TABLE I. Values of K^0-K^- mass difference for individual events.

Event number	K^0-K^- mass difference (MeV)
309 941	5.37±1.12
309 691	3.60±0.69
359 964	2.44±3.13
368 562	3.92±0.64
372 533	4.08±1.12
372 823	3.95±1.24
487 277	2.25±0.87
Weighted average	$\Delta=3.71\pm0.35$ (MeV)

average is $\Delta=3.71\pm0.35$ MeV. Combining this result with the assumed value[2] of the K^- mass

$$m_{K^-}=493.8\pm0.15 \text{ MeV}$$

gives $m_{K^0}=497.51\pm0.41$ MeV.

The determination of the mass from these events depends upon the K^- and pion momenta. Since both the $\bar{p}$ and K^- exposures were made in the same chamber, the momenta were corrected in the same manner as discussed in Sec. II. The 0.1% momentum uncertainty results in less than 0.01-MeV change in the K^0 mass.

[2] P. Schmidt, preceding paper, Phys. Rev. **140**, B1328 (1965).

The proton momentum is determined from a range-momentum table which has a scale factor known to 0.5%. The mass change due to this uncertainty is 0.03 MeV.

Both of our results are in agreement with the values of the mass obtained by Burnstein and Rubin,[3] $m_{K^0}=497.70\pm0.32$ MeV, and the value given in the compilation by Rosenfeld *et al.*,[4] $m_{K^0}=498.0\pm0.5$ MeV. The weighted average of these four values is $\bar{m}_{K^0}=497.62\pm0.18$ MeV.

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[3] R. A. Burnstein and H. A. Rubin, Phys. Rev. **138**, B895 (1965).
[4] A. H. Rosenfeld, A. Barbaro-Galtieri, W. H. Barkas, P. L. Bastien, J. Kirz, and M. Ross, Rev. Mod. Phys. **36**, 977 (1964).

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Faddeev Formalism for Four-Particle Systems*

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From the four-particle Lippmann-Schwinger equation a set of equations of the Faddeev type with properly connected kernels has been obtained. The kernels contain only the two- and three-particle scattering amplitudes, and do not involve the potentials.

I. INTRODUCTION

RECENTLY, there has been considerable interest in the three-particle problem[1–5] and in particular the Faddeev equation[1] for the three-body scattering amplitudes. The main advantages of the Faddeev approach are (1) only the off energy shell two-body scattering amplitudes rather than the potentials occur in the kernel, and (2) the square of the kernel is completely connected, consequently the kernel is likely to be square integrable since the troublesome delta functions arising from disconnected graphs are absent. Of course, the Faddeev method is only one of an infinite number of ways of obtaining L^2 kernels.[4] Weinberg[3] has given a

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[1] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)].
[2] C. Lovelace, Phys. Rev. **135**, B1225 (1964); references to Faddeev's other papers are cited here.
[3] S. Weinberg, Phys. Rev. **133**, B232 (1964).
[4] R. Sugar and R. Blankenbecler, Phys. Rev. **136**, B472 (1964).
[5] For a review of current methods for multiparticle systems see J. Gillespie, *Final State Interactions* (Holden-Day, Inc., San Francisco, California, 1965), Chap. 10.

general prescription for obtaining a completely connected kernel for the n-body system. The present treatment differs from his in that only amplitudes occur in the final equations. This feature appears to be useful in the relativistic formulation of multiparticle scattering for which several attempts are presently underway.[6,7] Also, it seems that the appearance of only the two- and three-body amplitudes is highly advantageous for numerical calculations. Thus it is useful to explicitly construct equations of the Faddeev type for the four-particle system.

II. FOUR-PARTICLE AMPLITUDES

We start from the four-body Lippmann-Schwinger equation

$$T=V+VG_0T\,, \tag{1}$$

where $G_0=(E-H_0)^{-1}$ is the free Green's function.

