Small-Angle Scattering of Neutrons by Nuclei

ROBERT F. REDMOND
Battelle Memorial Institute, Columbus Laboratories, Columbus, Ohio
(Received 16 June 1965; revised manuscript received 12 August 1965)

The partial-wave phase shifts for the scattering of neutrons by nuclei are considered with some emphasis on large values of orbital angular momentum l. It is shown that a rather simple general approximation technique for obtaining the phase shifts follows from the scattering integral equation for the partial-wave amplitudes. This method is used to study the electromagnetic scattering of neutrons by nuclei. It is shown that the first-order approximation for the phase shifts gives a small but non-negligible correction to the Born approximation (zero order) even for large l. However, it is shown that higher order approximations tend to reduce the correction significantly. The interference effects between the electromagnetic scattering and the nuclear scattering are also considered. For large l the scattering tends to be dominated by the electromagnetic scattering. However it is found that if the nuclear spin-orbit interaction is sufficiently strong then the nuclear scattering can become important even for large l. It is suggested that this effect may be significant in explaining the anomalous small-angle scattering of neutrons by Th, U, and Pu nuclei.

1. INTRODUCTION

THE scattering of neutrons by nuclei is influenced by the long-range electromagnetic interaction between a neutron and a scattering nucleus as well as by the short-range nuclear interaction.[1] The electromagnetic interaction is of spin-orbit form and can give rise to large polarization effects as well as large scattering cross sections at small scattering angles.[1] A part of the nuclear interaction is also of spin-orbit form. Thus, it is desirable to study the simultaneous effect of the nuclear and the electromagnetic interactions in order to interpret neutron scattering and polarization data. Indeed it appears to be not only desirable but necessary in order to interpret polarization data for scattering angles as large as 24 deg.[2]

In a recent paper Monahan and Elwyn[2] (hereafter referred to as ME) considered the influence of electromagnetic scattering upon the polarization of neutrons scattered by nuclei. An important result of their paper was the derivation of an approximate formula for the phase shifts which includes the nuclear potential as well as the electromagnetic interaction. In Sec. 2 this formalism is summarized and made somewhat stronger. It is then shown that a generalization of the approximation scheme given by ME can easily be made and that the result should be generally useful in scattering problems.

In Sec. 3 the formalism of Sec. 2 is applied to the electromagnetic scattering of neutrons. In particular the behavior of the phase shifts for large angular momentum l is studied. It is shown that the integrals for the first-order approximation can easily be evaluated explicitly. With this result the behavior of the phase shifts for large l is then established. It is shown that the first-order approximation for the phase shifts gives a non-negligible correction to the Born approximation (zero order) even for large l. However, it is also shown, using the more general approximation technique of Sec. 2, that this first-order correction to the Born approximation tends to disappear in higher order approximation.

In Sec. 4 the interference effects from nuclear scattering are discussed with emphasis upon the phase shifts for large l. The behavior of the phase shifts for the nuclear interaction with a spin-orbit term is studied. Using a simple model for the nuclear interaction it is shown that for a sufficiently strong nuclear spin-orbit interaction the phase shifts may be significant for large l. In particular a scattering resonance can occur for a large value of l ($j=l+\frac{1}{2}$). It is then discussed how the consequences of this unusual situation can perhaps account for the anomalous small-angle scattering observed with Th, U, and Pu nuclei (first observed by Aleksandrov[3]).

2. SCATTERING INTEGRAL EQUATION

Following ME consider the scattering of neutrons from a spherically symmetric potential $V(r)$ (with possibly spin-orbit terms) which can be written in the form

$$V(r)=V_1(r)+V_2(r)\,, \tag{1a}$$

where

$$V_1(r)=0\,,\quad r>r_c\,, \tag{1b}$$

$$V_2(r)=0\,,\quad r<r_c\,. \tag{1c}$$

The "cutoff" radius r_c is chosen such that $V_2(r)$ can be treated as a perturbation. The following integral equation for the partial waves $\Psi_{lj}(r)$ was obtained in ME:

$$\Psi_{lj}(r)=j_l(kr)\left[A_{lj}(r_c)-k\int_{r_c}^{r}dx\,x^2U_{lj}(x)n_l(kx)\Psi_{lj}(x)\right]$$
$$+n_l(kr)\left[B_{lj}(r_c)+k\int_{r_c}^{r}dx\,x^2U_{lj}(x)j_l(kx)\Psi_{lj}(x)\right],\tag{2}$$

[1] J. Schwinger, Phys. Rev. 73, 407 (1948).
[2] J. E. Monahan and A. J. Elwyn, Phys. Rev. 136, B1678 (1964).
[3] Y. A. Aleksandrov, Zh. Eksperim. i Teor. Fiz. 33, 294 (1957) [English transl.: Soviet Phys.—JETP 6, 228 (1958)].

where U_{lj} is the potential $2m\hbar^{-2}V_2(r)$ acting on the radial partial wave $\Psi_{lj}(r)$, m is the neutron mass, and

$$\Psi_{lj}(r_c)=A_{lj}(r_c)j_l(kr_c)+B_{lj}(r_c)n_l(kr_c) \quad (3)$$

is the value of the wave function $\Psi_{lj}(r)$ at the cutoff radius.

