and

$$d(p)=\int_0^\infty E(0,t)\exp(-pt)\,dt.$$

We can thus obtain the transform of the fluctuating part of the electron density in the following form:

$$n(p,q)=\frac{(e/m)n_0d(p)-a^2qc(p)}{\omega_1^2+p^2-a^2q^2},$$

where $\omega_1^2=4\pi n_0e^2/m$. We recognize at once in the denominator (when set equal to zero) the dispersion relation of the plasma expressed with the variables $p=-i\omega$ and $q=ik$.

(*b*) *Elimination of the superfluous boundary or initial values.* Our transform $n(p,q)$ should be holomorphic in two associated half-planes, defined by $\mathrm{Re}(p)>p_0$ and $\mathrm{Re}(q)>q_0$, p_0 and q_0 being fixed. The roots of the denominator are

$$q=\pm a^{-1}(p^2+\omega_1^2)^{1/2}.$$

It is clear that one of these two roots cannot be considered as a singularity of the function $n(p,q)$ for it tends to p/a and its real part increases indefinitely, when that of p increases. This shows that one of our boundary values is superfluous. We then choose to eliminate $d(p)$.

Let us designate by $q^+=+a^{-1}(p^2+\omega_1^2)^{1/2}$ that branch of the function $q(p)$ which is real and positive when p is real and positive.

We shall render the function $q(p)$ uniform (i.e., such that to every point in the p plane there corresponds a single point in the q plane), by making a cut in the p plane, such as shown in Fig. 1, joining the critical points $\pm i\omega_1$, and located in R^- (i.e., to the left of the imaginary axis of the p plane). Thus, whatever the value of p in R^+ of the p plane, the corresponding branch q^+ is in R^+ of the q plane and the corresponding branch q^- is in R^- of the q plane (see Fig. 2).

Now the root q^+ of the denominator should also be a root of the numerator of $n(p,q)$ when p is in R^+. This means that we must have

$$(e/m)n_0d(p)-a^2q^+(p)c(p)=0.$$

From this we find

$$d(p)=(a^2m/n_0e)(1/a)(p^2+\omega_1^2)^{1/2}c(p).$$

This enables us to write the expression for $n(p,q)$ as follows:

$$n(p,q)=c(p)/[q+a^{-1}(p^2+\omega_1^2)^{1/2}],$$

where the square root has the sign specified above. Thus the function $n(p,q)$ is uniform in the whole q plane and in the p plane comprising the indicated cut.

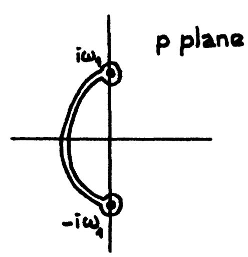

FIG. 1. Branch cut between critical points $\pm i\omega_1$, in the p plane.

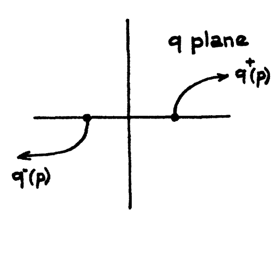

FIG. 2. The q plane with branches of $q(p)$.

If p is taken in R^+, the function is holomorphic when q is in R^+ and its only singularity is q^- in R^-.

Now if q is taken in R^+, the singularities of $n(p,q)$ are the poles of $c(p)$ and also the value of p such that $(p^2+\omega_1^2)^{1/2}=-aq$. The latter singularity is in R^-, if it exists, because of the sign we have chosen for the square root.

Thus $n(p,q)$ is holomorphic in p and q in the associated half-planes $\mathrm{Re}(p)>p_0$ and $\mathrm{Re}(q)>q_0$.

In the special case when the poles of $c(p)$ are imaginary, the function $n(p,q)$ is holomorphic in

$$\mathrm{Re}(p)>0\,, \qquad \mathrm{Re}(q)>0\,.$$

(*c*) *The inversion.* We shall obtain $n(x,t)$ by a double Mellin-Fourier inversion in the p and q planes, i.e., by a successive integration of $n(p,q)\exp(qx+pt)$ along the Bromwich contours of the p and q planes (Figs. 3 and 4).

$$n(x,t)=\frac{1}{(2i\pi)^2}\int_{B_2}\int_{B_1} dp\, dq\, n(p,q)\exp(qx+pt)\,,$$

where

$$n(p,q)=c(p)/[q+a^{-1}(p^2+\omega_1^2)^{1/2}]\,.$$

One should note that $n(p,q)$ is well defined on the Bromwich contours. Let us first integrate in the q plane. If p is given in R^+, the only singularity of $n(p,q)$ is the pole: $q=-a^{-1}(p^2+\omega_1^2)^{1/2}$ which is in R^-. Closing the Bromwich contour by the infinite semicircle in R^- and applying Cauchy's theorem, one obtains

$$\frac{1}{2i\pi}\int_{B_1} n(p,q)\exp qx\, dq=c(p)\exp[-(x/a)(p^2+\omega_1^2)^{1/2}]\,.$$

We shall now integrate in the p plane, assuming that $n(0,t)$ is of the form $\exp(-i\omega_0 t)$, i.e., that $c(p)$ is of the form $(p+i\omega_0)^{-1}$. We thus get

$$n(x,t)=\frac{1}{2i\pi}\int_{\alpha-i\infty}^{\alpha+i\infty} dp\exp pt\frac{\exp[-(x/a)(p^2+\omega_1^2)^{1/2}]}{p+i\omega_0}\,.$$

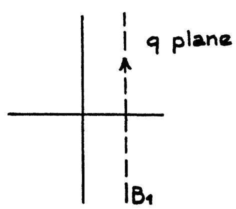

FIG. 3. The Bromwich contour in the q plane.

