Some Aspects of Complex Angular Momentum and Three-Particle States

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In this paper we try to gain some understanding of the problem of how complex singularities affect the continuation of multiparticle amplitudes to complex angular momentum J by examining a few simple pole contributions to a production amplitude. We find that in order to compute the effect of a three-particle state on a two-particle amplitude of definite signature the unitarity integral has to be considerably reorganized. This has the result of requiring us to deal with three different production amplitudes of definite signature, each of which provides a different continuation to complex J. We find also that the phase-space integrations which occur in unitarity-like equations must be performed over suitably deformed contours when J is complex. For integer angular momentum, of course, the deformation has no effect. We elucidate this situation by introducing doubly projected partial-wave amplitudes in which both the total and a subangular momentum are projected out.

INTRODUCTION

RECENTLY Omnes and Alessandrini[1,2] have analyzed in detail the problem of continuing the angular momentum of three-particle amplitudes to complex values. They base their analysis on the Faddeev[3] equations for a nonrelativistic system and on a particular generalization of the Froissart-Gribov continuation of a two-particle amplitude to arbitrary angular momentum. One of the difficulties they emphasize[2] is the existence of singularities of the three-particle amplitude at complex values of the cosines of the various scattering angles.

In this paper we attempt to illuminate this aspect of the complete problem by studying the effect of a few, simple, pole contributions to a multiparticle amplitude. In fact these contributions are so simple that we are able to avoid a discussion of the very difficult problem of nonsense channels and infinite helicity sums. Indeed, we simplify the problem yet further by restricting our discussion to the effect of a three-particle intermediate state on the imaginary part of an elastic two-body amplitude. Doing so has two advantages:

(i) We need consider only production amplitudes, with a consequent simplification of kinematics.

(ii) We can relate the more difficult three-particle problem to one we understand better, namely, that of continuing the angular momentum of two-particle amplitudes.

An additional reason for pursuing the following analysis is that we wish to use it in a later paper which discusses a very simple dynamical calculation of a three-particle Regge residue.

In spite of these simplifications we are left with a nontrivial problem the resolution of which implies that the continuation to arbitrary angular momentum must be performed in a manner slightly different from that suggested by Omnes and Alessandrini.[2] It becomes clear, also, that care must be taken in specifying the phase-space integration contours in unitarity-type equations and that the continued partial-wave amplitudes are not, in a straightforward sense, unitary.

I. KINEMATICS

In order to have definite amplitudes to study we consider a model in which two scalar particles A and B, masses m_A and m_B, can annihilate to produce three scalar particles, π mesons of mass m_π. We shall suppose that the particles are distinct though the problem is hardly changed by making them identical.

We are interested, then, in the processes

$$A+B\leftrightarrow A+B\,, \tag{1}$$

$$A+B\leftrightarrow \pi_1+\pi_2+\pi_3\,, \tag{2}$$

which correspond to the diagrams in Fig. 1. The amplitude describing process (1) is $T_{AB}(s,t,u)$ where

$$\begin{aligned} s&=(p_A+p_B)^2\,,\\ t&=(p_A-p_A')^2\,,\\ u&=(p_A-p_B')^2\,,\\ s+t+u&=2m_A^2+2m_B^2\,, \end{aligned} \tag{3}$$

and that describing process (2) is $T(s;s_1,s_2,s_3,t_1,t_2,t_3)$ where

$$\begin{aligned} s&=(p_1+p_2+p_3)^2\,,\\ t_i&=(p_A-p_i)^2\,,\quad i=1,2,3\,,\\ s_i&=(p_j+p_k)^2\,,\quad (i,j,k)=\mathrm{perm}(1,2,3)\,. \end{aligned} \tag{4}$$

It is also convenient to define

$$u_i=(p_B-p_i)^2\,. \tag{5}$$

Of course, not all of these variables are independent;

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[1] R. L. Omnes, Phys. Rev. **134**, B1358 (1964).
[2] R. L. Omnes and V. Alessandrini, Phys. Rev. **136**, B1137 (1964).
[3] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)].

they are subject to linear relations, for example,

$$\begin{aligned} s_1+s_2+s_3&=s+3m_\pi^2\,, \\ s+t_1+t_2+t_3&=2m_A^2+m_B^2+3m_\pi^2\,, \\ s+t_i+u_i&=s_i+m_A^2+m_B^2+m_\pi^2\,, \quad i=1,2,3\,. \end{aligned} \tag{6}$$

It is convenient to define the cosines of various scattering angles.

(1) In the over-all center-of-mass frame

$$\begin{aligned} z&=\hat{p}_A\cdot\hat{p}_A'\,, \\ z_i&=\hat{p}_A\cdot\hat{p}_i\,, \\ z_i'&=\hat{p}_A'\cdot\hat{p}_i\,, \\ z_{ij}&=\hat{p}_i\cdot\hat{p}_j\,. \end{aligned} \tag{7}$$

(2) In the (2,3) center-of-mass frame

$$\begin{aligned} x_{ij}&=\hat{q}_i\cdot\hat{q}_j\,, \\ x_i&=\hat{q}_A\cdot\hat{q}_i\,, \\ y_i&=\hat{q}_B\cdot\hat{q}_i\,, \end{aligned} \tag{8}$$

where we use the symbol q rather than p in order to emphasize that the energies and momenta are evaluated in the (2,3), rather than the over-all center-of-mass frame.

The energies and momenta of the particles in these frames are:

(1) In the over-all center-of-mass frame

$$\begin{aligned} p_{A0}&=(s+m_A^2-m_B^2)/2(s)^{1/2}\,, \\ p_A^2&=\lambda(s,m_A^2,m_B^2)/4s=p_B^2\,, \\ p_{B0}&=(s+m_B^2-m_A^2)/2(s)^{1/2}\,, \\ p_{i0}&=(s+m_\pi^2-s_i)/2(s)^{1/2}\,, \\ p_i^2&=\lambda(s,m_\pi^2,s_i)/4s\,, \end{aligned} \tag{9}$$

where

$$\lambda(x,y,z)=x^2+y^2+z^2-2(xy+yz+zx)\,; \tag{10}$$

(2) in the (2,3) center-of-mass frame

$$\begin{aligned} q_{A0}&=(s_1+m_A^2-u_1)/2(s_1)^{1/2}\,, \\ q_A^2&=\lambda(s_1,m_A^2,u_1)/4s_1\,, \\ q_{B0}&=(s_1+m_B^2-t_1)/2(s_1)^{1/2}\,, \\ q_B^2&=\lambda(s_1,m_B^2,t_1)/4s_1\,, \\ q_{20}&=q_{30}=(s_1)^{1/2}/2\,, \\ q_2^2&=q_3^2=s_1/4-m_\pi^2\,, \\ q_{10}&=(s-s_1-m_\pi^2)/2(s_1)^{1/2}\,, \\ q_1^2&=\lambda(s,s_1,m_\pi^2)/4s_1\,. \end{aligned} \tag{11}$$