A first-order approximation for the phase shifts was obtained in ME by use of the approximation

$$\Psi_{lj}(r)=A_{lj}(r_c)j_l(kr)+B_{lj}(r_c)n_l(kr), \quad r>r_c, \quad (4)$$

in the integrals of Eq. (2). Thus

$$\tan\delta_{lj}=\frac{(1+b_{lj})\tan\xi_{lj}-a_{lj}}{1-b_{lj}+c_{lj}\tan\xi_{lj}}, \quad (5)$$

where

$$\tan\xi_{lj}=-B_{lj}(r_c)/A_{lj}(r_c), \quad (6)$$

and

$$a_{lj}=k\int_{r_c}^{\infty}dx\, x^2U_{lj}(x)[j_l(kx)]^2. \quad (7)$$

The coefficients b_{lj} and c_{lj} are similarly defined by replacing $[j_l]^2$ in Eq. (7) with $[j_ln_l]$ and $[n_l]^2$, respectively. At this point it is worth noting (ME) that the Born approximation for the phase shifts is $\tan\delta_{lj}=-a_{lj}$.

It is then pointed out in ME that since the coefficients a_{lj}, b_{lj}, and c_{lj} in Eq. (5) depend only on the interaction $V_2(r)$, the phase shifts ξ_{lj} contain all the effect of the short-range potential required to determine $\tan\delta_{lj}$. It is also stated in ME that this is true of all higher approximations obtained by iteration of Eq. (2). This argument can be strengthened somewhat as follows. It is, of course, clear that the approximation Eq. (4) is exact if $V_2(r)=0$, $r\geq 0$, and that the ξ_{lj} are the phase shifts determined by the potential $V_1(r)$. However, even without approximation it follows from Eq. (2) that

$$\Psi_{lj}'(r_c)=A_{lj}(r_c)j_l'(kr_c)+B_{lj}(r_c)n_l'(kr_c), \quad (8)$$

where the primes denote differentiation with respect to r. Hence it follows from Eqs. (3) and (8) that $\tan\xi_{lj}$ is determined by the logarithmic derivative of Ψ_{lj} at $r=r_c$ but this is determined only by the potential $V_1(r)$. Thus the phase shifts ξ_{lj} are just those obtained with $V_2(r)=0$, $r\geq 0$. Further, from Eq. (2) the function $\Psi_{lj}(r)/A_{lj}$ is determined by $\tan\xi_{lj}$ and U_{lj} while $\tan\delta_{lj}$ is determined by $\tan\xi_{lj}$, U_{lj}, and $\Psi_{lj}(r)/A_{lj}$. Hence, except for the normalization of $\Psi_{lj}(r)$ the solution is completely determined by $\tan\xi_{lj}$ and U_{lj} for $r>r_c$. The normalization is fixed by the incident-plane-wave normalization.

Higher-order approximations for $\Psi_{lj}(r)$ can be obtained from Eq. (2) by iteration; i.e., Born expansion. Even in second order this iteration technique is very cumbersome when the potential $V_2(r)$ is nonzero over a large volume. However, a different type of iteration is suggested by Eq. (2). Thus, select a finite number of radii $r_c=r_0<r_1<\cdots<r_n=\infty$ and approximate the wave function in each region by

$$\Psi_{lj}^i(r)=A_{lj}(r_{i-1})j_l(kr)+B_{lj}(r_{i-1})n_l(kr), \quad r_{i-1}\leq r\leq r_i. \quad (9)$$

The first-order perturbation theory used to obtain Eq. (5) now gives

$$\tan\delta_{lj}(i)=\frac{[1+b_{lj}(i)]\tan\delta_{lj}(i-1)-a_{lj}(i)}{1-b_{lj}(i)+c_{lj}(i)\tan\delta_{lj}(i-1)}, \quad i=1,\cdots,n, \quad (10)$$

where $\tan\delta_{lj}(0)=\tan\xi_{lj}$ and the coefficients $a_{lj}(i)$, etc., are given by integrals of the form of Eq. (7) with $r_{i-1}(r_i)$ as the lower (upper) limit of integration. Clearly $\tan\delta_{lj}(n)$ will generally be a better approximation for $\tan\delta_{lj}$ than that given by Eq. (5). It is also clear that the approximation $\tan\delta_{lj}(n)$ given by Eq. (10) can be made as accurate as desired by selecting a sufficiently dense set $\{r_i\}$. The advantage of this method of higher order approximation is that only single integrations are involved and the choice of the r_i allows one to improve the approximation in the r range where it is most desired. The utility of this method will be shown later when it is applied to the electromagnetic scattering problem. However, the method should also be useful in other scattering problems.

