Influence of Multiple Virtual Transitions on the Reorientation Effect in Coulomb Excitation*

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Calculations of the reorientation effect in Coulomb excitation of the first 2^+ state of even-even nuclei are extended by employing the semiclassical approximation to take into account second- and third-order virtual $E2$ transitions to higher excited states. Deviations from first-order perturbation theory in the excitation probability are investigated numerically as a function of the bombarding energy and the scattering angle for heavy-ion projectiles. Results are presented for rotational and vibrational nuclei as a guide in the selection of conditions suitable for the determination of the static quadrupole moment of the lowest 2^+ state. Conditions for the practical convergence of the perturbation expansion are discussed.

I. INTRODUCTION

UNDER conditions of heavy-ion bombardment higher order effects in Coulomb excitation can be appreciable and the necessity for taking such effects into account was first pointed out by Breit and Lazarus[1] in connection with gamma-ray angular-distribution measurements and inelastic-scattering studies. These authors considered the second-order reorientation effect, which can be described as a change in direction of the nuclear spin axis during the excitation process. After the nucleus has first been excited from the ground state to one of the magnetic sublevels of the final state, a second transition to another magnetic sublevel of that state can take place on account of the interaction of the projectile with the static moment of the final state. Breit, Gluckstern, and Russell[2] studied effects which arise in the case of $E2(0\rightarrow 2)$ excitation from the cross product between the first- and second-order excitation amplitudes. Particular attention was paid to this term because it is linear in Q_{22}, the static quadrupole moment of the 2^+ state, and therefore offers the possibility for a determination of the sign as well as the absolute value of the static moment. The modification of the inelastic scattering and the angular distribution of the de-excitation gamma rays, as well as the effects in certain coincidence experiments involving the observation of inelastically scattered projectiles and of the de-excitation gamma rays, were worked out by them in the semiclassical (SC) approximation.[3] Numerical results indicated that measurable deviations from first-order theory were to be expected.

The possibility of multiple excitations to higher states in heavy-ion bombardment indicates that such transitions, which can affect a measurement of the reorientation effect in the first excited state, must also be taken into account. Second- and third-order virtual $E2$ transitions to intermediate states of spin 0, 2, and 4 were investigated in the work reported below with reference to selected even-even nuclei. The deviations expected from first-order theory are analyzed into second- and third-order components and studied as a function of the bombarding energy and the scattering angle. Studies of the inelastic scattering were treated first, since the interpretation of these measurements is not affected by complications which may affect the de-excitation gamma rays.[3,4]

The higher order SC probability amplitudes are introduced in Sec. II as a product of a nuclear-structure factor and an orbital integral. The contributions to the excitation probability are classified according to the order of the interaction energy. Some general features of the reorientation effect in the first 2^+ state of even-even nuclei are discussed briefly in Sec. III. Corrections due to virtual $E2$ transitions to higher excited states are considered for rotational nuclei in Sec. IV and vibrational nuclei in Sec. V.

II. OUTLINE OF THE SEMICLASSICAL THEORY OF HIGHER ORDER EFFECTS

Application of standard time-dependent perturbation theory yields the higher order SC probability amplitudes[3,5] for an electric transition from an initial state i with spin I_i and projection quantum number M_i to a final state f with spin I and projection M,

$$a_{IM}^{(1)}=-i\sum_{\lambda=1}^{\infty}t(E\lambda;i,f)R^{(1)}(I_iM_i(\lambda)IM),\qquad(1a)$$

$$a_{IM}^{(2)}=(-i)^2\sum_{s}\sum_{\lambda=1}^{\infty}\sum_{\lambda'=1}^{\infty}t(E\lambda;i,s)t(E\lambda';s,f)\times R^{(2)}(I_iM_i(\lambda)I_s(\lambda')IM),\qquad(1b)$$

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[1] G. Breit and J. P. Lazarus, Phys. Rev. **100**, 942 (1955).

[2] G. Breit, R. L. Gluckstern, and J. E. Russell, Phys. Rev. **103**, 727 (1956).

[3] G. Breit and R. L. Gluckstern, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1959), Vol. XLI/1, p. 496.

[4] (a) G. Breit, R. L. Gluckstern, and J. E. Russell, Phys. Rev. **105**, 1121 (1957); (b) G. Breit, in *Proceedings of the Third Conference on Reactions Between Complex Nuclei*, edited by A. Ghiorso, R. M. Diamond, and H. E. Conzett (University of California Press, Berkeley, 1963), pp. 273, 277.

$$a_{IM}^{(3)}=(-i)^3\sum_{r,s}\sum_{\lambda=1}^{\infty}\sum_{\lambda'=1}^{\infty}\sum_{\lambda''=1}^{\infty} t(E\lambda;i,r)\times t(E\lambda';r,s)t(E\lambda'';s,f)\times R^{(3)}(I_iM_i(\lambda)I_r(\lambda')I_s(\lambda'')IM). \quad (1c)$$

The summations over the intermediate states r and s include both the initial state i and the final state f. These equations contain a dimensionless factor, which measures the strength of the coupling between the states r and s due to the electric interaction, and is defined as

$$t(E\lambda;r,s)=(2I_r+1)^{-1/2}[Z_1e^2/(\hbar va'^\lambda)]T_{rs}^{(\lambda)*}, \quad (2)$$

where $T_{rs}^{(\lambda)}$ is the reduced nuclear matrix element[3] of the electric multipole operator between the states r and s. Here a' is half the classical distance of closest approach in a head-on collision; Z_1e and Z_2e are the charges of the projectile and target, respectively; and v is the asymptotic value of the relative velocity of the two particles. The reduced radiative transition probability[5] is related to the quantity $t(E\lambda;r,s)$ by

$$|t(E\lambda;r,s)|^2=[Z_1e/(\hbar va'^\lambda)]^2B(E\lambda,r\rightarrow s). \quad (3)$$