II. SIGNATURE AND TWO-PARTICLE UNITARITY

Before discussing signature and three-particle amplitudes we would like to emphasize those properties of the two-particle phase-space integral which permit the definition of unitary elastic scattering amplitudes of definite signature. If the elastic scattering amplitude is written as a function of energy and scattering angle, unitarity implies that

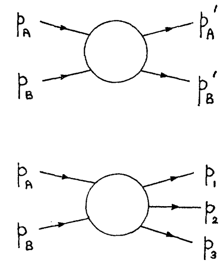

FIG. 1. Reactions considered in the text.

$$\Delta_2 T_{AB}^{(s)}(s,z) =i(2\pi)^4\int d\rho(2)\; T_{AB}^{(s)}(s_+,z')T_{AB}^{(s)}(s_-,z'')\,, \tag{12}$$

where

$$\begin{aligned} d\rho(2)&=\frac{1}{(2\pi)^6}d^4p_A''d^4p_B''\delta(p_A''^2-m_A^2) \\ &\quad\times\delta(p_B''^2-m_B^2)\delta^{(4)}(p_A+p_B-p_A''-p_B'')\,, \\ z'&=\hat{p}_A\cdot\hat{p}_A''\,,\quad z''=\hat{p}_A'\cdot\hat{p}_A''\,, \end{aligned} \tag{13}$$

and Δ_2 indicates the two-particle contribution to the s discontinuity of T_{AB}. Of course, Eq. (12) holds for the symmetric and antisymmetric parts of the amplitude separately. Since we intend to use the symmetric part for illustration we have taken advantage of this and added an s superscript (for symmetric) to T_{AB}. Thus

$$T_{AB}^{(s)}(s,z)=\tfrac{1}{2}[T_{AB}(s,z)+T_{AB}(s,-z)]\,. \tag{14}$$

If T_{AB} has the analytic structure required by the Mandelstam representation, then

$$T_{AB}^{(s)}(s,z)=\tfrac{1}{2}[T_{AB}^{(+)}(s,z)+T_{AB}^{(+)}(s,-z)]\,, \tag{15}$$

where

$$T_{AB}^{(+)}(s,z)=\frac{1}{\pi}\int_{z_0}^{\infty}\frac{dz'}{z'-z}f(s,z')\,, \tag{16}$$

so that the amplitude of definite signature $T_{AB}^{(+)}(s,z)$ has singularities only in the right-half z plane.

By substituting Eq. (15) into Eq. (12) and by identifying, on both sides, those terms giving rise to singularities in the right-half z plane, we can deduce that $T_{AB}^{(+)}(s,z)$ is unitary. That is,

$$\Delta_2 T_{AB}^{(+)}(s,z) =i(2\pi)^4\int d\rho(2)\; T_{AB}^{(+)}(s_+,z')T^{(+)}(s-,z'')\,. \tag{17}$$

In determining the position of the z-plane singularities of the various terms we make use of the result that the

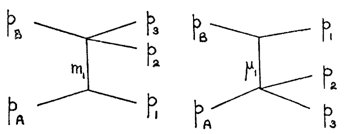

FIG. 2. Simple pole contributions to the production amplitude.

integrals

$$\int d\rho(2)\ T_{AB}^{(+)}(s_+, \pm z')T_{AB}^{(+)}(s_-, \pm z'') \tag{18}$$

produce right-half z-plane singularities, while the integrals

$$\int d\rho(2)\ T_{AB}^{(+)}(s_+, \pm z')T_{AB}^{(+)}(s_-, \mp z'') \tag{19}$$

produce left-half z-plane singularities. The upper and lower signs preceding z' and z'' are to be taken together in both (18) and (19). We can sum up these results by saying that when the factors in the integrand have either both right- or both left-half z-plane singularities then the unitarity integral itself has a right-half z-plane singularity. Whereas, a right-left or a left-right combination produces a left-half z-plane singularity. It is this property of the two-particle phase-space integral which allows us to construct unitary elastic scattering amplitudes of definite signature. Since it is these amplitudes we use to define the partial waves of definite signature it follows that they also are unitary for arbitrary angular momentum.

When we go on to discuss three-particle states we will find that singularities do *not* combine within the three-particle phase-space integral in such a simple way. In particular a left-right classification cannot be set up to predict the effect of combining two singularities. It is the lack of this property which causes some of the complication in continuing the angular momentum.

III. SIGNATURE AND THREE-PARTICLE UNITARITY

In this section we wish to examine the effect of those pole contributions to the production amplitude T represented by the diagrams in Fig. 2 together with those obtained by permuting (1,2,3). That is, we suppose,

$$T = A_1(t_1,u_1) + A_2(t_2,u_2) + A_3(t_3,u_3)\,, \tag{20}$$

where

$$A_i(t_i,u_i) = G_i/(m_i^2 - t_i) + \Gamma_i/(\mu_i^2 - u_i)\,. \tag{21}$$

If the particles were identical then, of course, the various masses and residues would be independent of i.

Perhaps it is worth emphasizing that we do not regard Eq. (20) as providing a good approximation to T. Our main point is that a correct theory of complex angular momentum should at least be able to take into account the contributions included in Eq. (20).

It is convenient to think of the amplitudes as functions of the cosines of scattering angles, so that, we will write

$$A_i(t_i,u_i) = A_i(z_i)\,. \tag{22}$$

Each A_i, of course, also depends on s and s_i and has a pole in both the left- and right-half z_i plane.

From unitarity we deduce that the contribution of the three-π-meson intermediate state to the s discontinuity of T_{AB} is

$$\Delta_3 T_{AB}(s,z) = i(2\pi)^4 \int d\rho(3)\ T(s_+,s_{1+},s_{2+},s_{3+},z_1,z_2,z_3) \times T(s_-,s_{1-},s_{2-},s_{3-},z_1',z_2',z_3')\,, \tag{23}$$

where

$$d\rho(3) = \frac{1}{(2\pi)^9} d^4p_1 d^4p_2 d^4p_3 \delta(p_1^2 - m_\pi^2)\delta(p_2^2 - m_\pi^2) \times \delta(p_3^2 - m_\pi^2)\delta^{(4)}(p_A + p_B - p_1 - p_2 - p_3)\,. \tag{24}$$