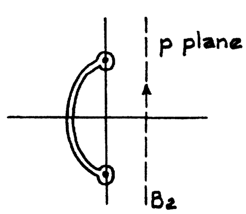

FIG. 4. The Bromwich contour in the p plane.

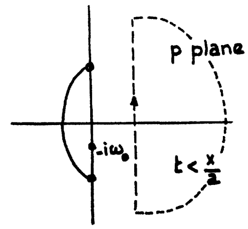

FIG. 5. Closing the integration contour in the p plane for $t<x/a$.

The integration technique will depend on the values of t considered as compared with x/a. Indeed, when p tends to infinity, the integrand tends to

$$p^{-1}\exp[p(t-(x/a))]\,.$$

Therefore, when $t<x/a$, the Bromwich contour should be closed by an infinite semicircle to the right, i.e., in R^+ (see Fig. 5); whereas for $t>x/a$, it should be closed by an infinite semicircle to the left (the integral must vanish on the semicircles) and in this case it should be joined to the contour of the cut, as shown in Fig. 6. For $t>x/a$, one has, by applying Cauchy's theorem:

$$n(x,t)=\exp[(-i\omega_0 t)-(x/a)(\omega_1^2-\omega_0^2)^{1/2}]-J(x,t)\,,$$

where $J(x,t)$ is the contribution due to the integration along the contour of the cut.

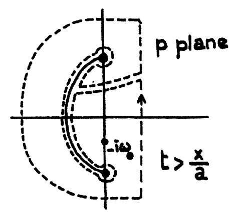

FIG. 6. Closing the integration contour in the p plane for $t>x/a$.

(*d*) *$J(x,t)$ a transient.* We shall now show that $J(x,t)$ represents a transient, i.e., that $J(x,t)\to 0$ when $t\to\infty$. One can see on Fig. 7 that

$$2i\pi J(x,t)=\int_{\Gamma_2} dp \times\exp pt\frac{\exp[-(x/a)(p^2+\omega_1^2)^{1/2}]}{p+i\omega_0}+\int_{\gamma_2}+\int_{\Gamma_1}+\int_{\gamma_1}\,.$$

The integrals along the paths γ_1 and γ_2 are equal to

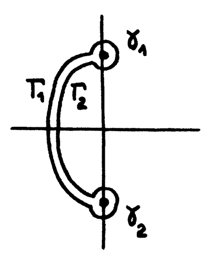

FIG. 7. Contributions to the integration along the cut.

zero according to Jordan's lemma. Indeed

$$\frac{p-i\omega_1}{p+i\omega_0}\exp\left(-\frac{x}{a}(p^2+\omega_1{}^2)^{1/2}\right)\exp pt\to 0$$

when $p\to i\omega_1$. Let us now associate Γ_1 and Γ_2, noting that the square root has opposite signs on the two sides of the cut; we get

$$2i\pi J(x,t)=\int_\Gamma dp\exp pt\frac{\exp[(x/a)(p^2+\omega_1{}^2)^{1/2}]-\exp[-(x/a)(p^2+\omega_1{}^2)^{1/2}]}{p+i\omega_0}.$$

Let us introduce a function

$$U(p,x)=(p+i\omega_0)^{-1}\{\exp[(x/a)(p^2+\omega_1{}^2)^{1/2}]-\exp[-(x/a)(p^2+\omega_1{}^2)^{1/2}]\}.$$

This function is continuous on Γ_1, bounded and is zero at the limits $p=\pm i\omega_1$. By an integration by parts, we obtain for $2i\pi J(x,t)$ the following expression:

$$2i\pi J(x,t)=\left[U(p,x)\frac{\exp pt}{t}\right]_{-i\omega_1}^{+i\omega_1}-\int_\Gamma dp\frac{\exp pt}{t}\frac{dU}{dp},$$

where the first term on the right-hand side is nil. The integral along Γ_1 is absolutely convergent, for dU/dp behaves like $(p\pm i\omega_1)^{-1/2}$ when p tends to $\mp i\omega_1$. Thus

$$2i\pi tJ(x,t)=-\int_{\Gamma_1}dp\exp pt\frac{dU}{dp}(p,x).$$

But on Γ_1, we have $|\exp pt|\leq 1$, for the cut is located in R^-. We thus can write, using the rule of the moduli majoration:

$$2i\pi t|J(x,t)|<\int_{\Gamma_1}\left|\frac{dU}{dp}\right|ds$$

where s is the line abscissa of p. This latter integral is independent of t. We thus can conclude that:

$$J(x,t)\to 0,\quad\text{when}\quad t\to\infty.$$

(*e*) *Conclusion*. We see that the wave front of the transient moves with the velocity a ($x=at$) equal to the thermal velocity of the plasma. Beyond this front ($x>at$), the medium is not disturbed. Behind it, the transient is followed by an evanescent wave (see Fig. 8) when $\omega_0<\omega_1$.

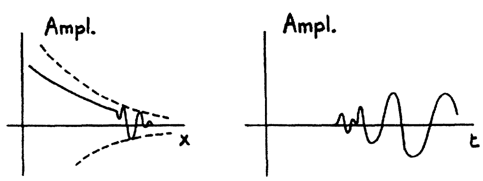

FIG. 8. (a) Wave front of transient followed by evanescent wave, at a given instant of time. (b) The wave as a function of time at a given point in space.