The quantity $R^{(n)}$ in Eq. (1) is also dimensionless. It is simply a linear combination of orbital integrals, which are functions of the scattering angle in the center-of-mass system θ and the set of adiabaticity parameters ξ_{rs} associated with the nuclear states r and s involved in the excitation process. The adiabaticity parameter is defined as usual by

$$\xi_{rs}=\eta_r-\eta_s, \quad (4)$$

where

$$\eta_s=Z_1Z_2/e^2(\hbar v_s),$$

and v_s is the velocity of relative motion at infinite separation, when the target nucleus is in state s. Working in the focal coordinate system[5] of the hyperbolic orbit, the explicit formulas for $R^{(n)}$ are

$$R^{(1)}(I_iM_i(\lambda)I_fM_f)=\sum_{\mu}A(I_f\lambda I_i;M_f\mu M_i)I_{\lambda\mu}^{(1)}(\xi_{fi};\theta), \quad (5a)$$

$$R^{(2)}(I_iM_i(\lambda)I_s(\lambda')I_fM_f)=\sum_{\mu}\sum_{\mu'}\sum_{M_s}A(I_f\lambda' I_s;M_f\mu' M_s)\times A(I_s\lambda I_i;M_s\mu M_i)I_{\lambda'\mu',\lambda\mu}^{(2)}(\xi_{fs},\xi_{si};\theta), \quad (5b)$$

$$R^{(3)}(I_iM_i(\lambda)I_s(\lambda')I_r(\lambda'')I_fM_f)=\sum_{\mu}\sum_{\mu'}\sum_{\mu''}\sum_{M_r}\sum_{M_s}A(I_f\lambda'' I_s;M_f\mu'' M_s)\times A(I_s\lambda' I_r;M_s\mu' M_r)A(I_r\lambda I_i;M_r\mu M_i)\times I_{\lambda''\mu'',\lambda'\mu',\lambda\mu}^{(3)}(\xi_{fs},\xi_{sr},\xi_{ri};\theta), \quad (5c)$$

[5] K. Alder, A. Bohr, T. Huss, B. Mottelson, and A. Winther, Rev. Mod. Phys. 28, 432 (1956).

with

$$A(I'\lambda I;M'\mu M)=4\pi(2\lambda+1)Y_{\lambda\mu}(\pi/2,0)C(I'\lambda I;M'\mu M),$$

where the standard spherical harmonics $Y_{\lambda\mu}$ and vector addition coefficients $C(j_1j_2j_3;m_1m_2m_3)$ have been used. The orbital integrals are defined as

$$I_{\lambda\mu}^{(1)}(\xi;\theta)=\int_{-\infty}^{\infty}K_{\lambda\mu}(w)dw,$$

$$I_{\lambda'\mu',\lambda\mu}^{(2)}(\xi',\xi;\theta)=\int_{-\infty}^{\infty}dw'K_{\lambda'\mu'}(w')\int_{-\infty}^{w'}dwK_{\lambda\mu}(w), \quad (6)$$

$$I_{\lambda''\mu'',\lambda'\mu',\lambda\mu}^{(3)}(\xi'',\xi',\xi;\theta)=\int_{-\infty}^{\infty}dw''K_{\lambda''\mu''}(w'')\int_{-\infty}^{w''}dw'K_{\lambda'\mu'}(w')\int_{-\infty}^{w'}dwK_{\lambda\mu}(w),$$

where the abbreviations

$$K_{\lambda\mu}(w)=\exp(i\omega t+i\mu\varphi)(\epsilon\cosh w+1)^{-\lambda},$$
$$\omega t=\xi(\epsilon\sinh w+w),$$
$$\varphi=\tan^{-1}\left[\frac{(\epsilon^2-1)^{1/2}\sinh w}{\cosh w+\epsilon}\right], \quad (7)$$
$$\epsilon=1/\sin(\tfrac{1}{2}\theta),$$

have been employed.

The excitation probability for the final state f may be expanded as

$$P=P^{(1,1)}+P^{(1,2)}+P^{(2,2)}+P^{(1,3)}+\cdots, \quad (8)$$

where the superscripts indicate the order of the amplitudes that are involved. The individual terms for electric excitation of the final state f are

$$P^{(1,1)}=\sum_{\lambda=1}^{\infty}[t(E\lambda;i,f)]^2F^{(1,1)}, \quad (9a)$$

$$P^{(1,2)}=\sum_{s}\sum_{\lambda}\sum_{\lambda',\lambda''}t(E\lambda;i,f)t(E\lambda';i,s)\times t(E\lambda'';s,f)F_s^{(1,2)}, \quad (9b)$$

$$P^{(2,2)}=\sum_{s}\sum_{\lambda_1,\lambda_2}\{[t(E\lambda_1;i,s)t(E\lambda_2;s,f)]^2F_s^{(2,2)}+\sum_{s'}{}'\sum_{\lambda_1',\lambda_2'}{}'t(E\lambda_1;i,s)t(E\lambda_2;s,f)t(E\lambda_1';i,s')\times t(E\lambda_2';s',f)F_{ss'}^{(2,2)}\}, \quad (9c)$$

$$P^{(1,3)}=\sum_{r,s}\sum_{\lambda}\sum_{\lambda_1}\sum_{\lambda_2}\sum_{\lambda_3}t(E\lambda;i,f)t(E\lambda_1;i,r)\times t(E\lambda_2;r,s)t(E\lambda_3;s,f)F_{rs}^{(1,3)}. \quad (9d)$$

The prime on the summation sign in Eq. (9c) indicates a restriction to those terms in which at least one index is different in the two sets of indices (s,λ_1,λ_2) and $(s',\lambda_1',\lambda_2')$. Since the summations over magnetic quantum numbers have been performed, the indices s, s', and r refer here only to a summation over different energy levels.

The functions F introduced in Eq. (9) are defined as

$$F^{(1,1)}=(2I_i+1)^{-1}\sum_{M_i,M_f}[R^{(1)}(I_iM_i(\lambda)I_fM_f)]^2, \quad (10a)$$

$$F_s^{(1,2)}=(2I_i+1)^{-1}\sum_{M_i,M_f}2R^{(1)}(I_iM_i(\lambda)I_fM_f) \times \mathrm{Im}R^{(2)}(I_iM_i(\lambda')I_s(\lambda'')I_fM_f), \quad (10b)$$

$$F_s^{(2,2)}=(2I_i+1)^{-1} \times \sum_{M_i,M_f}|R^{(2)}(I_iM_i(\lambda_1)I_s(\lambda_2)I_fM_f)|^2, \quad (10c)$$

$$F_{ss'}^{(2,2)}=(2I_i+1)^{-1} \times \sum_{M_i,M_f}2\,\mathrm{Re}[R^{(2)*}(I_iM_i(\lambda_1)I_s(\lambda_2)I_fM_f) \times R^{(2)}(I_iM_i(\lambda_1')I_{s'}(\lambda_2')I_fM_f), \quad (10d)$$

$$F_{rs}^{(1,3)}=-(2I_i+1)^{-1}\sum_{M_i,M_f}2R^{(1)}(I_iM_i(\lambda)I_fM_f) \times \mathrm{Re}R^{(3)}(I_iM_i(\lambda_1)I_r(\lambda_2)I_s(\lambda_3)I_fM_f). \quad (10e)$$

The quantities F are real functions of θ, the multipole order of the transitions, the spins of the intermediate states, and the adiabaticity parameters coupling these states. The phases of the nuclear wave functions have been chosen so that the reduced matrix elements are real. For a particular nucleus, once the energy levels and spins have been measured for the states which are involved in the Coulomb excitation process, the functions F can be computed numerically as a function of θ and the set of ξ's. All other nuclear structure information is contained in the t factors.