We will consider only two-body interactions so the potential is of the form

$$V=\sum_{i>j} V_{ij} \quad i,j=1,2,3,4. \tag{2}$$

Multiparticle forces cause no additional difficulty. It is convenient to introduce two-body amplitudes

$$t_{ij}=V_{ij}+V_{ij}G_0t_{ij} \tag{3}$$

and the three-body Faddeev amplitudes for particles i, j, k.

$$t_{ij}{}^k=t_{ij}+t_{ij}G_0(t_{ik}{}^j+t_{jk}{}^i)\equiv t_{ij}+R_{ij}{}^k. \tag{4}$$

The subscripts on the three-body amplitudes designate the particles which interacted last. It should be noted that G_0 is the four-particle free Green's function and the operators act on a four-particle Hilbert space; consequently the t_{ij} and $t_{ij}{}^k$ contain momentum conservation delta functions for the noninteracting particles.

We express Eq. (1) in the Faddeev form

$$T^{ij}=t_{ij}+t_{ij}G_0(T^{ik}+T^{jk}+T^{lk}+T^{il}+T^{jl})\,, \tag{5}$$

where

$$T=\sum_{i>j} T^{ij}, \quad ijkl \text{ distinct.} \tag{6}$$

Since there are two distinct three-body amplitudes for which the pair ij interacts last, we express T^{ij} in the equivalent forms

$$\begin{aligned} T^{ij}&=t_{ij}{}^k+t_{ij}{}^kG_0(T^{il}+T^{jl}+T^{kl})\,,\\ T^{ij}&=t_{ij}{}^l+t_{ij}{}^lG_0(T^{ik}+T^{jk}+T^{kl})\,, \end{aligned} \tag{7}$$

where we have used Eq. (4) with different inhomogeneous terms. Adding the two forms of Eq. (7) we

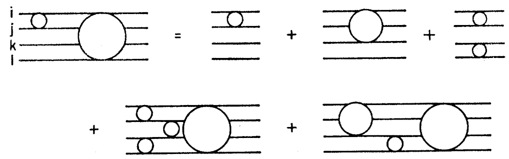

FIG. 1. Integral equation for T^{ij}. Note that the last diagram on the right of the first row stands for all iterations of t_{ij} and t_{kl}, and only looks like a product of the two t matrices in the sense of Eq. (12).

obtain

$$\begin{aligned} 2T^{ij}=t_{ij}&+R_{ij}{}^k[1+G_0(T^{il}+T^{jl}+T^{kl})]\\ &+R_{ij}{}^l[1+G_0(T^{ik}+T^{jk}+T^{kl})]\\ &+[t_{ij}+t_{ij}G_0(T-T^{ij})]+t_{ij}G_0T^{kl}. \end{aligned} \tag{8}$$

By use of Eq. (5) we cancel the fourth term with one T^{ij} and expand T^{kl} in the last term to obtain

$$\begin{aligned} T^{ij}=&[1-t_{ij}G_0t_{kl}G_0]^{-1}[t_{ij}+t_{ij}G_0t_{kl}]\\ &+R_{ij}{}^k[1+G_0(T^{il}+T^{jl}+T^{kl})]\\ &+R_{ij}{}^l[1+G_0(T^{ik}+T^{jk}+T^{kl})]\\ &+[1-t_{ij}G_0t_{kl}G_0]^{-1}t_{ij}G_0t_{kl}[1+G_0t_{ij}]\\ &\qquad\times G_0[T^{il}+T^{ik}+T^{jl}+T^{jk}]. \end{aligned} \tag{9}$$

In obtaining Eq. (9) we have used the identity

$$\begin{aligned} &[1-t_{ij}G_0t_{kl}G_0]^{-1}R_{ij}{}^k[1+G_0(T^{il}+T^{jl}+T^{kl})]\\ &=R_{ij}{}^k[1+G_0(T^{il}+T^{jl}+T^{kl})]\\ &\quad+[1-t_{ij}G_0t_{kl}G_0]^{-1}t_{ij}G_0t_{kl}G_0t_{ij}G_0(t_{ik}{}^j+t_{jk}{}^i)\\ &\quad\times[1+G_0(T^{il}+T^{jl}+T^{kl})]\\ &=R_{ij}{}^k[1+G_0(T^{il}+T^{jl}+T^{kl})]\\ &\qquad+[1-t_{ij}G_0t_{kl}G_0]^{-1}t_{ij}G_0t_{kl}G_0t_{ij}G_0(T^{ik}+T^{jk})\,, \end{aligned}$$

which can be obtained from Eq. (7) for T^{kl} and Eq. (4) for $R_{ij}{}^k$.