In either approximation scheme, Eq. (5) or Eq. (10), the cutoff radius r_c can be chosen with some arbitrariness. A larger cutoff radius improves the perturbation approximation, since more of the interaction $V(r)$ is handled exactly, but it also increases the amount of numerical work required to calculate $\tan\xi_{lj}$.

3. ELECTROMAGNETIC SCATTERING

The potential $V_2(r)$ due to the electromagnetic interaction between an incident neutron and a scattering nucleus is given by[1] V_e where

$$U_e=2m\hbar^{-2}V_e=2\gamma r^{-3}\mathbf{l}\cdot\boldsymbol{\sigma}, \quad r>r_c\geq R. \quad (11)$$

And where $\gamma=\frac{1}{2}\mu_nZ(e^2/\hbar c)(\hbar/mc)$, μ_n is the neutron magnetic moment in nuclear magnetons, Ze is the nuclear charge and R is the nuclear radius.

Schwinger[1] first studied the electromagnetic scattering of neutrons and obtained the well-known Born approximation for the scattering amplitude

$$f_e(\theta)=-i\gamma\boldsymbol{\sigma}\cdot\mathbf{n}\cot\tfrac{1}{2}\theta, \quad (12)$$

where $\mathbf{k}_0\times\mathbf{k}=\mathbf{n}k^2\sin\theta$ and $\hbar\mathbf{k}_0$ ($\hbar\mathbf{k}$) is the incident (scattered) momentum vector of the neutron. Schwinger then treated $f_e(\theta)$ as a perturbation to the nuclear scattering amplitude $f_0(\theta)$ to obtain the total scattering amplitude $f(\theta)=f_0(\theta)+f_e(\theta)$. In order to estimate the polarization, Schwinger considered both hard-sphere and black-nucleus approximations for $f_0(\theta)$.

Sample[4] treated the potential V_e as a first-order perturbation in the Schrödinger equation; hard-sphere scattering was assumed to obtain the zero-order wave function. Sample's method is not identical with the method employed in ME (allowing for the different zero-order approximation used) and hence these two first-order approximations are not directly comparable. However, Sample's method does not appear to be as direct as the method of ME when more general zero-order wave functions are employed.

Baz[5] first used an optical-model potential to obtain zero-order wave functions (a square-well complex potential was used). The potential V_e as well as the diffuseness of the nuclear potential were then treated as a first-order perturbation (the details of this calculation are not given in Baz's paper).

Monahan and Elwyn[2] (ME) have given the problem its most general treatment. As already noted their method allows the zero-order wave function to include a nuclear spin-orbit interaction in addition to other short-range interactions. The work of ME is therefore used as a starting point in what follows.

For the potential V_e given by Eq. (11) the coefficients a_{lj}, etc., of Eq. (7) are given by

$$a_{lj}=2\gamma k\beta_{lj}\int_{r_c}^{\infty}dx\,x^{-1}[j_l(kx)]^2, \tag{13a}$$

$$b_{lj}=2\gamma k\beta_{lj}\int_{r_c}^{\infty}dx\,x^{-1}j_l(kx)n_l(kx), \tag{13b}$$

$$c_{lj}=2\gamma k\beta_{lj}\int_{r_c}^{\infty}dx\,x^{-1}[n_l(kx)]^2, \tag{13c}$$

where $\beta_{lj}=l$, $j=l+\frac{1}{2}$, and $\beta_{lj}=-l-1$, $j=l-\frac{1}{2}$. In ME it is pointed out that these integrals can be evaluated by a recursion relation given by Watson[6] (this was also the procedure followed by Sample[4]). However, as is shown in Appendix A these integrals can be evaluated explicitly by using a relation also given by Watson.[7] This result is very useful in applying Eqs. (5) and (10), particularly for large l. Thus, using the results from Appendix A, the coefficients a_{lj}, etc., are given explicitly as

$$a_{lj}=D_{lj}\{1-z_c^2[(1-lz_c^{-2})j_l^2(z_c)+j_{l-1}^2(z_c)-2lz_c^{-1}j_l(z_c)j_{l-1}(z_c)]\}, \tag{14a}$$

$$b_{lj}=-D_{lj}z_c^2\{(1-lz_c^{-2})j_l(z_c)n_l(z_c)+j_{l-1}(z_c)n_{l-1}(z_c)-[j_l(z_c)n_{l-1}(z_c)+j_{l-1}(z_c)n_l(z_c)]lz_c^{-1}\}, \tag{14b}$$

$$c_{lj}=D_{lj}\{1-z_c^2[(1-lz_c^{-2})n_l^2(z_c)+n_{l-1}^2(z_c)-2lz_c^{-1}n_l(z_c)n_{l-1}(z_c)]\}. \tag{14c}$$