In searching for practical convergence conditions for the perturbation series, it is convenient to introduce the ratios of successive terms of different order

$$P/P^{(1,1)}=1+R_{21}+R_{31}+\cdots \quad (11a)$$

$$=1+R_{21}(1+R_{32}+\cdots), \quad (11b)$$

where

$$R_{21}=P^{(1,2)}/P^{(1,1)}, \quad (12)$$

$$R_{31}=[P^{(1,3)}+P^{(2,2)}]/P^{(1,1)}, \quad (13)$$

$$R_{32}=[P^{(1,3)}+P^{(2,2)}]/P^{(1,2)}. \quad (14)$$

The third-order effect $P^{(1,3)}$ is included with the term $P^{(2,2)}$ that arises from a product of second-order amplitudes, since both contributions are of fourth order in the interaction energy and, in typical cases, were found to be of the same order of magnitude. The probability ratios introduced above are investigated numerically as a function of θ and ξ.

III. REORIENTATION EFFECT

In order to avoid confusion with other processes the words "reorientation effect" will refer here exclusively to higher order virtual transitions between the magnetic sublevels of the final state. Preliminary results of a calculation of the third-order reorientation effect for $E2(0\rightarrow 2)$ excitation have been reported by Lin and Masso.[6] In this case, when all other higher order effects are neglected, the probability ratios take the form

$$R_{21}=\lambda D_{21}, \quad (15)$$

$$R_{32}=\lambda D_{32}, \quad (16)$$

where

$$D_{21}(\xi;\theta)=\left(\frac{35}{2\pi}\right)^{1/2}\frac{F_2^{(1,2)}}{F^{(1,1)}}, \quad (17)$$

$$D_{32}(\xi;\theta)=\left(\frac{35}{2\pi}\right)^{1/2}\frac{[F_2^{(2,2)}+F_{22}^{(1,3)}]}{F_2^{(1,2)}}, \quad (18)$$

and ξ without any subscripts refers to the $0\rightarrow 2$ transition, i.e., $\xi=\xi_{20}$. The energy separation between the magnetic sublevels of the 2^+ state has been neglected. The negative of the function D_{21} is plotted in Ref. 7, where it is denoted by $L(\theta,\xi)$. The reorientation effect for this special case is characterized by a dimensionless parameter,[2]

$$\lambda=-\frac{Q_{22}Z_1e^2}{4\hbar va'^2}, \quad (19)$$

which enters into a calculation of successive orders of the reorientation effect. The term $P^{(1,3)}$, which arises from the cross product between the first- and third-order amplitudes, and $P^{(2,2)}$, the direct second-order reorientation contribution, are both of order λ^2 and were found numerically to have opposite signs for $\xi\lesssim 1.5$; since these two terms are of the same order of magnitude for $\xi\lesssim 0.8$, a strong cancellation occurs for this range of ξ. This circumstance reduces the over-all contribution of order λ^2 to the excitation probability.

Numerical calculations were carried out for $0.05\leq\xi\leq 3$ and $10°\leq\theta\leq 180°$ employing an IBM 7094 computer for the projectiles He^4, C^{12}, N^{14}, O^{16}, S^{32}, Ne^{20}, and A^{40} bombarding Fe^{56}, Se^{76}, Cd^{114}, Sm^{152}, Er^{168}, and Pt^{194} in order to survey a variety of conditions. The ratio λD_{21} of the second-order reorientation probability $P_2^{(1,2)}$ to the first-order probability $P^{(1,1)}$ is a monotonically increasing function of the scattering angle in the center-of-mass system. The angular variation of λD_{21}

[6] D. L. Lin and J. F. Masso, in *Proceedings of the Third Conference on Reactions Between Complex Nuclei*, edited by A. Ghiorso, R. M. Diamond, and H. E. Conzett (University of California Press, Berkeley, 1963), p. 267.

[7] A. C. Douglas, W. Bygrave, D. Eccleshall, and M. J. L. Yates, in *Proceedings of the Third Conference on Reactions Between Complex Nuclei*, edited by A. Ghiorso, R. M. Diamond, and H. E. Conzett (University of California Press, Berkeley, 1963), p. 274.

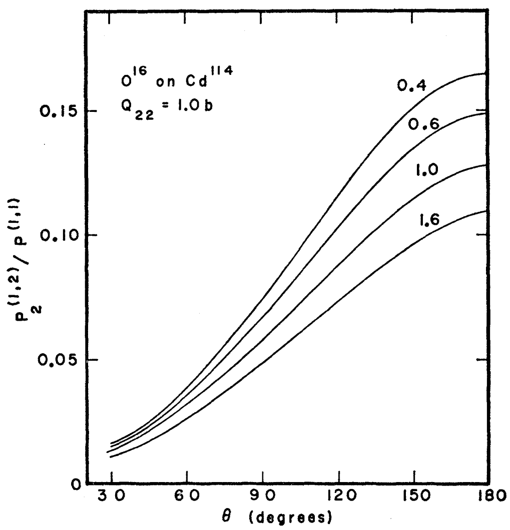

FIG. 1. The ratio λD_{21} of the second-order reorientation effect $P_2^{(1,2)}$ to the first-order probability $P^{(1,1)}$ plotted as a function of the scattering angle in the center-of-mass system θ for O^{16} bombarding Cd^{114}. The curves are designated by several fixed values of the adiabaticity parameter ξ and Q_{22}, the static quadrupole moment of the lowest 2^+ state, is taken as 1 b.

is illustrated for O^{16} on Cd^{114} in Fig. 1 for several values of ξ.