A similar study can be done for more complex problems, where several components (thermal electrons, beams, ions, etc) are present. But in that case the degree of the dispersion equation is higher than 2 and the expression of the roots is cumbersome or even impossible. It thus appears interesting to predict the form of the solution without actually evaluating it but by a mere inspection of the dispersion equation. This is the object of the following sections where a general study of the problem is made.

We shall follow a method similar to the one used in the preceding case, which is rather simple, and start with some preliminary remarks.

E. Preliminary Remarks

The case we have just analyzed suggests the following remarks:

(a) There exists a continuous solution in x for $0\leq x\leq at$; in other words, there exists a permanent regime. This property is due to the fact that we were able to provide a cut in R^-, which has led us to *a convective transient*, disappearing at every point after a finite time. If one of the critical points were located in R^+, the functions $J(x,t)$ would have been an increasing function of t, thus giving rise to a nonconvective instability, preventing the establishment of a permanent regime, and the solution would then have been discontinuous in space at $x=0$.

(b) We also can see why this permanent regime corresponds to a solution decreasing in time, i.e., why only the negative sign of the square root appears in the solution: This is because, when $p=-i\omega_0$, only the root q^- located in R^- can be retained as a singularity of the transform $n(p,q)$.

This root satisfies the following condition: There exist two numbers p_0 and q_0 such that $\mathrm{Re}(p)>p_0$ entails $\mathrm{Re}(q)<q_0$. If the root satisfying this condition were located in R^+ for $p=-i\omega_0$, we would have found a *growing solution*.

By considering step by step the different stages of this method and examining systematically the various eventualities, we shall find a criterion enabling us to predict the behavior of the medium by a mere inspection of the dispersion relation.

F. Semi-Infinite Medium in $x \geqslant 0$ Form of the Solution

(a) The system of differential equations to be solved is obtained as follows:

The medium is subdivided into a certain number of "components" 1, 2, $\cdots n$, adequately chosen (e.g., an isotropic plasma + a beam, etc.—see Fig. 9).

One then writes down the linearized macroscopic equations for each of the components. These equations are coupled by the Poisson equation. We thus obtain n equations of continuity, n equations of motion, and the Poisson equation.

(b) The double Laplace transform then gives the transform $F(p,q)$ of the required function $F(x,t)$ in the form

$$F(p,q)=A(p,q)/H(p,q).$$

A and H are algebraic in p and q. A usually contains the *superfluous* initial or boundary values, introduced by the transforms.

The singularities of $F(p,q)$ should be such that there exist two real fixed numbers p_0 and q_0 ensuring the holomorphy of F in the associated half-planes:

$$\mathrm{Re}(p)>p_0;\ \mathrm{Re}(q)>q_0. \tag{1}$$

These singularities are of two kinds: some of them are fixed; they are due to factors of the form $[g(p)]^{-1}$ or $[h(q)]^{-1}$. The others are "mobile"; they are due to factors of the form $[\xi(p,q)]^{-1}$. The boundary condition $\exp-i\omega_0 t$, whose transform appearing in A is $\tau(p)=(p+i\omega_0)^{-1}$, gives a fixed singularity $p=-i\omega_0$. One can show that $A(p,q)$ has no other singularities. On the other hand, the zeros of $H(p,q)$ are mobile singularities $q(p)$. The latter can violate the holomorphy condition; they must be such that there should be two real fixed numbers p_0 and q_0 satisfying the requirement

$$\mathrm{Re}(p)>p_0 \text{ entails } \mathrm{Re}(q)<q_0. \tag{2}$$

(c) Thus we have to find those roots $q(p)$ of $H(p,q)=0$, which do not satisfy the holomorphy condition of F. These roots are those whose real part increases indefinitely when the real part of p is increasing (see Figs. 10 and 11).

If all the roots $q(p)$ satisfy the condition (2), the function F is of course holomorphic in the associated half-planes and (1) is satisfied. Moreover, $F(p,q)$ is then uniform in the whole p plane, since the rotation of p around a critical point of the function $q(p)$, results in a simple permutation of the zeros $q_k(p)$ of the polynomial $H(p,q)=[q-q_1(p)][q-q_2(p)]\cdots[q-q_n(p)]$. Now, if certain roots $q(p)$ of H do not satisfy the condition (2),

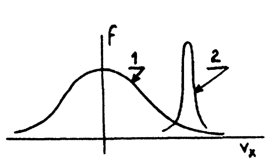

Fig. 9. Velocity distribution for two plasma components.

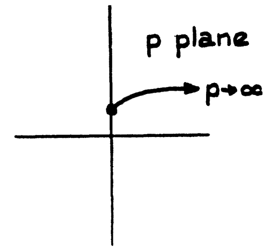

Fig. 10. A path with indefinitely increasing real part in the p plane.

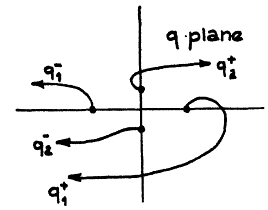

Fig. 11. Roots of different behavior in the q plane.

one must eliminate them. Such is the case of the root q_2^+ of Fig. 11, whose real part increases indefinitely when that of p increases. This elimination of certain singularities $q_\alpha(p)$ entails the fact that $F(p,q)$ is no longer a symmetric combination of the roots $q(p)$ of the equation $H(p,q)=0$, since certain factors of the form $[q-q_\alpha(p)]$ disappear in the product $H(p,q)=(q-q_1)(q-q_2)\cdots$.