[4] J. T. Sample, Can. J. Phys. 34, 36 (1956).
[5] A. I. Baz, Zh. Eksperim. i Teor. Fiz. 31, 831 (1956) [English transl.: Soviet Phys.—JETP 4, 704 (1957)].
[6] G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, London, 1944), 2nd ed., p. 136.
[7] See Ref. 6, pp. 447, 137.

In these expressions $z_c=kr_c$ and $D_{lj}=k\gamma\beta_{lj}/l(l+1)$. These formulas are very useful in determining the asymptotic behavior of the coefficients for large l which controls the small-angle scattering. Thus, asymptotically for large l ($l\gg z_c$)

$$a_{lj}\sim D_{lj}, \tag{15a}$$

$$b_{lj}\sim -D_{lj}lz_c^{-1}, \tag{15b}$$

$$c_{lj}\sim D_{lj}l[(2l-1)!!]^2z_c^{-2l-2}. \tag{15c}$$

As already noted, $-a_{lj}$ is just the Born approximation for $\tan\delta_{lj}$ and one expects this to dominate in Eq. (5) for large l. Indeed it can be shown by the summation technique of Sample[4] that the a_{lj} given by Eq. (15a) do in fact give the Born-approximation result, Eq. (12), for small θ.

Contrary to the statement made in ME, the coefficient b_{lj} can lead to a non-negligible correction to $\tan\delta_{lj}$ in Eq. (5) even for large l since by Eq. (15b)

$$b_{lj}\sim\gamma\mp r_c^{-1},\quad j=l\pm\tfrac{1}{2}. \tag{16}$$

For uranium $b_{lj}\sim\mp 0.017Rr_c^{-1}$ which is small but not negligible if $r_c\approx R$. Thus, the first-order approximation of Eq. (5) can modify the Born approximation for $\tan\delta_{lj}$ even when l is large. It will be seen, however, that the higher order approximation given by Eq. (10) suppresses the correction to $\tan\delta_{lj}$ obtained from Eq. (16). If the cutoff radius is made large enough in Eq. (5) it is also clear that the correction from Eq. (16) can be made negligible. However, in this case some of the effect is transferred to the $\tan\xi_{lj}$ terms in Eq. (5) and also the numerical work required to determine $\tan\xi_{lj}$ is increased.

A more accurate estimate for the phase shifts can be obtained from Eq. (10) as shown in Appendix B. Thus, if the $\tan\xi_{lj}$ terms are neglected

$$\tan\delta_{lj}\approx -a_{lj}-4\gamma k\beta_{lj}\int_{r_c}^{\infty}dx\,x^{-1}[j_l(kx)]^2b_{lj}(x) \tag{17}$$

where $b_{lj}(x)$ is defined by Eq. (13b) with x replacing r_c as the lower limit of integration. Since most of the value of a_{lj} in Eq. (13a) comes from the range of integration where $kx>l$ it can be shown from Eq. (17) and the behavior of $b_{lj}(x)$ that the first-order estimate

$$\tan\delta_{lj}\approx -a_{lj}(1-b_{lj})^{-1} \tag{18}$$

overcorrects the Born estimate $\tan\delta_{lj}\sim -a_{lj}$ to some extent. And in fact, the Born estimate for large l is better than the first-order estimate.

When the phase shifts $\tan\xi_{lj}$ are not negligible but $|c_{lj}\tan\xi_{lj}|\ll 1$, then Eq. (B7) can be used to obtain a more accurate estimate than Eq. (5) for the phase shifts. If the phase shifts are not small (in the above sense) then a more accurate estimate than Eq. (5) can be based upon the direct application of Eq. (10).