The absolute value of this second-order effect can be increased by employing even heavier ions, since λ varies approximately as the first power of the projectile's mass number A_1 for a fixed value of ξ or E/A_1, where E is the incident laboratory energy. The ratio $\lambda D_{21}(\xi,\theta)$ is illustrated in Figs. 2–4 for $E2$ excitation of the 558-keV level in Cd^{114}, the 122-keV level in Sm^{152}, and the 80-keV level in Er^{168}. It is found that the ratio

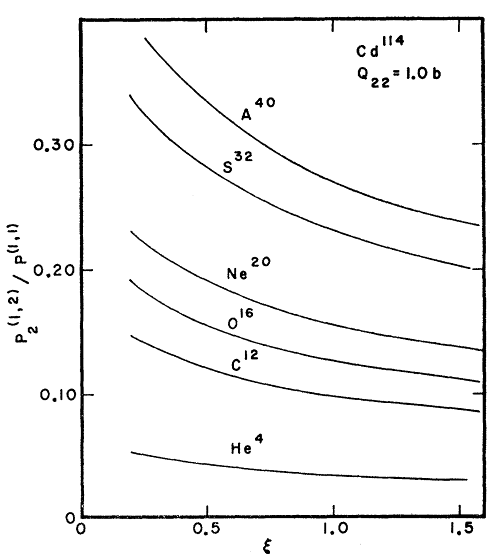

FIG. 2. The ratio $\lambda D_{21}=P_2^{(1,2)}/P^{(1,1)}$ plotted as a function of the adiabaticity parameter ξ for heavy ions bombarding Cd^{114} at $\theta=180°$ with $Q_{22}=1$ b.

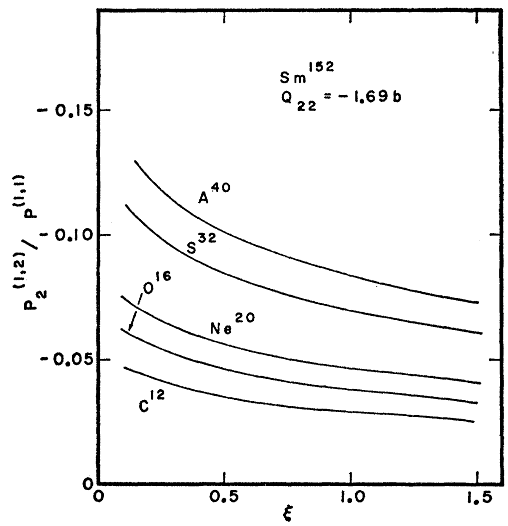

FIG. 3. The ratio $\lambda D_{21}=P_2^{(1,2)}/P^{(1,1)}$ plotted as a function of the adiabaticity parameter ξ for heavy ions bombarding Sm^{152} at $\theta=180°$ with $Q_{22}=-1.69$ b.

λD_{21} is approximately independent of the bombarding energy and decreases slowly as ξ is increased. This feature is important for a possible separation of the second-order reorientation effect, which depends linearly on Q_{22}, from other higher-order corrections.

The second-order reorientation effect can also be increased for any pair of nuclei by simply raising the beam energy. However, there are restrictions on the lower limit of the adiabaticity parameter. Numerical results show that corrections of higher order than the second are appreciable for ξ smaller than a limiting value depending on the target nucleus. In this region of ξ, calculations employing the perturbation approxi-

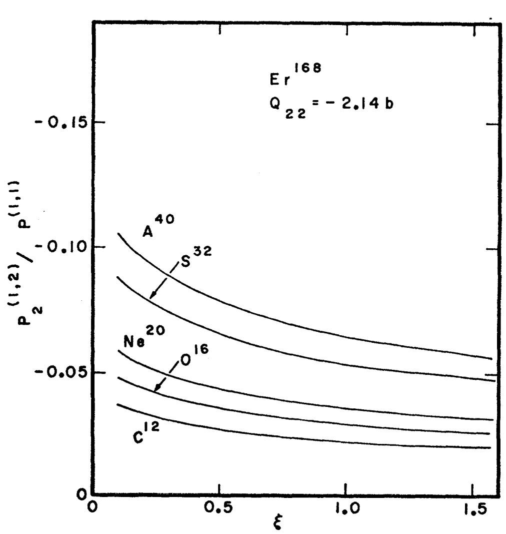

FIG. 4. The ratio $\lambda D_{21}=P_2^{(1,2)}/P^{(1,1)}$ plotted as a function of the adiabaticity parameter ξ for heavy ions bombarding Er^{168} at $\theta=180°$ with $Q_{22}=-2.14$ b.

mation may not be accurate. Secondly, the bombarding energy should be sufficiently below the Coulomb barrier so that nuclear reactions other than electric excitation are negligible. This requires that there be a large geometric separation or gap between the nuclear surfaces of the projectile and target.[6]

IV. CORRECTIONS DUE TO VIRTUAL $E2$ TRANSITIONS IN ROTATIONAL NUCLEI

Besides the reorientation effect other virtual transitions to higher excited states during the collision can populate the final state. In studying the reorientation effect in the first 2^+ level of even-even nuclei displaying a rotational energy spectra with the spin sequence 0^+, $2^+, 4^+, \cdots$ virtual third-order $E2$ transitions to the ground state and to the 4^+ state will introduce corrections to the term $P^{(1,3)}$,

$$P^{(1,3)}=P_{22}{}^{(1,3)}+P_{20}{}^{(1,3)}+P_{24}{}^{(1,3)}, \qquad (20)$$

where the subscripts indicate the intermediate states in the third-order excitation. The individual terms are

$$P_{22}{}^{(1,3)}=(T/e)^4B(E2,0\rightarrow 2)B(E2,2\rightarrow 2)F_{22}{}^{(1,3)}, \qquad (21a)$$

$$P_{20}{}^{(1,3)}=(T/e)^4[B(E2,0\rightarrow 2)]^2F_{20}{}^{(1,3)}/\sqrt{5}, \qquad (21b)$$

$$P_{24}{}^{(1,3)}=(0.6\sqrt{5})(T/e)^4B(E2,4\rightarrow 2) \times B(E2,0\rightarrow 2)F_{24}{}^{(1,3)}, \qquad (21c)$$

where

$$T=Z_1e^2/(\hbar va'^2). \qquad (22)$$

Virtual excitation to states beyond the 4^+ level is neglected. Measured $B(E2)$ values, which were taken from Ref. 8 for Er^{168} and from Refs. 9 and 10 for Sm^{152}, were used in the calculation. To obtain an estimate of the higher order corrections, the static moment is approximated by the value from the rotational model[5]

$$eQ_{22}=-(2/7)[(16\pi/5)B(E2,0\rightarrow 2)]^{1/2}, \qquad (23)$$

which gives $Q_{22}=-2.14$ b for Er^{168} and $Q_{22}=-1.69$ b for Sm^{152}.