In this case, we must prevent the roots which have been retained, $q_n(p)$, from interchanging with the rejected ones $q_\alpha(p)$. To do so we will have to provide appropriate cuts in the p plane (see Figs. 12 and 13), joining the critical points Ω around which certain retained roots $q_n(p)$ interchange with the rejected ones $q_\alpha(p)$. The cuts should be drawn in R^- wherever possible, i.e., where there are no critical points in R^+. In Fig. 12 we have a case in which the cuts can be provided in R^-. Figure 13 corresponds to the case in which the cuts cannot be made in R^-.

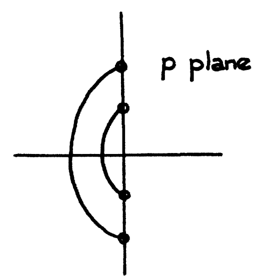

Fig. 12. p plane with cuts in the left half-plane.

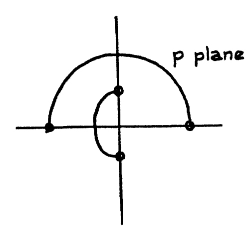

Fig. 13. p plane in the case when cuts cannot all be made in the left half-plane.

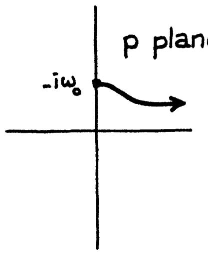

FIG. 14. p plane with p starting from $p=-i\omega_0$.

(d) Next we shall express the fact that the roots $q_\alpha(p)$ which do not satisfy the holomorphy condition, must nullify the numerator $A(p,q)$. These roots will not be poles of $F(p,q)$, and the superfluous initial or boundary values will be eliminated through the relations

$$A[p,q_\alpha(p)]=0.$$

The function $A(p,q)$ can then be written in the following form:

$$A(p,q)=A^1[p,q,q_\alpha(p)]\prod_\alpha [q-q_\alpha(p)],$$

so that the singularities $q_\alpha(p)$ of $F(p,q)$ will disappear:

$$F(p,q)=\frac{A(p,q)}{H(p,q)}$$
$$=\frac{A^1[p,q,q_\alpha(p)]\prod_\alpha [q-q_\alpha(p)]}{\prod_n [q-q_n(p)]\prod_\alpha [q-q_\alpha(p)]}=\frac{A^1[p,q,q_\alpha]}{\prod [q-q_n]}.$$

Let us note again that the new numerator A^1 is in general a function of the roots q_α and that it is uniform in the p plane, thanks to the cuts previously provided in that plane.

Now, when p assumes a pure imaginary value, let us say $-i\omega_0$ (Fig. 14) we have several possible cases of localization of the poles $q_n(p)$ of the function $F(p,q)$. Some of these may be purely imaginary (Fig. 15); some others may be in R^- or in R^+.

(e) Now we shall make the Mellin-Fourier inversion in the q plane, by an integration along the Bromwich contour in the q plane having to the left the poles $q(p)$ of F (Fig. 16). Suppose the function $F(q)$ is of the order of q^{-n} (with $n>0$ when $q\to\infty$ in R^+), we have

$$F(p,x)=\frac{1}{2i\pi}\int_{\mathrm{Br}} F(p,q)\exp qx\,dq$$
$$=\sum \text{Res. at the poles } q(p),$$

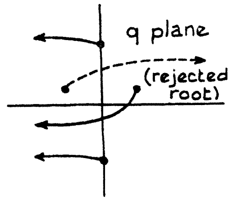

FIG. 15. q plane showing rejected root.

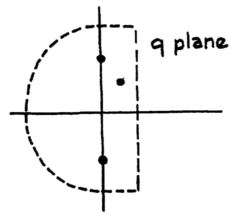

FIG. 16. q plane with integration contour, and poles.

or

$$F(p,x)=\frac{1}{2i\pi}\int_{\mathrm{Br}}\frac{A(p,q)}{H(p,q)}\exp qx\,dq$$
$$=\sum_n \frac{A(p,q_n)}{(\partial H/\partial q)(p,q_n)}\exp q_n x,$$

where A contains the factor $g(p)=(p+i\omega_0)^{-1}$, corresponding to the harmonic boundary value, whose frequency is ω_0, so that

$$F(p,x)=g(p)\textstyle\sum_n b(p,q_n)\exp q_n x,$$

where b has no poles in p, outside the cuts, since $\partial H/\partial p=0$ only at the critical points $q_n(p)$, these being the retained roots. The only pole p of $F(p,x)$ is that of the function $g(p)$, i.e., $p=-i\omega_0$.

(f) The Mellin-Fourier inversion in the p plane is somewhat more involved owing to the cuts we had to provide in that plane. We join the Bromwich contour to the cuts, thus establishing a path of integration to the left, of which $F(p,x)$ is holomorphic except at the pole $p=-i\omega_0$ (see Fig. 17). Thus

$$F(x,t)=\frac{1}{2i\pi}\int_{\mathrm{Br}} F(p,x)\exp pt\,dp$$
$$=\frac{1}{2i\pi}\int_{\mathrm{Br}}\frac{b(p,q_n)}{p+i\omega_0}\exp(q_n x+pt)\,dp,$$

$$F(x,t)=\sum b_n\exp(q_n x-i\omega_0 t)-J(x,t),$$

where

$$b_n=b[-i\omega_0,\, q_n(-i\omega_0)],$$

q_n are the poles of $F(p,q)$ when p has the value $p=-i\omega_0$, and $J(x,t)$ is the contribution of the cuts.