If $kR\ll l$ and if $\tan\xi_{lj}$ is near the hard-sphere value, then the l-wave scattering is dominated by the electromagnetic scattering and the l-wave phase shift is given by $\tan\delta_{lj}\sim -a_{lj}$. The Born approximation for the potential of Eq. (11) is actually $f_e(\theta)(KR)^{-1}\sin KR$, where $K=2k\sin\frac{1}{2}\theta$ and $f_e(\theta)$ is given by Eq. (12). Thus, $f_e(\theta)$ of Eq. (12) is the leading term from the Born approximation when $KR\ll 1$. The electromagnetic potential for $r<R$ is not very important. If U_e for $r<R$ is approximated by $2\gamma R^{-3}\mathbf{l}\cdot\boldsymbol{\sigma}$ then the scattering amplitude due to U_e for $r>0$ is given in Born approximation when $KR\ll 1$ by

$$f_e(\theta)=-i\gamma\boldsymbol{\sigma}\cdot\mathbf{n}[\cot\tfrac{1}{2}\theta-\tfrac{1}{5}(kR)^2\sin\theta], \qquad (19)$$

where the last term inside the brackets results from U_e for $r<R$ plus the second term in the expansion of $(KR)^{-1}\sin KR$. However, the $\sin\theta$ term in Eq. (19) is small compared with the term of similar form which results from the typical nuclear spin-orbit interaction. An indication of this can be seen from the Born approximation for the nuclear spin-orbit interaction of the form $-V_{so}R\delta(r-R)\mathbf{l}\cdot\boldsymbol{\sigma}$ for $KR\ll 1$; i.e.,

$$f_{so}=(iR/3)\alpha(kR)^2\sin\theta\boldsymbol{\sigma}\cdot\mathbf{n}, \qquad (20)$$

where $\alpha=2mV_{so}R^2\hbar^{-2}$ and typically α is of order unity. Thus, even for uranium the nuclear spin-orbit term is several orders of magnitude greater than the $\sin\theta$ term of Eq. (19).

4. INTERFERENCE WITH NUCLEAR SCATTERING

When the $\tan\xi_{lj}$ terms are not negligible in Eq. (5) or Eq. (10), interference effects between electromagnetic and nuclear scattering are expected. The $\tan\xi_{lj}$ will normally be negligible for $l\gg z_c$ unless $\tan\xi_{lj}$ is near a resonance. Generally, $\tan\xi_{lj}$ will not have a resonance if $l>KR$ where K is the wave number characteristic of the internal region $r<R$; e.g., for a real square-well potential $(V=-V_0, r<R)K^2=2mV_0\hbar^{-2}+k^2$. A possible exception to this which is of some interest may occur when the nuclear spin-orbit interaction is sufficiently strong. This will be discussed after the next paragraph.

When $|c_{lj}\tan\xi_{lj}|\ll 1$ the nature of the interference is apparent from Eqs. (B7) and (B8). When $\tan\xi_{lj}$ is not this small the interference is more complicated. In this regard Eq. (5) would be expected to give a poor estimate when $\tan\xi_{lj}$ is not this small. Again the violation of the inequality $|c_{lj}\tan\xi_{lj}|\ll 1$ will require $\tan\xi_{lj}$ to be near a resonance. As an extreme example if $\tan\xi_{lj}$ is at resonance, Eq. (5) indicates that $\tan\delta_{lj}=(1+b_{lj})/c_{lj}$. However, for $z_c\ll l$, c_{lj} can be quite large [see Eq. (15c)] and thus $\tan\delta_{lj}$ from Eq. (5) can be quite small. This, of course, is an erroneous result as can be shown by an application of Eq. (10).

As indicated above a resonance for $\tan\xi_{lj}$ may occur for a large l value $(l\gg KR)$ if the nuclear spin-orbit interaction is sufficiently strong. This can most readily be seen by considering a simple model for the nuclear interaction. Let the nuclear interaction be given by $V(r)$, where

$$V(r)=-V_0(r)-V_{so}R\delta(r-R)\mathbf{l}\cdot\boldsymbol{\sigma} \qquad (21)$$

and $V_0(r)=V_0$, $r<R$, $(=0, r>R)$. The phase shifts for this scattering potential can be written exactly[8] as

$$\tan\xi_{lj}=\frac{\tan\xi_{lj}^0+\alpha\beta_{lj}xj_l(x)[j_l(x)-\tan\xi_{lj}^0n_l(x)]}{1+\alpha\beta_{lj}xn_l(x)[j_l(x)-\tan\xi_{lj}^0n_l(x)]}, \qquad (22)$$

where ξ_{lj}^0 is the phase shift for the potential $V_0(r)$. This result can be obtained by means of the Wronskian theorem[8] or by means of Eq. (2).