Equation (9) is depicted in Fig. 1. The kernel is seen to consist of the connected part of the three-body amplitudes and separate pairs of two-body amplitudes. These latter terms represent the only new complication for systems of more than three particles. The terms in Eq. (9) corresponding to these graphs can be simplified by considering their matrix elements in momentum space.[8] They take the form

$$\begin{aligned} Y_{ij}{}^{kl}&=[1-t_{ij}G_0t_{kl}]^{-1}[t_{ij}+t_{ij}G_0t_{kl}]\\ &=[1-V_{ij}G_{kl}]^{-1}V_{ij}[1+G_{kl}V_{kl}]\,, \end{aligned} \tag{10}$$

where

$$G_{kl}=G_0+G_0V_{kl}G_{kl}=\int d^3Q\,d^3p_{ij}\,d^3p_{kl} \times\frac{|\mathbf{p}_{ij}\mathbf{Q}\rangle|\psi^+(\mathbf{p}_{kl})\rangle\langle\psi^+(\mathbf{p}_{kl})|\langle\mathbf{p}_{ij}\mathbf{Q}|}{E-Q^2-p_{ij}{}^2-p_{kl}{}^2}, \tag{11}$$

[6] D. Stojanov and A. N. Tavkhelidze, Phys. Letters 13, 76 (1964).

[7] R. Blankenbecler and R. Sugar, Princeton report, 1965 (to be published); V. A. Allessandrini and R. L. Omnes, Phys. Rev. 139, B167 (1965); S. Mandelstam, Phys. Rev. (to be published).

[8] This approach was suggested to us by Professor R. F. Sawyer.

so that

$$[1-V_{ij}G_{kl}]^{-1}V_{ij} = \int d^3Q\,d^3p_{kl}\,|\mathbf{Q}\rangle\,|\psi^+(\mathbf{p}_{kl})\rangle\langle\mathbf{Q}|\,\langle\psi^+(\mathbf{p}_{kl})| \times t_{ij}^+(E-Q^2-p_{kl}^2).$$

The relative momentum of the ith and jth particles is denoted by $\mathbf{p}_{ij}$; $\mathbf{Q}$ denotes the momentum of the ij system relative to the kl system. The ket $|\psi^+(\mathbf{p}_{kl})\rangle$ represents the scattering state for the potential V_{kl}; $|\mathbf{p}_{ij},\mathbf{Q}\rangle$ is a plane wave state. The momentum space matrix elements of Y_{ij}^{kl} are

$$\begin{aligned}
&\langle \mathbf{p}_{ij}'',\mathbf{p}_{kl}'',\mathbf{Q}''|Y_{ij}^{kl}|\mathbf{p}_{ij}',\mathbf{p}_{kl}',\mathbf{Q}'\rangle \\
&=\delta^3(\mathbf{Q}''-\mathbf{Q}')\int d^3p_{kl}\langle \mathbf{p}_{ij}''|t_{ij}^+(E-Q'^2-p_{kl}^2)|\mathbf{p}_{ij}'\rangle\langle \mathbf{p}_{kl}''|\psi^+(\mathbf{p}_{kl})\rangle\langle\psi^+(\mathbf{p}_{kl})|[1+(E-Q'^2-p_{ij}'^2-p_{kl}^2+i\epsilon)^{-1}V_{kl}]|\mathbf{p}_{kl}'\rangle \\
&=\delta^3(\mathbf{Q}''-\mathbf{Q}')\Big[\langle \mathbf{p}_{ij}''|t_{ij}^+(E-Q'^2-p_{kl}'^2)|\mathbf{p}_{ij}'\rangle\{\delta^3(\mathbf{p}_{kl}''-\mathbf{p}_{kl}')+\langle \mathbf{p}_{kl}''|t_{kl}^+(p_{kl}'^2)|\mathbf{p}_{kl}'\rangle(p_{kl}'^2-p_{kl}''^2+i\epsilon)^{-1}\} \\
&\quad+(E-Q'^2-p_{ij}'^2-p_{kl}'^2)\Big\{\langle \mathbf{p}_{ij}''|t_{ij}^+(E-Q'^2-p_{kl}''^2)|\mathbf{p}_{ij}'\rangle \\
&\quad\times\langle \mathbf{p}_{kl}''|t_{kl}^-(p_{kl}''^2)|\mathbf{p}_{kl}'\rangle[(p_{kl}''^2-p_{kl}'^2-i\epsilon)(E-Q'^2-p_{ij}'^2-p_{kl}''^2+i\epsilon)]^{-1} \\
&\quad-\frac{1}{2\pi i}\int_0^\infty dp_{kl}^2\frac{\langle \mathbf{p}_{kl}''|t_{kl}^+(p_{kl}^2)|\mathbf{p}_{kl}'\rangle-\langle \mathbf{p}_{kl}''|t_{kl}^-(p_{kl}^2)|\mathbf{p}_{kl}'\rangle}{(p_{kl}^2-p_{kl}''^2+i\epsilon)(p_{kl}^2-p_{kl}'^2-i\epsilon)(E-Q'^2-p_{ij}'^2-p_{kl}^2+i\epsilon)}\Big\}\Big] \\
&\equiv\langle \mathbf{p}_{ij}'',\mathbf{p}_{kl}'',\mathbf{Q}''|(t_{ij}+X_{ij}^{kl})|\mathbf{p}_{ij}',\mathbf{p}_{kl}',\mathbf{Q}'\rangle, \qquad (12)
\end{aligned}$$