The ratio

$$R_{32}=[P_{22}{}^{(2,2)}+P_{22}{}^{(1,3)}+P_{20}{}^{(1,3)}+P_{24}{}^{(1,3)}]/P^{(1,2)} \qquad (24)$$

is illustrated as a function of ξ for S^{32} bombarding Er^{168} in Fig. 5, where curves for several scattering angles are included to show how the lower limit for ξ must be raised, if measurements are made at smaller angles. The rather strong ξ dependence of R_{32}, especially at the lowest ξ, is in contrast to the slow variation of R_{21} with ξ for all energies. The value of ξ at which the curve for R_{32} increases rapidly is observed to become larger as θ is reduced from 180° to 60°. For ξ below approximately 0.4 for $\theta\geq 60°$, the ratio R_{32} increases quite sharply in absolute value, becoming much greater than unity, indicating that to this order the perturbation expansion cannot be expected to be accurate for this range. Small values of ξ correspond to high bombarding energies, since ξ depends on $1/E^{3/2}$. When $\xi=0.4$ the incident laboratory energy for a S^{32} projectile on Sm^{152} is about 30 MeV and on Er^{168} about 24 MeV. Some estimates are presented in Table I. For low energies

TABLE I. Estimates for rotational nuclei at $\xi=0.4$ and $\theta=180°$. The static moment Q_{22} was taken as -1.69b for Sm^{152} and -2.14b for Er^{168}.

Nucleus	Projectile	E (MeV)	R_{21}	R_{32}
Sm^{152}	C^{12}	10.4	−0.037	−0.093
	O^{16}	14.1	−0.049	−0.12
	Ne^{20}	17.9	−0.060	−0.14
	S^{32}	29.9	−0.089	−0.21
	A^{40}	35.9	−0.11	−0.26
Er^{168}	C^{12}	8.3	−0.029	−0.066
	O^{16}	11.3	−0.037	−0.086
	Ne^{20}	14.3	−0.046	−0.11
	S^{32}	23.8	−0.069	−0.16
	A^{40}	28.4	−0.080	−0.19

when $\xi\gtrsim 0.4$ and $Z_1\leq 16$, the results for Er^{168} show that $|R_{21}|\lesssim 0.07$ and $|R_{31}|\lesssim 0.01$, if $Q_{22}=-2.14$ b. Since the ratio R_{32} is less than 16% in absolute value, measurement of the second-order reorientation effect may not be seriously complicated by these third-order corrections. In the case of Sm^{152}, the limits on the ratios are $|R_{21}|\lesssim 0.09$, $|R_{31}|\lesssim 0.02$, and $|R_{32}|\lesssim 0.21$ for all angles, when $\xi\gtrsim 0.4$ and $Z_1\leq 16$, if $Q_{22}=-1.69$ b.

V. CORRECTIONS DUE TO VIRTUAL $E2$ TRANSITIONS IN VIBRATIONAL NUCLEI

Excitations of higher excited states of spin 2 have been measured in many cases of even-even medium weight[11] and heavy[12] nuclei. Second-order excitation of the lowest 2^+ state via the second and third spin-2 states contributes to the term

$$P^{(1,2)}=P_2{}^{(1,2)}+P_{2'}{}^{(1,2)}+P_{2''}{}^{(1,2)}, \qquad (25)$$

where the subscript indicates the intermediate state in the second-order excitation; here $2'$ and $2''$ designate the second and third 2^+ states, respectively. This term may be represented as

$$P^{(1,2)}=P_2{}^{(1,2)}C_2, \qquad (26)$$

where C_2 represents the correction factor to $P^{(1,2)}$ due

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[9] R. Graetzer and E. M. Bernstein, Phys. Rev. **129**, 1772 (1963).
[10] G. Goldring, J. de Boer, and H. Winkler, in *Proceedings of the Third Conference on Reactions Between Complex Nuclei*, edited by A. Ghiorso, R. M. Diamond, and H. E. Conzett (University of California Press, Berkeley, 1963), p. 278.

[11] P. H. Stelson and F. K. McGowan, Phys. Rev. **121**, 209 (1961); F. K. McGowan, R. L. Robinson, P. H. Stelson, J. L. C. Ford, and W. T. Milner, Bull. Am. Phys. Soc. **9**, 107 (1964); F. K. McGowan, R. L. Robinson, P. H. Stelson, and J. L. C. Ford, Jr., Nucl. Phys. **66**, 97 (1965).
[12] F. K. McGowan and P. H. Stelson, Phys. Rev. **122**, 1274 (1961).

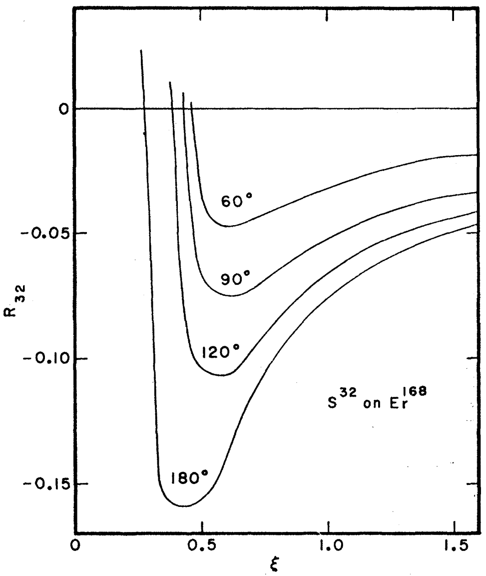

FIG. 5. The ratio $R_{32}=[P^{(1,3)}+P^{(2,2)}]/P^{(1,2)}$ plotted as a function of ξ for S^{32} bombarding Er^{168} at several scattering angles in the center-of-mass system with $Q_{22}=-2.14$ b.