Let $x=kR$, $X_0^2=2mV_0R^2\hbar^{-2}$, and $X^2=X_0^2+x^2$. Then for $x\ll l$ and X away from where $j_{l-1}(X)=0$, $\tan\xi_{lj}^0\approx j_l(x)/n_l(x)+A$ where $A=j_l(X)[Xj_{l-1}(X)xn_l^2(x)]^{-1}$. With this approximation Eq. (22) gives

$$\tan\xi_{lj}\approx\frac{j_l(x)/n_l(x)+A[1-\alpha\beta_{lj}xj_l(x)n_l(x)]}{1-\alpha\beta_{lj}j_l(X)/Xj_{l-1}(X)}. \qquad (23)$$

It is clear that $\tan\xi_{lj}$ will have a resonance when

$$Xj_{l-1}(X)/j_l(X)=\alpha\beta_{lj}. \qquad (24)$$

If $X\ll l$ then $Xj_{l-1}(X)/j_l(X)\sim 2l+1$ and Eq. (24) indicates a resonance for $j=l+\frac{1}{2}$ if $l=(\alpha-2)^{-1}$, while no resonance can occur for $j=l-\frac{1}{2}$. Thus a resonance can occur for $j=l+\frac{1}{2}$ at a large l value if $\alpha\approx 2$. Further a resonance of this type may occur for $l>X$ if $0<\alpha\leq 2+l^{-1}$ since

$$0<Xj_{l-1}(X)/j_l(X)<2l+1, \quad l>X>1. \qquad (25)$$

For example, if $l=X$ then $\tan\xi_{lj}$ will be resonant when $\alpha_0=j_{l-1}(l)/j_l(l)$ which gives the resonant combinations (l,α_0): (6,1.66); (8,1.61); (10,1.55); (12,1.51); and (13,1.49). In each case the resonant value of l will be increased (keeping X fixed) as α is increased from α_0 to 2. It might be noted here that α values inferred from scattering data[9] are typically 0.5–1.5.

The shifting of the resonances of $\tan\xi_{lj}$, $j=l+\frac{1}{2}$, to large l values when the nuclear spin-orbit interaction is strong (i.e., $1.5<\alpha<2$) may have some significance with regard to the small-angle scattering of neutrons. In particular, the anomalously large small-angle scattering of neutrons from Th, U, and Pu targets (observed first by Aleksandrov[3]) might be explained by this effect. In this connection, for $l\gg X$ Eq. (23) has the asymptotic form

$$\tan\xi_{lj}\sim\alpha\beta_{lj}x^{2l+1}[(2l+1)!!]^{-2}[1-\alpha\beta_{lj}(2l+1)^{-1}]^{-1}. \qquad (26)$$

From this expression it is clear that if $\tan\xi_{lj}$ is near

[8] R. F. Redmond, Phys. Rev. **136**, B112 (1964); **139**, AB1(E) (1965).

[9] P. A. Moldauer, Nucl. Phys. **47**, 65 (1963); E. H. Auerbach and S. O. Moore, Phys. Rev. **135**, B895 (1964).

resonance for a large l value there will be a strong energy dependence from the x^{2l+1} factor. Further, the partial waves for nearby l values will also be important and their importance should increase with increasing neutron energy. At small scattering angles these partial waves interfere constructively while for larger angles they tend to interfere destructively. Thus, the scattering cross section would tend to be peaked at the small scattering angles and increasingly so with increasing neutron energy.

The value for the α parameter is rather crucial for the occurrence of a resonance at a large l value. Typically the α parameter has a value near 1 as determined by optical-model parameter fits to scattering data.[9] However, unless polarization data are included the data fittings are not very sensitive to the spin-orbit interaction strength. (The interference effects from electromagnetic scattering appear to be important in the interpretation of the polarization data.[2]) It might be noted that Auerbach and Moore[9] found for U^{238} a spin-orbit parameter comparable to $\alpha=1.5$, although they did caution against drawing conclusions from this finding and further they did not find as large a value for Th.

Several complicating features will tend to smear out the resonance effect noted above. The spin-orbit potential has some spatial extent and the spherically symmetric potential is not appropriate for the deformed nuclei (such as Th, U, and Pu). In addition, a more realistic complex optical-model potential will modify the resonance behavior to some extent. Nevertheless, one might expect that the resonance-like behavior for large l will still be evident if the spin-orbit interaction is sufficiently strong.

ACKNOWLEDGMENTS

The author is grateful to Dr. E. M. Baroody for helpful comments regarding the manuscript and to Battelle Memorial Institute for encouragement and financial support.