where we have made use of the off-energy shell unitarity condition to simplify the last integral. [Note that in Eq. (12) the expressions in parentheses following t_{ij} and t_{kl} are the arguments of the t's, not factors.]

Equation (9) now assumes the simple form

$$\begin{aligned}
T^{ij}=t_{ij}&+R_{ij}^k[1+G_0(T^{il}+T^{jl}+T^{kl})] \\
&+R_{ij}^l[1+G_0(T^{ik}+T^{jk}+T^{kl})] \\
&+X_{ij}^{kl}[1+G_0(T^{il}+T^{ik}+T^{jl}+T^{jk})]. \qquad (13)
\end{aligned}$$

III. DISCUSSION AND APPLICATIONS

The final Eq. (13) simplifies considerably in the case of identical particles when amplitudes of the appropriate symmetry are used. Furthermore, separable two-body interactions, which are known to reduce three-body calculations to ones which are no more difficult than two-body calculations,[9–11] should reduce the four-body equations to the level of a three-body problem. In fact a preliminary attempt on similar lines at understanding the α particle as a bound state of four nucleons[12] already indicates the possibility of a tractable numerical scheme when the appropriate symmetries are taken into account in a four-particle Schrodinger equation.

One might also follow the methods of Lovelace[2] and approximate the connected parts of the three-particle as well as the two-particle amplitudes by a number of bound states and resonances. When such extensive approximations are justified, the four-body problem would reduce to the level of two-body calculations.

Since the formalism is nonrelativistic, it would be premature to think of immediate applications to elementary particles.[13] However, some applications of this formalism to d-d scattering and stripping phenomena are currently being investigated.

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Note: After completion of this work we received an unpublished report from V. Alessandrini which closely parallels our work. We wish to thank Dr. Alessandrini for making his results available to us before publication, thereby enabling us to correct an error in the matrix elements of X_{ij}^{kl}. We also received an unpublished report from L. Rosenberg in which multi-particle Faddeev equations are obtained by iterating the N-particle analog of Eq. (5). Our four-particle equation, which has been obtained algebraically, is identical to his if one accounts properly for the necessary convolutions[3] such as in our Eq. (12). The generalization of Eq. (12) to the N-particle case is straightforward.

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[10] A. N. Mitra and V. S. Bhasin, Phys. Rev. **131**, 1265 (1963).
[11] R. D. Amado, Phys. Rev. **132**, 485 (1963).
[12] N. Panchapakesan, Phys. Rev. **140**, B20 (1965).
[13] For application of separable interactions to a model "four-pion problem" taking isospin and symmetry effects into effects, see A. N. Mitra and S. Ray, Phys. Rev. **137**, B982 (1965).