to the addition of the last two terms in Eq. (25). The ratio of $P^{(1,2)}$ to the first-order probability $P^{(1,1)}$ may be expressed as

$$R_{21}=T\{(35/32\pi)^{1/2}Q_{22}F_2^{(1,2)}+S'F_{2'}^{(1,2)}+S''F_{2''}^{(1,2)}\}/F^{(1,1)}, \quad (27)$$

where

$$S'=\pm[B(E2,2'\rightarrow 0)B(E2,2'\rightarrow 2)/B(E2,2\rightarrow 0)e^2]^{1/2},$$

$$S''=\pm[B(E2,2''\rightarrow 0)B(E2,2''\rightarrow 2)/B(E2,2\rightarrow 0)e^2]^{1/2},$$

and T is defined in Eq. (22). The factors S' and S'' depend on the relative signs of the combination of reduced nuclear matrix elements contained in the above brackets. Contributions due to excitation of higher spin-2 states have been neglected, since the corresponding matrix elements may be expected to be small relative to the matrix elements for the states which are included. The quantities F are positive for $0.2\leq\xi\leq 2$, so that the correction contained in C_2 is quite sensitive to the signs of the matrix elements involved, which are generally unknown. For larger values of ξ, R_{21} becomes less dependent on these signs, because the correction due to virtual excitation of higher spin-2 states decreases relative to the reorientation effect $P_2^{(1,2)}/P^{(1,1)}$, which is approximately independent of E. The factor C_2 is approximately independent of the scattering angle; it changes by only a few percent as θ is varied. When both S' and S'' have the same sign, the correction due to virtual excitation of higher spin-2 states is largest, amounting to 12% of the second-order reorientation effect in the lowest 2^+ state of Cd^{114}, if $Q_{22}=-0.6$ b and $\xi=0.6$. This correction decreases, when ξ is increased, as illustrated in Fig. 7 for O^{16} bombarding Cd^{114}.

Virtual third-order $E2$ excitation of intermediate states of spin 0, 2, and 4 introduces the correction

$$\begin{aligned}P^{(1,3)}=&P_{22}^{(1,3)}+P_{20}^{(1,3)}+P_{24}^{(1,3)}+P_{20'}^{(1,3)}+P_{2'0}^{(1,3)}\\&+P_{22'}^{(1,3)}+P_{2'2}^{(1,3)}+P_{2'2'}^{(1,3)}+P_{2''2}^{(1,3)}\\&+P_{22''}^{(1,3)}+P_{2'4}^{(1,3)}+P_{2''4}^{(1,3)}+P_{2''0'}^{(1,3)}\\&+P_{2''2''}^{(1,3)}+P_{2'0'}^{(1,3)}+\cdots, \quad (28)\end{aligned}$$

where the subscripts indicate the sequence of intermediate states in the third-order transition. The excited states of spin 0 and 4 are designated by 0′ and 4, respectively. In order to study contributions to the excitation probability of fourth order in the interaction energy, one must add to $P^{(1,3)}$ the term

$$P^{(2,2)}=P_2^{(2,2)}+P_{2'}^{(2,2)}+P_{2''}^{(2,2)}+P_{22'}^{(2,2)}+P_{22''}^{(2,2)}+P_{2'2''}^{(2,2)}, \quad (29)$$

where the last three terms arise from cross products between second-order transition amplitudes involving different intermediate states.

Numerical results will be discussed for the reorientation of the 558-keV 2^+ level in Cd^{114}; the excitation energies of the $0'^+$, $2'^+$, $4'^+$, and $2''^+$ levels are 1133, 1208, 1282, and 1363 keV, respectively. Second- and third-order transitions involving six intermediate states are computed. The last five terms in Eq. (28) are neglected along with terms referring to transitions involving a second excited spin-0 state, since these contributions are expected to be small relative to the principal terms represented by $P_{24}^{(1,3)}$, $P_{20}^{(1,3)}$, $P_{22}^{(2,2)}$, $P_{22}^{(1,3)}$, $P_{22'}^{(1,3)}$. All nuclear matrix elements are obtained from measured $B(E2)$ values,[11] with the exception of Q_{22}, which was assigned different values in the calculation. The unknown signs of the matrix elements are not important in the third-order correction, because they enter only the terms in $P^{(1,3)}$ and $P^{(2,2)}$ which are relatively small.

The ratio of $P^{(1,3)}+P^{(2,2)}$ to $P^{(1,1)}$ can be written as

$$\begin{aligned}R_{31}=T^2\{&(35/32\pi)Q_{22}^2[F_2^{(2,2)}+F_2^{(1,3)}]+S'^2F_{2'}^{(2,2)}+S''^2F_{2''}^{(2,2)}\\&+(0.6\sqrt{5})B(E2,4\rightarrow 2)e^{-2}F_{24}^{(1,3)}+(\sqrt{5})B(E2,2\rightarrow 0)e^{-2}F_{20}^{(1,3)}+B(E2,2'\rightarrow 2)e^{-2}F_{22'}^{(1,3)}\\&+0.2(\sqrt{5})B(E2,0'\rightarrow 2)e^{-2}F_{20'}^{(1,3)}+(\sqrt{5})B(E2,2'\rightarrow 0)e^{-2}F_{2'0}^{(1,3)}+(35/32\pi)^{1/2}Q_{22}[S'(F_{22'}^{(2,2)}\\&+F_{2'2}^{(1,3)}+Q_{2'2'}Q_{22}^{-1}F_{2'2'}^{(1,3)})+S''(F_{22''}^{(2,2)}+F_{2''2}^{(1,3)})]+S'S''F_{2'2''}^{(2,2)}\}[F^{(1,1)}]^{-1}. \quad (30)\end{aligned}$$

This form of expression emphasizes that R_{31} can be regarded simply as a product of T^2 and the quantity contained in curly braces, which is an explicit function only of ξ, θ, and the nuclear matrix elements, but not of the projectile. On the other hand, the parameter T defined in Eq. (22) does depend on the charge, mass, and incident energy of the projectile. The ratio R_{21} has been represented in Eq. (27) as a similar product of T and another quantity, which is an explicit function of ξ, θ, and the target nucleus. It is possible to distinguish at least partly between second- and third-order effects by investigating the variation of the excitation probability with ξ, θ, Z_1, A_1, and E.[13] For example, it may be possible to separate out the quantities discussed above by keeping ξ and θ fixed in measurements employing different projectiles.

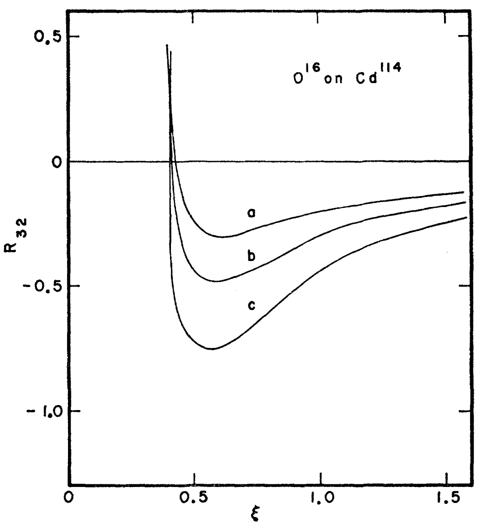

FIG. 6. The ratio $R_{32}=[P^{(1,3)}+P^{(2,2)}]/P^{(1,2)}$ plotted as a function of ξ for O^{16} bombarding Cd^{114} at $\theta=180°$ for several values of Q_{22}. Curves a, b, and c correspond to the assigned values for Q_{22} of -1.0, -0.6, and -0.4 b, respectively.