Equation (23) holds for the symmetric and antisymmetric parts of the various amplitudes separately. Adding an s (for symmetric) superscript, we deduce that

$$\Delta_3 T_{AB}^{(s)}(s,z) = i(2\pi)^4 \int d\rho(3)\ T^{(s)}(s_+; s_{1+},s_{2+},s_{3+},z_1,z_2,z_3) \times T^{(s)}(s_-; s_{-1},s_{2-},s_{3-},z_1',z_2',z_3')\,, \tag{25}$$

where

$$T^{(s)}(s; s_1,s_2,s_3,z_1,z_2,z_3) = \tfrac{1}{2}[T(s; s_1,s_2,s_3,z_1,z_2,z_3) + T(s; s_1, s_2, s_3, -z_1, -z_2, -z_3)]\,. \tag{26}$$

In our case because of Eq. (20) we have

$$T^{(s)}(s; s_1,s_2,s_3,z_1,z_2,z_3) = A_1^{(s)}(z_1) + A_2^{(s)}(z_2) + A_3^{(s)}(z_3)\,, \tag{27}$$

where

$$A_i^{(s)}(z_i) = \tfrac{1}{2}[A_i(z_i) + A_i(-z_i)]\,. \tag{28}$$

The analytic structure of A allows us to write

$$A_i^{(s)}(z_i) = \tfrac{1}{2}[A_i^{(+)}(z_i) + A_i^{(+)}(-z_i)]\,, \tag{29}$$

where $A_i^{(+)}(z_i)$ contains only right-half z_i-plane singularities. That is,

$$A_i^{(+)}(z_i) = \frac{G_i}{m_i^2 - t_i(z_i)} + \frac{\Gamma_i}{\mu_i^2 - u_i(-z_i)}\,, \tag{30}$$

where

$$t_i(z_i) = m_A^2 + m_\pi^2 - 2p_{A0}p_{i0} + 2p_A p_i z_i\,, \tag{31}$$
$$u_i(z_i) = m_B^2 + m_\pi^2 - 2p_{B0}p_{i0} - 2p_A p_i z_i\,.$$

If we substitute Eq. (29) into the right side of Eq. (25) and identify those terms which produce singularities in the right-half z plane, we find,

$$\Delta_3 T_{AB}^{(+)}(s,z) = i(2\pi)^4 \left\{ \int d\rho(3)\ A_1^{(+)}(z_1) T_1^{(+)}(z_1',z_2',z_3') + \int d\rho(3)\ A_2^{(+)}(z_2) T_2^{(+)}(z_1',z_2',z_3') + \int d\rho(3)\ A_3^{(+)}(z_3) T_3^{(+)}(z_1',z_2',z_3') \right\}, \tag{32}$$

where we have suppressed the dependence on (s,s_1,s_2,s_3) and

$$T_1^{(+)}(z_1,z_2,z_3)=A_1^{(+)}(z_1)+A_2^{(+)}(-z_2)+A_3^{(+)}(-z_3),$$
$$T_2^{(+)}(z_1,z_2,z_3)=A_1^{(+)}(-z_1)+A_2^{(+)}(z_2)+A_3^{(+)}(-z_3), \quad (33)$$
$$T_3^{(+)}(z_1,z_2,z_3)=A_1^{(+)}(-z_1)+A_2^{(+)}(-z_3)+A_3^{(+)}(z_3).$$

In obtaining this result we have made use of the fact that the integrals

$$\int d\rho(3)\, A_i(z_i)A_i(z_i'), \quad (34a)$$

$$\int d\rho(3)\, A_i(z_i)A_j(-z_j'), \quad i\neq j \quad (34b)$$

produce right-half z-plane singularities and that the integrals

$$\int d\rho(3)\, A_i(z_i)A_i(-z_i'), \quad (35a)$$

$$\int d\rho(3)\, A_i(z_i)A_j(z_j'), \quad i\neq j \quad (35b)$$

produce left-half z-plane singularities.

To prove these statements it is simplest to regard the contributions to the integrals as the s discontinuities of various Feynman amplitudes.[4] Diagrams representing typical contributions to (33a) and (33b) are shown in Fig. 3. The amplitudes corresponding to these diagrams have been intensively studied.[5–7] In general they satisfy the Mandelstam representation with only an (s,t) double spectral function. We shall ensure that this is so by requiring that

$$m_i>m_A+m_\pi, \quad \mu_i>m_B+m_\pi \quad (36)$$

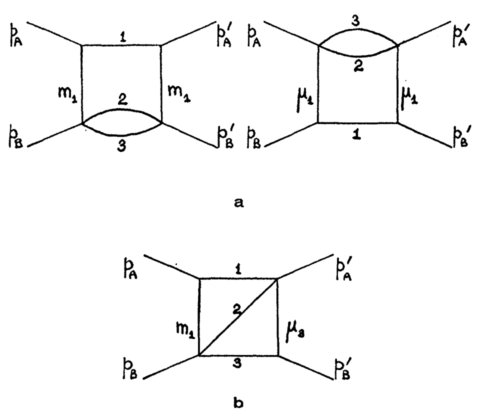

FIG. 3. Unitarity diagrams.

[4] R. E. Cutkosky, J. Math. Phys. 1, 429 (1960).
[5] R. J. Eden, P. V. Landshoff, J. C. Polkinghorne, and J. C. Taylor, J. Math. Phys. 2, 656 (1961).
[6] V. N. Gribov and I. T. Dyatlov, Zh. Eksperim. i Teor. Fiz. 42, 196 (1962) [English transl.: Soviet Phys.—JETP 15, 140 (1962)].
[7] J. N. Islam and Y. S. Kim, Phys. Rev. 138, B1222 (1965).

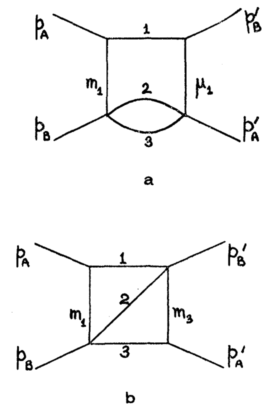

FIG. 4. Unitarity diagrams.

though these conditions are not crucial to our analysis. It follows therefore, that the Feynman amplitudes and their s discontinuities have singularities only in the right-half t plane. Since

$$t(z)=-2p_A^2(1-z), \quad (37)$$

the same statement holds for the z-plane analytic structure. The remaining contributions to (34a) and (34b) are subject to a similar analysis except that it turns out that they are functions of $(s,u(-z))$. Typical diagrams are shown in Fig. 4. Again, the fact that the corresponding Feynman amplitudes and their s discontinuities have singularities only in the right-half u plane implies that the same is true of the z-plane singularities, since

$$u(-z)=m_A^2+m_B^2-2p_{A0}p_{B0}+2p_A^2z. \quad (38)$$

Of course the integrals (35a) and (35b) can be analyzed in exactly the same way with the results stated above. The reason for having to use an equation such as (31) was mentioned at the end of the previous section, namely, that the singularities of the production amplitude cannot be classified into two types. For example, when either $A_2(-z_2)$ or $A_3(-z_3)$ is taken along with $A_1(z_1)$ in the three-particle phase-space integral the result is a function with right-half z-plane singularities. Consequently, singularities of both these amplitudes might be thought to be of the same type as those of $A_1(z_1)$ and, therefore, as each other. However, when taken together in a phase-space integral they produce a *left*-half z-plane singularity which would suggest that they were of opposite type. Clearly, then, a simple classification is not possible.