This is the solution of the problem in the general case. Now the behavior of this solution depends on the localization of the q_n and of the behavior of $J(x,t)$. In the following section we shall examine the various cases to be considered.

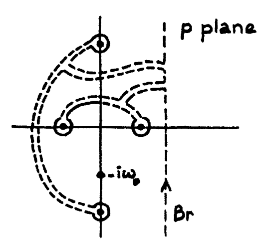

FIG. 17. Bromwich integration contour in p plane joined to cuts, and one pole.

G. The Various Cases to Be Considered

We have seen that the solution of the problem has the following general form:

$$F(x,t)=\sum_n b_n \exp(q_n x-i\omega_0 t)-J(x,t).$$

The first term may represent a permanent regime, whose frequency is ω_0 and which may be of growing or decreasing amplitude in space depending on the localization of the q_n. The second term represents a nonpermanent state which can be either a transient if $J(x,t)\to 0$ when $t\to\infty$ or a nonconvective instability if $J(x,t)\to\infty$ when $t\to\infty$.

Thus we have to examine two points in order to ascertain the form of $F(x,t)$: the localization of the $q_n(p)$ when $p=-i\omega_0$, and the behavior of $J(x,t)$ when $t\to\infty$.

(*a*) *Localization of the* q_n. Remembering that the q_n are those roots $q(p)$ of the equation $H(p,q)=0$ which satisfy the condition $\mathrm{Re}(p)>p_0\to\mathrm{Re}(q)<q_0$, we can conclude that if among these roots there is at least one in R^+ when $p=-i\omega_0$, the permanent part of the solution is growing in space. It is not growing in space if none of the q_n are in R^+, and is even decreasing in space if all the q_n are in R^-.

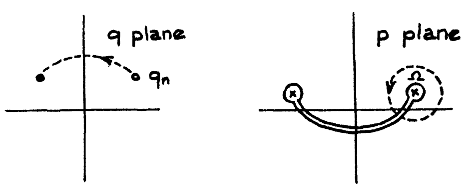

FIG. 18. Near the critical point Ω, the contribution $J(t)$ along the path is an increasing function of time.

(*b*) *Behavior of* $J(x,t)$. Remembering that $J(x,t)$ is the contribution due to the integration along the contour enclosing the cuts, we can see that its form depends on the localization of the critical points. The following situations may be encountered:

There is no critical point of $q(p)$ in R^+; it can then be shown that $J(x,t)\to 0$ when $t\to\infty$ for $|\exp pt|\leq 1$ on the cuts, so that the product $tJ(x,t)$ has an upper limit independent of t; in this case J represents *a convective transient regime*.

At least one of the critical points is in R^+. Because of the fact that at this point the roots q_n which have been retained interchange with those rejected q_α, the integrand $\sum_n [b(p,q_n)/(p+i\omega_0)]\exp(q_n x+pt)$ has different values on opposite sides of the cut, and since $\mathrm{Re}(p)>0$ it is an increasing function of t along a certain portion of the cut (cf. Fig. 18). The contribution of the cut is thus an increasing function of the time and $J(x,t)$ represents in this case a *nonconvective instability*.

(*c*) *Conclusion*. Table I presents in a compact form the various cases to be considered and the corresponding behavior of a semi-infinite medium in $x\geq 0$, when the only perturbation is an excitation at its boundary.

TABLE I. Stability criteria for waves moving to the right: q_n satisfying $R(p)>p_0\to R(q_n)<q_0$.

		Localization of the retained roots q_n when $p=-i\omega_0$: No q_n in R^+	Some q_n in R^+
Localization of the critical points in the p plane	None in R^+	The medium is stable—no growing waves	Convective instability, growing waves
	Some in R^+	nonconvective instability—no permanent regime	

H. Semi-Infinite Medium in $x\leqslant 0$

The above study cannot predict possible convective instabilities going to the left. For it is necessary to make a similar study for a semi-infinite medium in $x\leq 0$. The calculation is identical to that of Sec. IIG, but the holomorphy condition (2) is no longer the same. The retained roots $q_m(p)$ are such that

$$\mathrm{Re}(p)>p_0 \text{ entails } \mathrm{Re}(q)>q_0. \tag{3}$$

Noting that the waves growing to the left are associated with q_m in R^-, we obtain Table II, showing the behavior of a semi-infinite medium in $x\leq 0$ when the only perturbation is an excitation at its boundary.

III. BEHAVIOR OF AN INFINITE MEDIUM

A. The Basic Physical Idea

Section II describes the behavior of a semi-infinite medium in rather restrictive conditions. But, as we will see, this behavior reflects the intrinsic character of the medium. Indeed, if the medium is stable, that is, if perturbations of any kind within it die out, then an oscillation at its boundary generates a wave which does not grow in space, neither for $x>0$ nor for $x<0$. If an oscillation at the boundary $x=0$ generates a wave growing in space $x>0$ (the medium being semi-infinite in $x\geq 0$), then the medium exhibits a convective instability going to the right; in other words, any kind of perturba- such as initial perturbation, thermal noise, etc., will generate a disturbance which grows while propagating

TABLE II. Stability criteria for waves moving to the left: q_m satisfying $R(p)>p_0\to R(q)>q_0$.

