Approximate Supermultiplet Model Consistent with Relativity and Unitarity

N. J. PAPASTAMATIOU AND J. C. TAYTOR
Clarendon Laboratory, University of Oxford, Oxford, England
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We construct a relativistic model field theory which yields an approximate $SU(6)$ invariance. The Hamiltonian is H_0+H', where H_0 is invariant under an inhomogeneous $SL(6)$ with 36 translation generators P_A. H' is small, but it does not commute with the 32 unphysical P_A; and so its presence abolishes the problem of giving them a physical interpretation. Because of H', physical states involve a wave function of the unphysical momenta which, to a zeroth approximation, has no "spin-orbit" mixing and gives pure $SU(6)$ states. Treating the small "spin-orbit" term as a perturbation leads to deviations from $SU(6)$ symmetry. Other qualitative predictions of the model are: (a) that matrix elements should have an approximate $SL(6)$ structure except insofar as they are sensitive to the detailed spectrum of intermediate states; (b) that the matrix elements should include "irregular" terms even in zeroth-order approximation; and (c) that there should be excited states which behave like approximately degenerate sets of incomplete $SU(6)$ multiplets, and whose presence as intermediate states makes the model consistent with unitarity. The model does not include a parity operation, but the possibility of generalization to an inhomogeneous $S\tilde{U}(12)$ group is briefly and incompletely discussed. The Wigner-Bargmann equations seem to arise naturally.

1. INTRODUCTION

In spite of the successes of $SU(6)$[1] and its relativistic generalizations,[2–5] it seems impossible to construct a local, relativistic field theory which exhibits this type of invariance. The difficulty has been pinpointed by Wyld.[6] Even without demanding locality, there tends to be trouble with unitarity.[7]

In terms of a Lagrangian formulation, the trouble shows up in the Dirac kinetic energy $\boldsymbol{\alpha}\cdot\mathbf{p}$, which is not $SU(6)$-invariant.[8] One way out is to suppose that this term, for some reason or other, may be regarded as small, and used as a perturbation to give small corrections to $SU(6)$.[8] Alternatively, it is not difficult to make the kinetic energy invariant by increasing the number of momenta from 3 to 35 or more[4]; but one then encounters formidable problems of physical interpretation. The present paper seeks to answer the question: Can one break the translational symmetry connected with the extra, unphysical momenta, and so abolish the interpretation problem, but at the same time retain an approximate $SU(6)$ invariance? The answer suggested by our model is "yes".

The model is as follows: The underlying broken symmetry is that of an inhomogeneous $SL(6)$ group, the generators of the translations being chosen to transform as an Hermitian **36** representation of $SL(6)$. The Hamiltonian is H_0+H'. H_0 is constructed to be invariant under the whole inhomogeneous $SL(6)$ group. The natural way to attain this is to write H_0 in terms of fields which are functions of 36-dimensional coordinate "vectors". Presumably most of the ordinary features of quantum mechanics may be generalized to this situation. This is so because the "vectors" have two important properties: (a) there is a single (sixth-order) invariant, and (b) "time-like" vectors may be defined. The little-group of a "time-like" 36-momentum is $SU(6)$. Therefore single-particle eigenstates of H_0 may be designated $|p,s\rangle$, where p is a "time-like" 36-momentum whose invariant is fixed, and s stands for quantum numbers labelling the components of a particular $SU(6)$ representation.

H' has to break the 32 unphysical translational invariances. It inevitably breaks $SL(6)$ invariance as well. But it can preserve the $SL(2)\times SU(3)$ subgroup, where $SL(2)$ corresponds to the Lorentz group. For any 36-momentum p, it is convenient to write $p=(p_0,\mathbf{p},p_{\mu a})$, where p_0 is the energy, $\mathbf{p}$ the physical 3-momentum, and $p_{\mu a}(\mu=0,\cdots 3,\ a=1,\cdots 8)$ the 32 unphysical momenta. We shall also write q to stand for the $p_{\mu a}$. Assume that H' is small, in the sense that its matrix elements are small compared with some mass M which is characteristic of H_0 and which determines the spacing of the particle states described by H_0. Then, to a zeroth approximation, the chief effect of H' will be to mix neighboring eigenstates; that is to say, ones which differ only in their values of q. Thus an eigenstate of

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[2] R. P. Feynman, M. Gell-Mann, and G. Zweig, Phys. Rev. Letters **13**, 678 (1964); B. Sakita and K. C. Wali, *ibid.* **14**, 404 (1965); K. Bardakci, J. M. Cornwall, P. G. O. Freund, and B. W. Lee, *ibid.* **13**, 698 (1964); **14**, 48 (1965).