It follows immediately that there is *no* way of organizing the complete production amplitude into one of definite signature which may be used along with its complex conjugate to compute the contribution of a three-particle state to the imaginary part of $T_{AB}^{(+)}(s,z)$. In other words it is not possible to construct unitary production amplitudes of definite signature. The analysis of this section suggests, instead, that it is necessary to construct three such production amplitudes $T_1^{(+)}$, $T_2^{(+)}$, $T_3^{(+)}$, which are to be used along with incomplete amplitudes of definite signature, $A_1^{(+)}$, $A_2^{(+)}$, $A_3^{(+)}$ in computing the effect of three-particle unitarity. Furthermore, the natural method of projecting out partial waves varies from one amplitude to another. We discuss this in the next section.

It is worth pointing out that the results of this section imply that, whenever three-particle states are taken into account, there are necessarily two-particle amplitudes (and therefore trajectories) of *both* signatures. To assume the opposite leads immediately to the existence of a unitary production amplitude of definite signature. This state of affairs we expect to remain true even in the nonrelativistic limit.

An exception to these remarks does arise, however, when one of the three incomplete amplitudes A_i vanishes, for example $G_3=\Gamma_3=0$, and the other two have poles of opposite kinds, for example, $\Gamma_1=G_2=0$. In these circumstances it is easy to show that all of the allowed unitarity integrals have singularities only in the right-half z plane. When the π mesons are identical, of course, this situation does not arise.

IV. PARTIAL WAVES

For integer angular momentum we define partial waves of the production amplitude T in a manner consistent with the definition of Omnes and Alessandrini.[1,2,8] That is, we have

$$T_M(J)=\frac{1}{8\pi^2}\int T\mathfrak{D}_{0M}{}^J(R)dR. \tag{39}$$

In our case, of course, there is at most one nonzero helicity index M because the production amplitude is invariant under rotations about the initial-state space momentum, which is chosen to lie along the space-fixed z axis.

The rotation R which carries the space axes into a set fixed in the final state may be parametrized by Euler angles φ, θ, ψ. The meaning of these angles depends on how the final-state axes are chosen. Guided by the structure of the unitarity-like equation (31), we shall choose the final-state z axis along $\mathbf{p}_i$ for the amplitudes $T_i^{(+)}$ and $A_i^{(+)}$ ($i=1$, 2, 3). In each case we shall choose the x axis to lie in the plane of the three final-state space momenta.

As an example, we discuss the projection of $T_1^{(+)}$. In this case we have

$$\begin{aligned} z_1&=\cos\theta\,,\\ z_2&=z_1z_{12}-(1-z_1^2)^{1/2}(1-z_{12}^2)^{1/2}\cos\psi\,,\\ z_3&=z_1z_{13}+(1-z_1^2)^{1/2}(1-z_{13}^2)^{1/2}\cos\psi\,. \end{aligned} \tag{40}$$

The amplitude does not, of course, depend on φ.

The first step in the projection is to define an amplitude which describes the production of particles 2 and 3 with a helicity M along $\mathbf{p}_1$. That is we define

$$T_{1,M}{}^{(+)}(z_1)=\frac{1}{2\pi}\int_0^{2\pi}d\psi\, e^{-iM\psi}T_1{}^{(+)}(z_1,z_2,z_3)\,. \tag{41}$$

If we apply this procedure to $A_1^{(+)}$ we obtain zero unless $M=0$. It follows then from the structure of Eq. (31) that we shall only need the case $M=0$ for $T_1^{(+)}$ also. In calculating this function we encounter integrals of the form

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{d\psi}{\xi_2+z_1z_{12}-(1-z_1^2)^{1/2}(1-z_{12}^2)^{1/2}\cos\psi}=[k(\xi_2,\,-z_{12},\,z_1)]^{-1/2}, \tag{42}$$

where $\xi_2>1$ and

$$k(x,y,z)=x^2+y^2+z^2-2xyz-1\,. \tag{43}$$

Using this result, we find on performing the projection that

$$\begin{aligned} T_{1,0}{}^{(+)}(z_1)=&\frac{1}{2p_Ap_1}\left\{\frac{G_1}{\xi_1-z_1}+\frac{\Gamma_1}{\zeta_1-z_1}\right\}\\ &+\frac{1}{2p_Ap_2}\{G_2[k(\xi_2,\,-z_{12},\,z_1)]^{-1/2}\\ &+\Gamma_2[k(\zeta_2,\,-z_{12},\,z_1)]^{-1/2}\}\\ &+\frac{1}{2p_Ap_3}\{G_3[k(\xi_3,\,-z_{13},\,z_1)]^{-1/2}\\ &+\Gamma_3[k(\zeta_3,\,-z_{13},\,z_1)]^{-1/2}\}\,, \end{aligned} \tag{44}$$

where

$$\begin{aligned} \xi_i&=(m_i^2-m_A^2-m_\pi^2+2p_{A0}p_{i0})/2p_Ap_i\,,\\ \zeta_i&=(\mu_i^2-m_B^2-m_\pi^2+2p_{B0}p_{i0})/2p_Ap_i\,, \end{aligned}\quad i=1,\,2,\,3\,. \tag{45}$$

From Eq. (44) we can see that the z_1-plane singularities of $T_{1,0}{}^{(+)}$ are at the points

$$\begin{aligned} z_1&=\xi_1\,,\\ z_1&=-z_{12}\xi_2\pm i(1-z_{12}^2)^{1/2}(\xi_2^2-1)^{1/2},\\ z_1&=-z_{13}\xi_3\pm i(1-z_{13}^2)^{1/2}(\xi_3^2-1)^{1/2} \end{aligned} \tag{46}$$

together with those obtained by replacing ξ by ζ. Clearly, then, this function has the complex singularities referred to by Omnes and Alessandrini.[2] Their position in the complex plane is controlled mainly by z_{12} and z_{13}, which in turn depend on the energy and sub-energy variables (s,s_1,s_2,s_3). When z_{12} and z_{13} are negative they lie in the right-half z_1-plane. When either z_{12} or z_{13} becomes positive they lie in the left-half z_1 plane and when the value $+1$ is attained at least some of the singularities come together on the negative real axis. These singularities are illustrated in Fig. 5.