The ratio $R_{32}=[P^{(1,3)}+P^{(2,2)}]/P^{(1,2)}$, which includes these higher order corrections, is illustrated in Fig. 6 as a function of ξ for O^{16} on Cd^{114} at $\theta=180°$ for several choices of Q_{22}. The relative signs of the matrix elements have been chosen as positive and ξ refers again to the $0\rightarrow 2$ excitation. The results indicate that for ξ below 0.4, which corresponds in this case to an incident energy of 33.3 MeV, the absolute value of R_{32} increases sharply. This implies that for the $\xi\lesssim 0.4$ the perturbation expansion to this order cannot be expected to be accurate. At larger ξ, R_{32} decreases in absolute value and these corrections due to virtual excitation of higher states become less important relative to the second-order reorientation effect. When employing projectiles much heavier than Ne^{20}, R_{32} can exceed unity even at low incident energies. For example, this difficulty occurs over

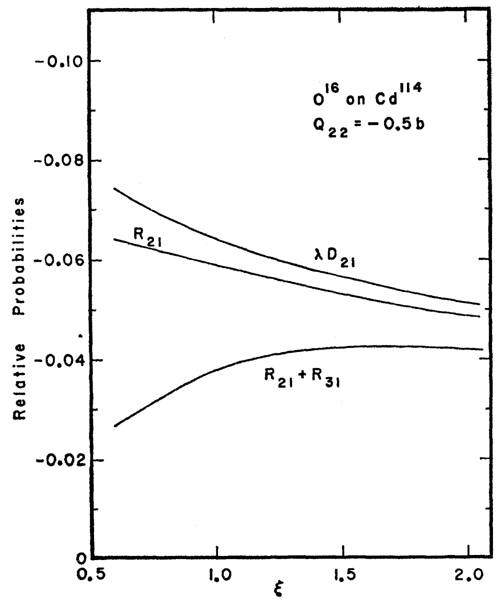

FIG. 7. The quantities $\lambda D_{21}=P_2^{(1,2)}/P^{(1,1)}$, $R_{21}=P^{(1,2)}/P^{(1,1)}$, and $(R_{21}+R_{31})=[P^{(1,2)}+P^{(2,2)}+P^{(1,3)}]/P^{(1,1)}$ plotted as a function of ξ for O^{16} on Cd^{114} at $\theta=180°$ with $Q_{22}=-0.5$ b.

a wide range of ξ in the case of A^{40} bombarding Cd^{114}, if $|Q_{22}|\leq 0.6$ b. These results imply that the perturbation approximation is less useful the heavier the projectile.

The quantity $R_{21}+R_{31}$, which represents the sum of second- and third-order effects relative to the first-order excitation probability, is illustrated in Figs. 7 and 8 as a function of ξ for O^{16} on Cd^{114} at $\theta=180°$. Deviations from first-order theory are indicated even at large ξ. Generally, when $\xi\gtrsim 0.6$ and Q_{22} is negative, the ratio

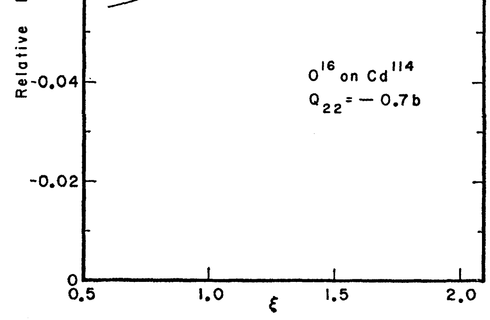

FIG. 8. The quantities λD_{21}, R_{21}, and $(R_{21}+R_{31})$ plotted as a function of ξ for O^{16} on Cd^{114} at $\theta=180°$ with $Q_{22}=-0.7$ b.

[13] This point has been emphasized frequently by G. Breit.

R_{31} is positive, while R_{21} is negative. As the adiabaticity parameter is increased above 0.6, the ratio R_{31} becomes smaller in comparison to the absolute value of R_{21}. This may permit an approximate separation of the second-order reorientation effect from the other higher order effects at the lower incident energies. Similar results were found for other heavy ions. When $\xi=0.6$, the bombarding energy in the laboratory is approximately 1.6 MeV/A_1 for C^{12}, N^{14}, O^{16}, S^{32}, and Ne^{20} and the gap distance between the nuclear surfaces in a head-on collision with Cd^{114} is about 14 F.

VI. DISCUSSION

As a check on the applicability of the perturbation approximation in heavy-ion Coulomb excitation, some second- and third-order excitation amplitudes have been computed. The relative magnitude of successive terms in the expansion of the excitation probability in powers of the interaction potential has been investigated as a function of the scattering angle and of the incident energy for heavy ions bombarding selected even-even nuclei. In searching for practical convergence conditions, a range of values of the adiabaticity parameter ξ was found for which the double requirement that the ratio R_{21} of second-order to first-order effects and the ratio R_{32} of third-order to second-order corrections be small in comparison to unity can be satisfied under suitable circumstances. The procedure employing the perturbation approximation may therefore still be used in studies of low-energy heavy-ion Coulomb excitation to provide some reasonable estimates, if the corresponding values of ξ for the transition are restricted by the lower limits indicated above.

The primary objective of the work has been to find conditions for which the second-order reorientation effect is appreciable, and corrections due to higher order effects can be applied in the analysis of experiments. In the cases studied, a range of ξ was found for which $|R_{32}|<1$, while R_{21} is appreciable. The second-order reorientation effect relative to the first-order probability always increases slowly as ξ is reduced. On the other hand, higher order effects increase much more sharply as ξ is reduced, exceeding the second-order effect usually when $\xi\approx 0.4$. These higher order corrections can be reduced relative to the second-order effect by increasing ξ or lowering the incident energy. However, R_{21} itself becomes smaller in absolute value for larger ξ. The total excitation probability P also decreases rapidly as ξ is increased. These circumstances affect the accuracy to which Q_{22} can be determined from a perturbation calculation.