		Localization of the retained roots q_m when $p=-i\omega_0$: No q_m in R^-	Some q_m in R^-
Localization of the critical points in the p plane	None in R^+	The medium is stable—no growing waves	Convective instability, growing waves
	Some in R^+	Nonconvective instability—no permanent regime	

TABLE III. General stability criteria. Ω = branch point of $k(\omega)$ where some k_n interchange with some k_m; k_n = roots satisfying $\mathrm{Im}(\omega)>a \rightarrow \mathrm{Im}(k)>b$; k_m = roots satisfying $\mathrm{Im}(\omega)>c \rightarrow \mathrm{Im}(k)<d$. ($a$, b, c, d, being given fixed numbers).

		Localization of the roots k_n and k_m when ω is real			
		No k_n in Z^- No k_m in Z^+	Some k_n in Z^- No k_m in Z^+	Some k_m in Z^+ No k_n in Z^-	Some k_n in Z^- Some k_m in Z^+
Localization of the critical points Ω	No Ω in Z^+	Stability No growing waves	Convective instability going to the right Growing waves in $x>0$	Convective instability to the left Growing waves in $x<0$	Convective instabilities going in both directions Growing waves in $x<0$ as well as in $x>0$
	Some Ω in Z^+	Nonconvective instability—no permanent regime			

to the right. If an oscillation at the boundary $x=0$ generates a wave growing in space $x<0$ (the medium being semi-infinite in $x\leq 0$), then the medium exhibits a convective instability going to the left. If the medium exhibits a nonconvective instability, in other words if perturbations of any kind such as initial perturbation, thermal noise, etc., generate a disturbance which grows in time at every point, then an excitation at the boundary will start this same instability which will invade progressively the whole medium, growing indefinitely at every place, and will prevent the establishment of a permanent regime, either for $x>0$ or for $x<0$. This connection between the behavior of a medium when it is excited at its boundary or by a source within it, has already been implicitly considered as evident by several authors and has been clearly stated by Sturrock.[1]

Thus, the results we have obtained with our special semi-infinite model can readily be used to establish a general criterion of the intrinsic character of the medium, as shown in the next section. This criterion establishes a connection between the stable or unstable, convective or nonconvective nature of a perturbation and the damped or amplified form of the progagating waves, on the one hand, and certain topological properties of the dispersion equation on the other, involving the location of the critical points and the behavior of the roots in the complex plane.

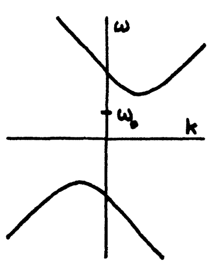

FIG. 19. Dispersion curve $\omega(k)$, case III.

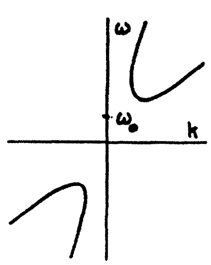

FIG. 20. Dispersion curve $\omega(k)$, case II.

B. General Form of the Criterion

In order to formulate our criterion and to compare it with that suggested by Sturrock, it is convenient to return to the classical parameters ω and k, setting

$$p=-i\omega \quad \text{and} \quad q=ik.$$

Thus, to the q in R^+ (right-hand-side half-plane) will correspond the k in Z^- (lower half-plane); and to the critical points in the R^+ part of the p plane correspond critical points in the Z^+ part of the ω plane.

Let us consider a medium whose properties are described by an algebraic dispersion relation $D(\omega,k)=0$. We designate by $k_n(\omega)$ the roots satisfying the condition

$$\mathrm{Im}(\omega)>a \text{ entails } \mathrm{Im}(k_n)>b, \tag{2'}$$

and by $k_m(\omega)$ the roots satisfying the condition

$$\mathrm{Im}(\omega)>c \text{ entails } \mathrm{Im}(k_m)<d \tag{3'}$$

(where a, b, c, d, are given fixed numbers). Then, if the Ω points are the critical points of the function $k(\omega)$, where some k_n interchanges with some k_m:

(a) The medium is stable and cannot support growing waves in space, if at the same time none of the $k_n(\omega)$ is in Z^- (the lower half-plane) for real ω, none of the $k_m(\omega)$ is in Z^+ for real ω, and no critical points Ω are in Z^+.

(b) The medium is subject to convective instabilities and can support growing waves in space, if at the same time some of the $k_n(\omega)$ are in Z^- for real ω, or some of the $k_m(\omega)$ are in Z^+ for real ω, and no critical points Ω are in Z^+.

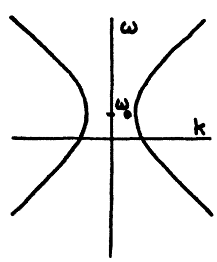

FIG. 21. Dispersion curve $\omega(k)$, case I.

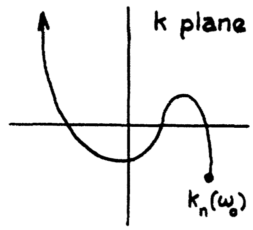

FIG. 22. Root k_n initially (for $\omega=\omega_0$) in the lower half-plane.

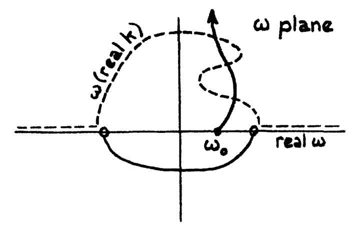

FIG. 24. Path in ω plane corresponding to real k.