Having performed the helicity projection the next step is to multiply by $d_{0M}{}^J(z_1)$ and integrate from -1 to $+1$. Since we have $M=0$, the completely projected

[8] D. Branson, P. V. Landshoff, and J. C. Taylor, Phys. Rev. **132**, 902 (1963).

partial wave of interest is, for integer J,

$$T_{1,0}^{(+)}(J)=\frac{1}{2}\int_{-1}^{1} dz_1 T_{1,0}^{(+)}(z_1)P_J(z_1)\,. \quad (47)$$

In order to continue to complex J we use essentially the same generalization of the Froissart-Gribov prescription suggested by Omnes and Alessandrini,[2] namely,

$$T_{1,0}^{(+)}(J)=\frac{1}{\pi i}\int_C dz_1 T_{1,0}^{(+)}(z_1)Q_J(z_1)\,, \quad (48)$$

whether the contour C is indicated in Fig. 5. Note that we have arranged the cuts attached to the singularities of $T_{1,0}^{(+)}(z_1)$ so that this definition is possible so long as neither z_{12} nor z_{13} is $+1$. Of course, we can deform the contour C so that it encloses the singularities and cuts of $T_{1,0}^{(+)}(z_1)$. It follows immediately that

$$|T_{1,0}^{(+)}(J)|<Ke^{\lambda|J|}J\to\infty\,, \quad (49)$$

where K is a constant and $\lambda\leqq\pi$. Furthermore λ only attains the value π when either z_{12} or z_{13} is $+1$, at which points the singularities of $T_{1,0}^{(+)}(z_1)$ come down onto the real axis and pinch the contour C.

The definition adopted by Omnes and Alessandrini[2] differs from this one in that they, in effect, choose $T_{1,0}^{(+)}(z_1)$ so that its discontinuities all lie in the right-half z_1 plane. Under certain circumstances (z_{12}, z_{13} both negative) this definition coincides with the one we have adopted. As z_{12} or z_{13} increases, however, we permit the singularities of $T_{1,0}^{(+)}(z_1)$ to penetrate the left-half z_1 plane while they, effectively, define a new function with these left-half plane singularities folded back into the right-half plane. Of course, both continuations coincide with the physical amplitude when J is an *even* integer.

The advantage of the Omnes-Alessandrini definition is that the resulting amplitude has asymptotic behavior, for large J, which is less divergent (in general) than the amplitude we define. Our reason for preferring the definition we have adopted is that it is this continuation which enters, via unitarity-like equations derived from (31), into the evaluation of the three-particle discontinuity of the continued two-particle partial-wave amplitude.

Finally we obtain an explicit formula for $T_{1,0}^{(+)}(J)$. We express it in terms of the partial waves of the incomplete amplitudes $A_i^{(+)}(z_i)$. Define, for integer J,

$$A_i^{(+)}(J)=\frac{1}{2}\int_{-1}^{1} dz_i A_i^{(+)}(z_i)P_J(z_i)\,, \quad (50)$$

which can be continued to complex J by means of the equation

$$A_i^{(+)}(J)=\frac{1}{2p_Ap_i}[G_iQ_J(\xi_i)+\Gamma_iQ_J(\zeta_i)]\,. \quad (51)$$

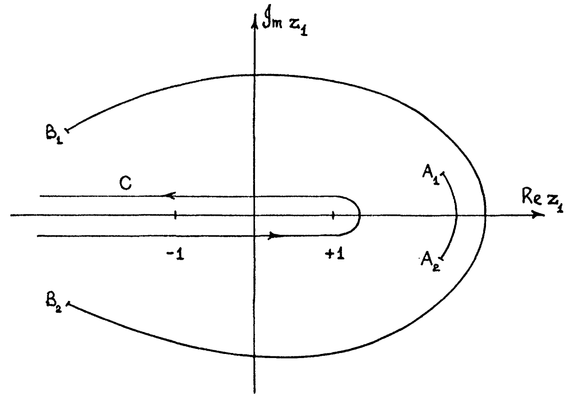

FIG. 5. Complex singularities in the z_1 plane. When $z_{12}\approx-1$ their position is A_1, A_2 and when $z_{12}\approx+1$ their position is B_1, B_2.

It is now a simple matter of applying the theorems about analytically continued rotation matrices, developed by Omnes and Alessandrini,[2] to verify that

$$T_{1,0}^{(+)}(J)=A_1^{(+)}(J,s,s_1)+A_2^{(+)}(J,s,s_2)P_J(-z_{12}) +A_3^{(+)}(J,s,s_3)P_J(-z_{13})\,, \quad (52)$$

where we have explicitly exhibited the dependence on (s,s_1,s_2,s_3). The presence, in Eq. (52), of the P_J functions, makes clear what we already know from the analytic structure of $T_{1,0}^{(+)}(z_1)$, namely, that this continuation has bad asymptotic behavior ($\sim e^{i\pi J}$) when either z_{12} or z_{13} is $+1$.

Formulas for the partial waves of the other production amplitudes can be obtained similarly, for example,

$$T_{2,0}^{(+)}(J)=A_1^{(+)}(J,s,s_1)P_J(-z_{12})+A_2^{(+)}(J,s,s_2) +A_3^{(+)}(J,s,s_3)P_J(-z_{23})\,, \quad (53)$$

the final-state z axis being chosen, in this case, along $\mathbf{p}_2$.

V. THREE-PARTICLE DISCONTINUITY FOR PARTIAL WAVES

For integral values of the angular momentum the discontinuity formula becomes

$$\begin{aligned}\Delta_3T_{AB}^{(+)}(J,s) &= i(2\pi)^4\bigg\{\int d\bar{\rho}(3)\;A_1^{(+)}(J,s,s_1)T_{1,0}^{(+)}(J,s;\,s_1,s_2,s_3)\\ &\quad+\int d\bar{\rho}(3)\;A_2^{(+)}(J,s,s_2)T_{2,0}^{(+)}(J,s;\,s_1,s_2,s_3)\\ &\quad+\int d\bar{\rho}(3)\;A_3^{(+)}(J,s,s_3)T_{3,0}^{(+)}(J,s;\,s_1,s_2,s_3)\bigg\}\,, \end{aligned} \quad (54)$$

where

$$T_{AB}^{(+)}(J,s)=\frac{1}{2}\int_{-1}^{1} dz\; T_{AB}^{(+)}(s,z)P_J(z) \quad (55)$$

and

$$d\bar{\rho}(3)=\frac{1}{(2\pi)^9}\frac{\pi^2}{4s}ds_1ds_2\,. \quad (56)$$

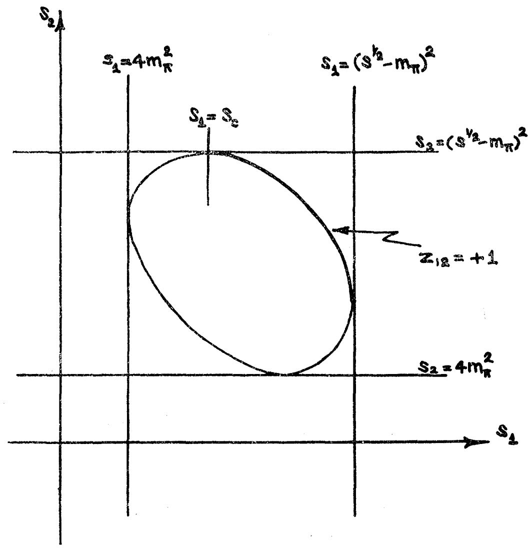

FIG. 6. Boundary of the (s_1,s_2) physical region.