APPENDIX A. INTEGRAL EVALUATION

The following integral is given by Watson[7]:

$$(4\nu^2-1)\int_x^\infty C_\nu^2(t)t^{-2}dt$$
$$=4\pi^{-1}(a^2+b^2)-x\{[x^{-1}C_\nu(x)+C_\nu'(x)]^2$$
$$+2(1-\nu^2x^{-2})C_\nu^2(x)+C_\nu'^2(x)\}, \quad \text{(A1)}$$

where $C_\nu=aJ_\nu(x)+bN_\nu(x)$ with a and b arbitrary constants. This relation can be put in terms of spherical Bessel functions by means of the definitions $j_l(x)=(\pi/2x)^{1/2}J_{l+1/2}(x)$ and $n_l(x)=(\pi/2x)^{1/2}N_{l+1/2}(x)$. The expressions for a_{lj} and c_{lj} given by Eqs. (14a) and (14c) can be obtained from Eq. (A1) by taking ($a=1$, $b=0$) and ($a=0$, $b=1$), respectively, and by using some Bessel function identities. The relationship for b_{lj} given by Eq. (14b) can be obtained from Eq. (A1) by letting $C_\nu=J_\nu(x)+N_\nu(x)$ and then subtracting the contribution of $J_\nu^2(x)+N_\nu^2(x)$ to the integral.

APPENDIX B. PHASE-SHIFT FORMULA

When the terms a_{lj}, b_{lj}, $\tan\xi_{lj}$, and $c_{lj}\tan\xi_{lj}$ are sufficiently small in magnitude it is possible to apply Eq. (10) to obtain an accurate integral formula for the phase shifts. Thus, let $|a_{lj}|\ll1$, $|b_{lj}|\ll1$, $|\tan\xi_{lj}|\ll1$ and $|c_{lj}\tan\xi_{lj}|\ll1$. For convenience, in using Eq. (10) let $a_i=a_{lj}(i)$, etc., and $\tan\delta_i=\tan\delta_{lj}(i)$. Then as a good approximation

$$\tan\delta_i=-A_i+B_i\tan\delta_{i-1}-C_i\tan^2\delta_{i-1}, \quad \text{(B1)}$$

where $A_i=a_i(1+b_i)$, $B_i=(1+2b_i+a_ic_i)$, and $C_i=c_i\times(1+b_i)$. Starting with $i=1$ in Eq. (B1) the calculation of $\tan\delta_n$ over some set $\{r_i\}$, $i=1,\cdots,n$ can be carried out. Since the set $\{r_i\}$ can be made arbitrarily dense, terms in the expression for $\tan\delta_n$ which involve products of A's and C's become negligible. For example,

$$\tan\delta_1=-A_1+B_1\tan\delta_0-C_1\tan^2\delta_0, \quad \text{(B2)}$$

and

$$\tan\delta_2=-A_2+B_2[-A_1+B_1\tan\delta_0-C_1\tan^2\delta_0]$$
$$-C_2[-A_1+B_1\tan\delta_0-C_1\tan^2\delta_0]^2. \quad \text{(B3)}$$

Since A_1, A_2, C_1, and C_2 can be made arbitrarily small by choosing r_1-r_0 and r_2-r_1 sufficiently small to a good approximation, Eq. (B3) gives

$$\tan\delta_2=-A_2-A_1B_2+B_1B_2\tan\delta_0$$
$$-(B_2C_1+B_1^2C_2)\tan^2\delta_0. \quad \text{(B4)}$$

Continuing in this fashion, one obtains

$$\tan\delta_n=-\sum_{i=1}^n A_i\prod_{k=i+1}^n B_k+\prod_{k=1}^n B_k\tan\delta_0$$
$$-\sum_{i=1}^n C_i\prod_{k=1}^{i-1}B_k^2\prod_{s=i+1}^n B_s\tan^2\delta_0. \quad \text{(B5)}$$

In the product expressions of Eq. (B5), if the upper limit is less than the lower limit the product is interpreted as unity. The product expressions can be written as a good approximation as

$$\prod_{k=1}^n B_k\approx1+2\sum_{k=1}^n b_k=1+2b_{lj}, \quad \text{etc.} \quad \text{(B6)}$$

In this way one obtains finally

$$\tan\delta_{lj}=-a_{lj}-4\gamma k\beta_{lj}\int_{r_c}^\infty dx\, x^{-1}[j_l(kx)]^2b_{lj}(x)$$
$$+(1+2b_{lj})\tan\delta_0+\{-c_{lj}(1+4b_{lj})$$
$$+4\gamma k\beta_{lj}\int_{r_c}^\infty dx\, x^{-1}[n_l(kx)]^2b_{lj}(x)\}\tan^2\delta_0, \quad \text{(B7)}$$

where $b_{lj}(x)$ in the integrals is defined by Eq. (13b) with r_c replaced by x in the lower limit of integration. The expression Eq. (B7) should be compared with the analogous expression obtained from Eq. (5):

$$\tan\delta_{lj} \approx -a_{lj}(1+b_{lj})+(1+2b_{lj}+a_{lj}c_{lj})\tan\delta_0 - c_{lj}(1+b_{lj})\tan^2\delta_0. \quad \text{(B8)}$$