(c) The medium is subject to nonconvective instabilities and cannot support a permanent regime if certain critical points Ω are in Z^+.

Table III presents the connection between the character of the medium and the topological properties of the dispersion equation in a compact form.

C. Simplified Form. Connection with Sturrock's Criterion

When the dispersion equation has real coefficients, one can plot the dispersion curve $\omega(k)$ in real coordinates and examine the influence of the different pairs of branches (Figs. 19, 20, 21).

In case I (Fig. 21), it can be shown that the critical points of $k(\omega)$ are complex. One of them is in R^+. One of the $k(\omega)$ does not satisfy the holomorphy condition (2)′, for it is negative when $\omega\rightarrow\infty$ this $k(\omega)$ interchanges with the other $k_n(\omega)$ which satisfies the condition (2′). We thus have here a nonconvective instability.

In the two other cases the critical points are real. Consequently we must examine the behavior of the $k_n(\omega)$. We note that the conditions of IIIB(b),

$$k_n(\omega_0) \text{ in } Z^-, \quad \mathrm{Im}(\omega)>a\rightarrow \mathrm{Im}(k_n)>b, \tag{4}$$

as well as

$$k_m(\omega_0) \text{ in } Z^+, \quad \mathrm{Im}(\omega)>c\rightarrow \mathrm{Im}(k_m)<d, \tag{5}$$

can be expressed quite simply; indeed, the condition (4) means that the root k_n (Fig. 22) initially in Z^-, when $\omega=\omega_0$ is real (Fig. 23), passes into Z^+ when ω follows an arbitrary path in Z^+, starting from ω_0, avoiding the cut and leaving behind it the real axis. In the same way, condition (5) means that the root k_m initially in Z^+ passes into Z^-. Consequently, conditions (4) or (5) entail that all paths originating in ω_0 (Fig. 23) and proceeding upward in Z^+, must contain an odd number of points ω (real k). This means that the path ω (real k) plotted in the ω plane, and passing through the critical points, must join these critical points by a continuous line (Fig. 24).

This happens only in cases I and II (Figs. 21 and 20). Thus case II (Fig. 20) corresponds to a convective instability, and a wave growing in space can exist. Case III, on the other hand, corresponds to a stable medium, which cannot support growing waves.

This is exactly Sturrock's criterion,[1] as proposed for the case when the dispersion relation has real coefficients. Nevertheless if one applies the latter criterion it seems necessary to ascertain that no complex critical points have escaped the analysis, which is not self-evident.

D. Connection with Bers and Briggs' Criterion

Recently, Bers and Briggs[4] have given an original method which solves elegantly the problem of instabilities and growing waves in an infinite medium submitted to a well-localized source of excitations. They use the Fourier-Laplace transform, and they apply the principle of causality. This procedure appears to be equivalent to the use of our holomorphy condition. Considering that their contour F is a boundary between the roots of the class k_n and that of the class k_m, and that their "true singularities" (resulting from two k-plane poles that merge through the contour $\tilde{F}$) correspond to our Ω points where some k_n interchanges with some k_m, one sees that their procedure leads in fact to the same form of criterion as ours.

F. Illustrative Examples: Electrostatic Waves

(*a*) *Maxwellian plasma*. The dispersion relation is

$$1-\omega_1^2/(\omega^2-a^2k^2)=0,$$

where a is the thermal velocity and ω_1 is the plasma frequency. (See Figs. 25–27.) One can see that this medium is stable. The wave of frequency ω_0 is damped.

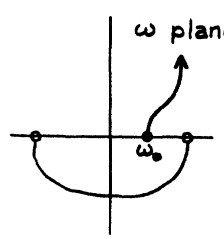

FIG. 23. ω plane with path starting from ω_0 and avoiding branch cut.

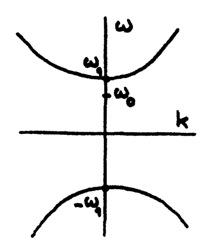

FIG. 25. Dispersion curve.

(*b*) *Crossed beams.* The dispersion relation is (see Fig. 28)

$$1-\omega_2^2/(\omega-kV)^2-\omega_2^2/(\omega+kV)^2=0,$$

where ω_2 is the plasma frequency of the beams; or, in reduced form:

$$1-(y-x)^{-2}-(y+x)^{-2}=0,$$

where $x=kV/\omega_2$, $y=\omega/\omega_2$. The critical point $i/2$ in Z^+ (Fig. 29) causes the interchange of roots according to the scheme

$$1\rightleftarrows 4,\quad 2\rightleftarrows 3.$$

The roots 1 and 3, which are of the class k_n, interchange with the roots 4 and 2, which are of the class k_m. (See Fig. 30.) We conclude that there is a nonconvective instability.

(*c*) *Beam-plasma.* The dispersion relation is (see Fig. 31)

$$1-\omega_1^2/(\omega^2-a^2k^2)-\omega_2^2/(\omega-kV)^2=0$$

where ω_1 and ω_2 are, respectively, the plasma frequency of the plasma and that of the beam. The root $k_1(\omega_0)$ in Z^- is of the class k_n (Fig. 33); thus we have a convective instability going to the right. When the medium is semi-infinite in $x\geq 0$ and is excited at $x=0$ the wave of frequency ω_0 (Fig. 31) is amplified in space.