For complex J, of course, $T_{AB}^{(+)}(J,s)$ is continued by means of the usual Froissart-Gribov prescription.

The phase-space integration is carried out over the inside of the curve

$$4m_\pi^6-m_\pi^2[(s_1-2m_\pi^2)^2+(s_2-2m_\pi^2)^2+(s_3-2m_\pi^2)^2]+(s_1-2m_\pi^2)(s_2-2m_\pi^2)(s_3-2m_\pi^2)=0 \quad (57)$$

which is illustrated in Fig. 6.

Unfortunately it is not possible to continue Eq. (54) to arbitrary J as it stands because, while the left side satisfies the conditions of Carlson's theorem and in fact vanishes as J becomes infinite, the right side contains contributions which do not satisfy the conditions of the theorem. This is because z_{12}, z_{13}, z_{23} attain the value $+1$ on regions of the phase-space boundary. For example, that part of the curve (57), for which $z_{12}=+1$, is indicated in Fig. 6.

The resolution of the difficulty involves replacing the simple, flat, phase-space integration contour by one which is suitably deformed around some of the singularities of the integrand. When J is an integer these singularities vanish and the integration contour can resume its standard shape. It is convenient to understand this resolution by considering as we do in the next section, a doubly projected partial-wave amplitude.

VI. DOUBLY PROJECTED PARTIAL WAVES

The three-particle phase-space integration may be written

$$\int d\rho(3)=\frac{1}{(2\pi)^9}\int_{4m_\pi^2}^{(s^{1/2}-m_\pi)^2} ds_1\left(\frac{p_1}{4s^{1/2}}\right)\left(\frac{q_2}{4s_1^{1/2}}\right)\int d\Omega_1 d\Omega_{23}\,, \quad (58)$$

where $d\Omega_1$ is the differential solid angle for $\mathbf{p}_1$ in the overall center-of-mass frame and $d\Omega_{23}$ is the same for $\mathbf{q}_2$ in the (2,3) center-of-mass frame. In terms of the Euler angle (φ,θ,ψ) we can write

$$d\Omega_1=d\cos\theta d\varphi=dz_1d\varphi\,,\quad d\Omega_{23}=dx_{12}d\psi\,. \quad (59)$$

The fact that the Euler angle ψ can be used in both frames of reference is due to the fact that they are related by a Lorentz transformation along $\mathbf{p}_1$ which leaves ψ invariant. Notice that $d\Omega_{23}$ involves an integration over x_{12} and therefore over one of the subenergies.

If we use Eq. (58) in order to evaluate the first term on the right side of Eq. (31) we find that it becomes

$$i(2\pi)^4\int d\rho(3)\, A_1^{(+)}(s,s_1,z_1)T_1^{(+)}(s;\,s_1,s_2,s_3,z_1',z_2',z_3')$$
$$=\frac{i}{(2\pi)^5}\int_{4m_\pi^2}^{(s^{1/2}-m_\pi)^2} ds_1\left(\frac{p_1}{4s^{1/2}}\right)\left(\frac{\pi q_2}{s_1^{1/2}}\right)$$
$$\times\int d\Omega_1\, A_1^{(+)}(s,s_1,z_1)M_1^{(+)}(s,s_1,z_1')\,, \quad (60)$$

where

$$M_1^{(+)}(s,s_1,z_1)=\frac{1}{4\pi}\int T_1^{(+)}(s;\,s_1,s_2,s_3,z_1,z_2,z_3)d\Omega_{23}\,. \quad (61)$$

We can interpret Eq. (61) as creating an amplitude $M_1^{(+)}$ which describes the production of the pair (2,3) not only with zero helicity along $\mathbf{p}_1$ but also with zero relative angular momentum in their own center-of-mass frame. The most general amplitude of this type with helicity M and angular momentum L would be generated by inserting the spherical harmonic $Y_{LM}(x_{12},\psi)$ as a factor in the integrand of Eq. (61).

The reason for considering this type of amplitude is that the positions of its z_1-plane singularities are relatively stable under variations of s_1. For the particular mass conditions we have imposed, Eq. (36), they remain on the positive real z_1 axis for physical values of s_1.

We can verify this by examining the effect of the projection just described on the various terms of $T_1^{(+)}$. It has, of course, no effect on $A_1^{(+)}(s,s_1,z_1)$ which contributes, unmodified, to $M_1^{(+)}$. The effect on the other incomplete amplitudes may be illustrated by applying the projection to one of the poles of $A_2^{(+)}(s,s_2,z_2)$, for example the term,

$$\Gamma_2/(\mu_2^2-u_2)\,. \quad (62)$$

Since the projection is made in the (2.3) center-of-mass frame, it is convenient to express the pole term as

$$\frac{1}{2q_Bq_2}\frac{\Gamma_2}{\eta_2+y_2}\,, \quad (63)$$

where

$$\eta_2=(\mu_2^2-m_B^2-m_\pi^2+2q_{B0}q_{20})/2q_Bq_2\,. \quad (64)$$

The result of doing the projection is $F(s_1,t_1)$, given by

$$F(s_1,t_1)=(\Gamma_2/2q_Bq_2)Q_0(\eta_2)\,. \quad (65)$$

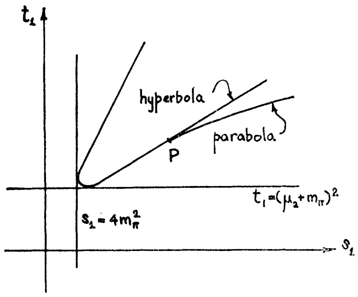

FIG. 7. Singularity curves of $F(s_1,t_1)$. The touching point between the curves is P.