For heavy deformed even-even nuclei, as long as the bombarding energy corresponds to a value of ξ above the approximate lower limit of 0.4, effects of higher order than the second are not expected to introduce significant corrections. Some quantitative estimates are given in Table I for Sm^{152} and Er^{168}. Although these corrections are appreciable, they may not prohibit an estimate for Q_{22}.

TABLE II. Estimates for Cd^{114} with $Q_{22}=-0.6$b and $\theta=180°$. The measured $B(E2)$ values were taken from Ref. 11, and the relative signs of the matrix elements were assumed to be positive.

Projectile	ξ	E (MeV)	$P_2^{(1,2)}/P^{(1,1)}$	R_{32}	$(R_{21}+R_{31}-\lambda D_{21})$
O^{16}	0.6	25.8	−0.089	−0.52	0.048
	0.8	21.3	−0.082	−0.43	0.037
	1.0	18.4	−0.077	−0.32	0.027
	1.2	16.3	−0.073	−0.25	0.020
	1.4	14.8	−0.069	−0.20	0.016
	1.6	13.5	−0.066	−0.17	0.013
	2.0	11.7	−0.062	−0.13	0.010
S^{32}	0.6	55.0	−0.159	−0.92	0.140
	0.8	45.7	−0.147	−0.76	0.109
	1.0	39.4	−0.137	−0.58	0.079
	1.2	35.0	−0.130	−0.45	0.059
	1.4	31.6	−0.124	−0.36	0.046
	1.6	28.9	−0.119	−0.30	0.038
	2.0	25.0	−0.111	−0.22	0.028

For vibrational nuclei, higher order corrections are expected to be more important relative to the second-order reorientation effect than for deformed even-even nuclei, especially if Q_{22} is small. However, a larger range of ξ may be accessible to experiment. Some estimates for Cd^{114} are presented in Table II as an illustration of the magnitude of the effects under discussion. Caution is necessary in applying the results presented here, since they are sensitive to the choice of the relative signs of the reduced matrix elements and to the $B(E2)$ values employed. As ξ is increased from 0.6 to 1.2, R_{32} is reduced by 50%, while $P_2^{(1,2)}/P^{(1,1)}$ is reduced by only 20%. In general, as the incident energy is lowered, R_{32} decreases much faster than R_{21}; the ratio of the second-order effect $P_2^{(1,2)}$ to the first-order probability $P^{(1,1)}$ is nearly independent of energy so that it might be possible to achieve an approximate separation of this effect from other higher order corrections. For sufficiently low bombarding energies, detection of the reorientation effect, which is linear in the quadrupole moment of the excited 2^+ state, may not be seriously complicated by virtual $E2$ excitation of higher states.

Since it is possible to determine the dependence of some of the higher order effects on energy, angle, and charge of the projectile in the bombardment of the same target, even if the transition moments needed for the complete calculation are not available, it should be helpful to have experiments in which the bombarding energy and projectile charge are varied so as to enable the determination of the proportionality constant to be made empirically.[13] A partial step in this direction has been made in the recent measurements of the reorientation effect in Cd^{114} by de Boer *et al.*[14] and Stelson *et al.*[15]

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In this preliminary survey of higher order corrections, particular attention has been paid to virtual $E2$ excitation, since the low-energy transitions in even-even nuclei undergoing collective excitations are known to be predominantly of the electric quadrupole type.[5] However, the possibility of virtual transitions of different multipolarity is not ruled out. Estimates[16] of a second $M1$ or $E4$ transition in the sequences $0^+(E2)2^+(M1)2^+$ and $0^+(E4)4^+(E2)2^+$ indicate that such effects are not expected to be serious in comparison to the $E2$ effects, at least in studies of the reorientation effect in even-even nuclei under conditions of low-energy heavy-ion bombardment. The possibility of second-order $E1$ transitions via the giant dipole resonance has been pointed out by Eichler[17] in connection with the reorientation effect in Cd^{114}. Recent estimates by MacDonald[18] indicate, however, that such effects may not be serious.

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Quadrupole Deformation in Li^7†

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The quadrupole deformation of Li^7 is calculated by generating many-particle wave functions from deformed single-particle orbitals. In order to account for the $E2$ properties of Li^7 and retain the dominance of the basic shell-model configuration, excitation of the $(1s)^4$ core must be included. The admixtures of higher configurations are appreciable.

I. INTRODUCTION

RECENT detailed calculations[1,2] of the electric-field gradients in LiH have led to an accurate value for the quadrupole moment of Li^7, namely $Q(Li^7) = -0.043$ b. van der Merwe showed[3] that this is much greater than one obtains with the usual $(1s)^4(1p)^3$ configuration. He showed that an effective charge of $\frac{1}{2}$ was needed to get the experimental result and also produce agreement with the measured $B(E2)$ strength[4] between the ground state and the first excited state. Recently, Present[5] reproduced the experimental value for $Q(Li^7)$ by mixing ^2P states from $(1p)^3$, $(1p)^2(2p)$, and $(1p)^2(1f)$—leaving the $(1s)^4$ core intact. The intensity found for the $(1p)^3$ component was only 35%.

The method of generator coordinates offers a direct procedure[6] for calculation of such configuration mixtures, starting from single-particle orbitals in a field of quadrupole deformation. It is also simple to include deformation of the $(1s)^4$ core with this method. Such an effect should certainly be included since the experimental value for $Q(Li^7)$ is about 1.8 times the value computed for $Q(Li^7)$ with the $(1s)^4(1p)^3$ configuration. As is seen from Present's results, such a large effect is difficult to obtain by deforming only the $1p$ orbitals. The objective of the calculation is to see whether one can account for the large quadrupole effects while keeping the $(1s)^4(1p)^3$ configuration dominant in Li^7.

II. PROCEDURE

The application of the generating procedure to Li^7 is considerably simplified by the fact that the energy spectrum and magnetic-dipole properties are well described by a model with negligible spin-orbit coupling. It has been shown[3] that for Li^7 the inclusion of spin-orbit coupling within the $1p$ shell increases the quadrupole moment not more than 10% above its value at the Wigner supermultiplet limit

$$Q(Li^7, {}^2P[4+3]) = -(6/25)e\langle r^2\rangle_{1p,1p}. \quad (1)$$

Here $\langle r^2\rangle_{1p,1p}$, the expectation value of r^2 evaluated with $1p$ radial functions, has a magnitude $\langle r^2\rangle_{1p,1p} \approx 10^{-25}$ cm^2. Therefore the desired enhancement must come from outside the $1s$ and $1p$ shells, and thus is most

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