(*d*) *Crossed beams, including collisions.* This time, the coefficients of the dispersion relation are complex, the dispersion relation cannot be plotted in real coordinates; it is thus difficult to apply Sturrock's criterion. The dispersion relation is

$$1-2\omega_2^2\frac{\omega(\omega+i\nu)+k^2V^2}{(\omega^2-k^2V^2)[(\omega+i\nu)^2-k^2V^2]}=0.$$

There are 6 critical points of $k(\omega)$:

$$\omega=0,$$
$$\omega=-i\nu,$$
$$\omega=-\tfrac{1}{2}i\nu\pm(2\omega_2^2-\tfrac{1}{4}\nu^2)^{1/2},$$
$$\omega=-\tfrac{1}{2}i\nu\pm\tfrac{1}{2}i[\omega_2^2/(\omega_2^2-\tfrac{1}{4}\nu^2)^{1/2}],$$

The localization of the critical points is shown by Figs. 34 and 35 and the behavior of the roots by Fig. 36. Only k_1 and k_3 are of the class k_n.

When $\nu<2\omega_2$, one of the critical points is in Z^+. At this point, the roots 1 and 3 interchange with the roots 4 and 2. Thus we have a nonconvective instability when $\nu<2\omega_2$.

When $\nu>2\omega_2$, there are no critical points in Z^+, none of the roots $k_n(\omega_0)$ is in Z^-; in this case the medium is stable and the waves are decreasing in space.

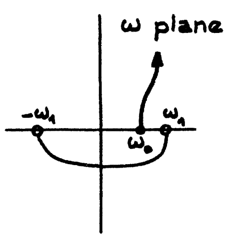

FIG. 26. Two real critical points $\pm\omega_1$ in ω plane.

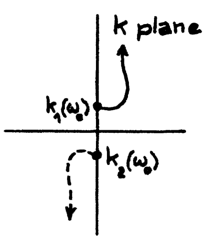

FIG. 27. Behavior of the roots $k(\omega)$; when $\omega=\omega_0$, the root k_1 belonging to the class k_n is in Z^+, the root k_2 belonging to the class k_m is in Z^-.

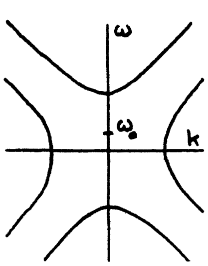

FIG. 28. Dispersion curve for crossed beams.

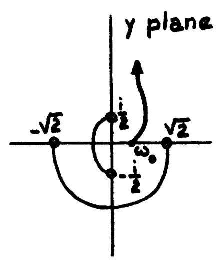

FIG. 29. Critical points of $x(y)$.

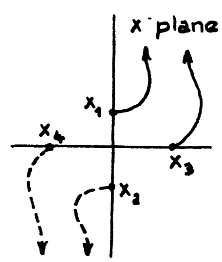

FIG. 30. Behavior of the roots $x(y)$.

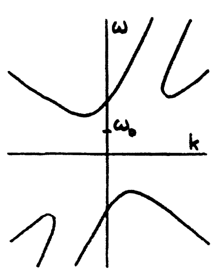

FIG. 31. Dispersion curve for a beam-plasma interaction when $\omega_2^2/\omega_1^2<(V^2/a^2)-1$.

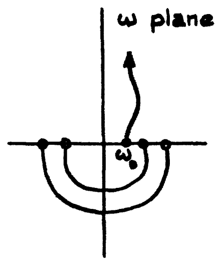

FIG. 32. Critical points of $k(\omega)$: 4 real points.

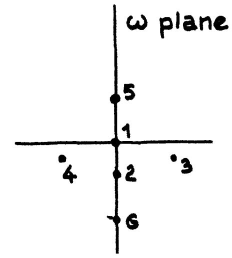

FIG. 34. Location of 6 critical points for the case of crossed beams, including collisions, for $\nu<2\omega_2$, in ω plane.

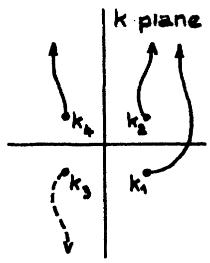

FIG. 33. Behavior of the roots $k(\omega)$.

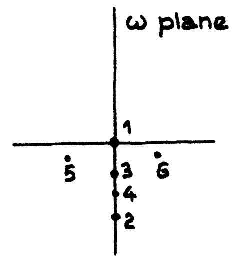

FIG. 35. The ω plane with location of the critical points for $\nu>2\omega_2$.

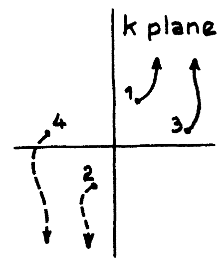

FIG. 36. Behavior of the roots in the k plane.

IV. CONCLUSION

It is encouraging to see that identical results are obtained by several independent approaches. None of these methods is free from difficulties, but since they deal with quite different models (initial disturbance in an infinite medium, boundary conditions in a semi-infinite medium, localized source in an infinite medium), it can be concluded that Sturrock's ideas are well confirmed.

From a practical point of view, it must be noted again that if we want $D(\omega,k)$ to be algebraic, the medium must be divided into a certain number of well defined components, adequately chosen, which is not always easy. Furthermore, the classification of the branch points may be laborious, mostly when the degree of $k(\omega)$ is higher than 4, or when $D(\omega,k)$ is no longer algebraic.

All the methods based on Fourier or Laplace transforms deal with analytic transforms of the disturbance. They can be directly applied to the multifluid theory, but they should be re-examined in the microscopic theory, where the presence of trapped particles entails analytic difficulties.