The singularities of $F(s_1,t_1)$ as a function of t_1 occur when

$$\eta_2=\pm 1 \tag{66}$$

and

$$q_B=0. \tag{67}$$

In terms of s_1 and t_1 these equations become

$$m_\pi^2(s_1+m_B^2-t_1)^2+s_1(\mu_2^2-m_B^2-m_\pi^2)(s_1+m_B^2-t_1) +s_1[\lambda(\mu_2^2,m_B^2,m_\pi^2)+s_1m_B^2]=0, \tag{66'}$$

$$\lambda(s_1,m_B^2,t_1)=0. \tag{67'}$$

The former curve (66′) is a hyperbola in the (s_1,t_1) plane and the latter (67′) is a parabola. They touch at the point

$$s_1=(\mu_2^2-m_B^2-m_\pi^2)^2/m_B^2, \quad t_1=(\mu_2^2-m_\pi^2)^2/m_B^2. \tag{68}$$

Those parts of the curves which are singular on the sheet of interest are illustrated in Fig. 7. We can see that for $s_1<(\mu_2^2-m_B^2-m_\pi^2)^2/m_B^2$, $F(s_1,t_1)$ has singularities at $t_1=t_1^{(\pm)}$, the two roots of Eq. (66′),

$$t_1^{(\pm)}=s_1+m_B^2+s_1(\mu_2^2-m_B^2-m_\pi^2)/2m_\pi^2 \pm[s_1(s_1-4m_\pi^2)\lambda(\mu_2^2,m_B^2,m_\pi^2)]^{1/2}/2m_\pi^2. \tag{69}$$

The attached cut may be chosen to lie between the singularities. For $s_1>(\mu_2^2-m_B^2-m_\pi^2)^2/m_B^2$, these two points remain singular and the larger root of Eq. (67′)

$$t=(s_1^{1/2}+m_B)^2 \tag{70}$$

also becomes singular. The additional cut may be chosen to lie between this singularity and $t=t_1^{(-)}$. It follows, then, that for $s_1>4m_\pi^2$, the singularities of $F(s_1,t_1)$ lie on the positive real t_1 axis and therefore on the positive real z_1 axis.

The remaining contributions to $M_1^{(+)}(s,s_1,z_1)$ may be analyzed in the same way and the same conclusions drawn. Consequently, we have verified the statement made above about the positions of the singularities and cuts of $M_1^{(+)}$.

For integer values of J we can complete the double-projection procedure and define

$$M_1^{(+)}(J,s,s_1)=\frac{1}{2}\int_{-1}^{1}dz_1M_1^{(+)}(s,s_1,z_1)P_J(z_1). \tag{71}$$

It can be continued to arbitrary values of J by the Froissart-Gribov prescription

$$M_1^{(+)}(J,s,s_1)=\frac{1}{\pi i}\int_C dz_1\,M_1(s,s_1,z_1)Q_J(z_1). \tag{72}$$

Again C may be deformed around the singularities and cuts of $M_1^{(+)}(s,s_1,z_1)$ and from what we have pointed out about their position we can deduce that, for large J, the partial wave $M_1^{(+)}(J,s,s_1)$ is bounded by $Ke^{-\gamma\,\mathrm{Re}J}$, where K is a constant and $\gamma>0$.

It is clear from Eq. (60) and corresponding transformations of the other contributions to the right side of Eq. (31) that we can write, for integral J,

$$\Delta_3T_{AB}^{(+)}(J,s)=\frac{i}{(2\pi)^5}\int_{4m_\pi^2}^{(s^{1/2}-m_\pi)^2}ds_1\frac{\pi^2p_1q_2}{(ss_1)^{1/2}} \times A_1^{(+)}(J,s,s_1)M_1^{(+)}(J,s,s_1)+\text{other similar terms}. \tag{73}$$

Furthermore, since both sides have continuations to arbitrary J which obey the conditions of Carlson's theorem, Eq. (73) remains correct for arbitrary J. However, the comparison between the two types of partial wave, which we undertake in the next section, shows how we can use Eq. (73), which uses doubly projected amplitudes, to deduce the correct modification of Eq. (31), which uses singly projected amplitudes.

VII. COMPARISON BETWEEN SINGLY AND DOUBLY PROJECTED PARTIAL WAVES

For integral J, the relationship is straightforward, we have

$$M_1^{(+)}(J,s,s_1)=\frac{1}{16\pi^2}\int d\Omega_1d\Omega_{23} \times T_1^{(+)}(s;s_1,s_2,s_3,z_1,z_2,z_3)P_J(z_1). \tag{74}$$

Using expressions (59) for the differential solid angles we find

$$M_1^{(+)}(J,s,s_1)=\frac{1}{2}\int_{-1}^{1}dx_{12}\,T_{1,0}^{(+)}(J,s;s_1,s_2,s_3), \tag{75}$$

so that the (2,3) relative angular momentum can be projected out last. The relationship between x_{12} and (s_1,s_2) is

$$x_{12}=(s_2+s_1-2q_{10}q_{20}-2m_\pi^2)/2q_1q_2 \tag{76}$$

so that

$$dx_{12}=ds_2/2q_1q_2. \tag{77}$$

Therefore,

$$M_1^{(+)}(J,s,s_1)=\frac{1}{4q_1q_2}\int_{s_2^{(-)}}^{s_2^{(+)}}ds_2T_{1,0}^{(+)}(J,s;s_1,s_2,s_3), \tag{78}$$

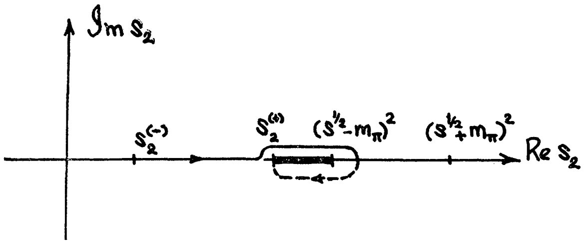

FIG. 8. Integration contour deformed around singularities of the integrand.

where $s_2^{(\pm)}$ are the largest and smallest roots of Eq. (57), that is,

$$s_2^{(\pm)}=2m_\pi^2+\tfrac{1}{2}(s-s_1-m_\pi^2) \pm\tfrac{1}{2}[(s_1-4m_\pi^2)\lambda(s,s_1,m_\pi^2)/s_1]^{1/2}. \quad (79)$$

If s_c is the value of s_1 for which $s_2^{(+)}$ attains a maximum, then, as indicated in Fig. 6, when $4m_\pi^2<s_1<s_c$, neither z_{12} nor z_{13} attain the value $+1$ for any of the physical values of s_2. It follows, therefore, that, when $4m_\pi^2<s_1<s_c$, Eq. (78) may be continued to arbitrary values of J since both sides satisfy the conditions of Carlson's theorem. However, if $s_1>s_c$, z_{12} becomes $+1$ when $s_2=s_2^{(+)}$ and z_{13} becomes $+1$ when $s_2=s_2^{(-)}$. Under these circumstances the right side of Eq. (78) fails to satisfy the conditions of Carlson's theorem so that it cannot be used, as it stands, for arbitrary J.