The limitations put on the magnitudes of the parameters of Eq. (10) to obtain Eq. (B7) are probably more restrictive than necessary and can probably be relaxed. The more general restriction appears to be that $\tan\delta(i)$ should not pass through a resonance at any step of the recursion process for solving Eq. (10). Of course, if this more general restriction is violated it is still possible to use Eq. (10) in difference equation form. However, in this instance it is evident that the points r_k where $\tan\delta(k)$ passes through resonance give some difficulty, i.e., that the set $\{r_i\}$ must be dense at r_k. Also it is clear that the accuracy of this method will then depend on the accuracy in locating the largest r_i where a zero for $\tan\delta(i)$ occurs and perhaps the largest r_i where a resonance occurs. Thus, if a zero occurs at r_k [$\tan\delta(k)=0$] and no zeros or resonances occur for $r_i>r_k$, then only the potential for $r>r_k$ contributes to the scattering, but that part nearest to r_k will frequently be the most important. If, however, a resonance occurs for $r_i>r_k$ then the potential beyond r_i as well as the actual location of r_i becomes of greatest importance.

PHYSICAL REVIEW VOLUME 140, NUMBER 5B 6 DECEMBER 1965

Calculation of the (π^-, π^0) Reaction on Complex Nuclei*

SHELDON B. KAUFMAN AND CHARLES O. HOWER
Department of Chemistry and Princeton-Pennsylvania Accelerator, Princeton University, Princeton, New Jersey

(Received 3 May 1965; revised manuscript received 12 August 1965)

The excitation function for the reaction ${}_ZA(\pi^-,\pi^0)_{Z-1}A$ has been calculated using the Fermi-gas model of the nucleus and the impulse approximation. Experimental data on the differential cross section of the free-particle reaction $\pi^-+p\rightarrow\pi^0+n$ was used, and the effect of the momentum distribution of the nucleons on the kinematics was included. The predicted cross section shows a minimum near 200 MeV and maxima near 100 and 350 MeV. This structure, if confirmed experimentally, would lend support to this simple model, which is frequently used to interpret high-energy nuclear reactions.

INTRODUCTION

MANY features of high-energy nuclear reactions can be successfully predicted by calculations based on a simple model: the nucleus is represented by a degenerate Fermi gas of protons and neutrons with which the incident particle interacts. The impulse approximation is assumed to hold, and free-particle cross sections are used. Extensive Monte Carlo calculations[1,2] have been performed using essentially this model and have given good agreement with experiment, with the notable exception of those for simple nuclear reactions such as the (p,pn) reaction. It is possible to treat such reactions in explicit calculations without resorting to the Monte Carlo method, as was done by Benioff,[3] who used a shell model with harmonic-oscillator wave functions to calculate (p,pn) cross sections. Ericson, Selleri, and Van de Walle[4] have calculated the excitation function for (p,$p\pi^+$) reactions, using a Fermi-gas model of the nucleus, and Remsberg[5] has improved their calculation and obtained good agreement with experiment.

The present calculation has been done in order to extend this model to pion-induced reactions. One of the simplest of such reactions is charge exchange:

$$\pi^\pm + {}_ZA \rightarrow \pi^0 + {}_{Z\pm1}A\,. \quad (1)$$

According to the model under discussion, reaction (1) occurs by a charge exchange involving a single nucleon[6]

$$\pi^- + p \rightarrow \pi^0 + n\,, \quad (2)$$

in which the neutron remains in the nucleus. In order that the final nucleus not evaporate any particles, its excitation energy must be low, and therefore the momentum transfer to the nucleon must also be low. The incident and outgoing pions are required not to undergo any interactions with other nucleons, since in that case the nucleus would gain too much excitation energy. Using experimental values for the cross section of reaction (2) with the limitations imposed by the low energy transfer to the nucleus, we can calculate the cross section for reaction (1).

* Work supported by the U. S. Atomic Energy Commission.
[1] N. Metropolis, R. Bivins, M. Storm, A. Turkevich, J. M. Miller, and G. Friedlander, Phys. Rev. **110**, 185 (1958); **110**, 204 (1958).
[2] H. Bertini, Phys. Rev. **131**, 1801 (1963).
[3] P. Benioff, Phys. Rev. **119**, 324 (1960).
[4] T. Ericson, F. Selleri, and R. T. Van de Walle, Nucl. Phys. **36**, 353 (1962).
[5] L. Remsberg, Phys. Rev. **138**, B572 (1965).
[6] In the following we consider incident π^- mesons; the calculation also holds for π^+ mesons if Z is replaced by N.