The way to overcome the difficulty is to notice that $M_1^{(+)}(J,s,s_1)$ is an analytic function of s_1. If we continue both sides of (78), therefore, from values of s_1 near $4m_\pi^2$ to values greater than s_c we should be able to deduce an expression for $M_1^{(+)}(J,s,s_1)$ with s_1 in this latter region. In making the continuation we will have to allow for the necessity of deforming the integration contour around the singularities of $T_1^{(+)}(J,s;s_1,s_2,s_3)$.

If we refer back to the definition of $A_i^{(+)}(J,s,s_1)$ [Eq. (51)] we see that it can be written

$$A_i^{(+)}(J,s,s_1)=(p_Ap_i)^JB_i^{(+)}(J,s,s_i), \quad (80)$$

where $B_i^{(+)}(J,s,s_i)$ is an analytic function of s_i at points where p_i vanishes. It follows, then, from Eq. (52), that $T_{1,0}^{(+)}(J,s;s_1,s_2,s_3)$ is singular when

$$s_2=(s^{1/2}-m_\pi)^2, \quad (81a)$$

$$s_3=(s^{1/2}-m_\pi)^2 \quad (81b)$$

and also at points on the boundary curve, Eq. (57), where $z_{12}=+1$ and $z_{13}=+1$. The singular curves (81) both touch the edge of the physical region when $s_1=s_c$, as shown in Fig. 6.

As s_1 is continued towards the value s_c, the end point of the integration contour in Eq. (78) $s_2^{(+)}$ moves towards the singularity (81a) while $s_2^{(-)}$ moves towards (81b). In order to avoid a coincidence of the end points with singularities of the integrand we reach values of s_1 beyond s_c by continuing clockwise in the complex s_1 plane round this point. Both ends of the integration contour become deformed round their respective singularities. For simplicity we discuss only the deformation of the contour attached to $s_2^{(+)}$ which is illustrated in Fig. 8. The deformation at the other end is, of course, quite similar.

In Fig. 8, the cut which lies between $s_2=s_2^{(+)}$ and $s_2=(s^{1/2}-m_\pi)^2$ arises from the cut $1\leqq z_{12}<\infty$ of the function $P_J(-z_{12})$. The cut lying between $s_2=(s^{1/2}-m_\pi)^2$ and $s_2=(s^{1/2}+m_\pi)^2$ arises from the singular factor $(p_2)^J$ of $A_2^{(+)}(J,s,s_2)$. The integration contour passes above the former of these cuts through the latter onto the second sheet of p_2 and back down to $s_2^{(+)}$, which is nonsingular on this sheet. From an examination of the effect of continuing z_{12} and p_2 along the integration contour it can be verified that part of the integration contour, which lies outside the physical region, can be reduced to a simple integration from $s_2^{(+)}$ to $(s^{1/2}-m_\pi)^2$ with an integrand

$$B_2^{(+)}(J,s,s_2)p_A{}^J[p_2{}^JP_J(-z_{12})-(e^{-i\pi}p_2)^JP_J(z_{12})] =-A_2^{(+)}(J,s,s_2)2/\pi\,\sin\pi J Q_J(z_{12}). \quad (82)$$

If we had continued anticlockwise round s_c, we would have obtained an s_2-contour with the complex-conjugate deformation. However, we would have been able to reduce the result to the same simple integration as above, thus confirming what we already know, namely, that $M_1^{(+)}(J,s,s_1)$ does not have a singularity at $s=s_c$.

We continue to use Eq. (78), then, to compute $M_1^{(+)}(J,s,s_1)$ even for values of $s_1>s_c$, provided we take into account the deformations at the ends of the s_2 contour. It is obvious, particulaly from Eq. (82), that when J is an integer the deformation has no effect and is, therefore, optional.

If we interpret the phase-space integration [Eq. (56)] as

$$\int d\bar{p}(3)=\frac{1}{(2\pi)^9}\left(\frac{\pi^2}{4s}\right)\int_{4m_\pi^2}^{(s^{1/2}-m_\pi)^2}ds_1\int_{s_2^{(-)}}^{s_2^{(+)}}ds_2 \quad (83)$$

and insert it into the first term on the right side of Eq. (54) with the s_2 contour deformed in the above manner we see, immediately, that this term reduces to the first term on the right side of Eq. (73) *even for complex J*. If, then, we deform the phase-space integrations of the other contributions to this equation similarly in an appropriate way we will have constructed the desired continuation to arbitrary values of angular momentum. This completes our discussion.

VIII. CONCLUSIONS

In this paper we have tried to gain some understanding of the problem of continuing multiparticle partial-wave amplitudes to arbitrary angular momentum. To simplify the problem we consider the effect of a three-particle intermediate state on the imaginary part of a two-particle elastic scattering amplitude.

We were led immediately to consider how this three-particle state affects the two-particle amplitudes of definite signature. Of course, an extensive understanding of the analytic structure of the relevant production amplitudes is necessary for a complete solution of this problem. However, we considered only the effect of a few simple pole contributions to these amplitudes. The result was still nontrivial for we found that it was not possible to construct a unitary production amplitude of definite signature. It follows that taking into account three-particle states makes it necessary to discuss amplitudes (and, presumably, trajectories) of both signatures.

Nevertheless, unitarity-like equations were obtained but they involved three different production amplitudes of definite signature matched, in the phase-space integral, with related incomplete amplitudes also of definite signature. In the exact case we would not expect to achieve a reorganization of unitarity quite as simple as our equations. However, we do believe that the idea of there being three different production amplitudes of definite signature will be useful in this case also.

We projected out partial-wave production amplitudes (for integer angular momentum) using the same techniques as Omnes and Alessandrini. The natural choice for the z axis in the three-particle state was along the space momentum of one of the particles but a different particle was appropriate for each of the three amplitudes.

Continuation in angular momentum was achieved by using a generalization of the Froissart-Gribov technique very similar to that suggested by Omnes and Alessandrini. Of course, *we* obtained three different analytic continuations and they were all involved in the continuation of the unitarity-like equations for the two-particle partial waves.

All three of the continuations suffer from the defect that there are regions on the boundary of the physical region for which they fail to satisfy the conditions of Carlson's theorem. However, we were able to overcome this difficulty by showing that they are to be used along with carefully deformed phase-space integrations when the angular momentum is complex. For integer values the deformation is irrelevant. The necessity for using deformed phase-space contours we expect to remain in the general case.

We demonstrated the above result by introducing doubly projected partial-wave amplitudes. These were defined by projecting out, first, not only the helicity but also the orbital angular momentum of one pair of the three particles. Afterwards the total angular momentum is projected out. The reason for introducing this type of amplitude was that performing the first two projections creates an amplitude whose analytic properties as a function of the remaining scattering angle are rather satisfactory. They permit the doubly projected amplitude to be continued to complex angular momentum in a manner which satisfies the conditions of Carlson's theorem. For this reason we believe, also, that these doubly projected amplitudes will have a role to play in a complete theory.

ACKNOWLEDGMENT

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