[3] A. Salam, R. Delbourgo, and J. Strathdee, Proc. Roy. Soc. **284** A, 146 (1965); A. Salam, J. Strathdee, J. C. Charap, and P. T. Matthews, Phys. Letters **15**, 184 (1965); J. M. Charap and P. T. Matthews, *ibid.* **13**, 34 (1964).

[4] T. Fulton and J. Wess, Phys. Letters **14**, 57 (1965); *ibid.* **14**, 334 (1965); H. Bacry and J. Nuyts, CERN Report 10068/TH509 (unpublished); W. Rühl, Nuovo Cimento **37**, 301, 319 (1965); W. Rühl, Phys. Letters **14**, 346 (1965).

[5] W. Rühl, Phys. Letters **15**, 340 (1965); R. Oehme, Phys. Rev. Letters **14**, 664 (1965).

[6] H. W. Wyld (unpublished).

[7] M. A. B. Bég and A. Pais, Phys. Rev. Letters **14**, 509 (1965). However, with "irregular terms" (see Sec. 5 and Ref. 5) the trouble is not so acute. We are indebted to Professor T. Fulton for a discussion on this and other points.

[8] M. Gell-Mann, Phys. Rev. Letters **14**, 77 (1965); K. T. Mahanthappa and E. C. G. Sudarshan, *ibid.* **14**, 163 (1965).

H_0+H' is approximately expressible as

$$\sum_s \int d^{32}q f_s(q) |\mathbf{p},q;s\rangle. \tag{1.1}$$

In Sec. 3 we choose a simple form for H' and deduce a sort of Schrödinger equation for the wave function $f_s(q)$. The equation is the same as that for an harmonic oscillator in 32 dimensions, with the addition of small "spin-orbit" coupling terms. Aside from the scaling mass M, there is a dimensionless parameter λ which we assume to be small, since H' is small. Neglecting the spin-orbit coupling, the ground-state wave function is

$$\exp[-(p_{0a}^2+p_{ia}^2)/(6^{1/2}\lambda^2M^2)], \tag{1.2}$$

where $i=1, 2, 3$. If $\lambda\ll 1$, the "spin-orbit" terms are of order λ compared to the main term, and their neglect to zeroth order is justified. To this approximation, the states constructed from $f_s(q)$ are pure degenerate members of an $SU(6)$ multiplet.

If the "spin-orbit" corrections are treated by perturbation theory, "spin-orbit" mixing [that is, $SU(6)$ impurity] is found to order λ. Mass splitting of the $SU(6)$ multiplets is of order $M\lambda^4$.

The "Schrödinger" equation for $f_s(q)$ also has excited solutions, separated from the ground state by energies of order $M\lambda^2$. To zeroth approximation, the excited states look like degenerate sets of incomplete $SU(6)$ multiplets. There are "orbital" contributions to their spin and $SU(3)$ quantum numbers, and they are not pure $SU(6)$ states.

In Sec. 5, we suggest rules expressing the restrictions placed on physical matrix elements by the approximate $SL(6)$ invariance of the underlying theory. Symmetry-breaking terms involving the (physical) 4-momenta ("spurions," "kinetons" or "irregular" couplings[5]) appear automatically in zeroth approximation. They are in no way expected to be small. The restrictions on matrix elements are not expected to hold in regions which are sensitive to the exact spectrum of intermediate states (near thresholds, etc.), but elsewhere they are expected to hold. This does not contradict unitarity, since excited intermediate states simulate the continua of states (depending on q) which would be $SL(6)$-invariant.

The model discussed up to this point does not contain a parity operation, so only its qualitative features are expected to be physically relevant. To include parity, one must enlarge the inhomogeneous $SL(6)$ group. One possibility is to go to an inhomogeneous $S\tilde{U}(12)$.[3] But the underlying space then is not sufficiently like Lorentz space for it to be clear how to formulate quantum mechanics. One may nevertheless ignore this lack of understanding, and attempt to formally generalize some features of the $SL(6)$ theory to $S\tilde{U}(12)$. In Sec. 6 this line of thought is pursued far enough to show that the Wigner-Bargmann equations[3] (as well as "irregular" couplings) appear naturally in such a generalization.

The consequences of our models are summarized in Sec. 7.

2. THE INHOMOGENEOUS $SL(6)$ GROUP

We are going to adopt an inhomogeneous $SL(6)$ group as an underlying symmetry, broken by a small term H' in the Hamiltonian. We summarize a few properties of this group, which are mostly to be found in the papers of Fulton and Wess, Rühl, and Bacry and Nuyts.[4]

$SL(6)$ is the group of 6-by-6 complex matrices of determinant 1. To construct its representations, one must distinguish between upper and lower, dotted and undotted indices. To define an inhomogeneous group we need a set of real commuting generators with specified $SL(6)$ transformation properties. The usual choice is the 36-fold Hermitean representation $P_{\alpha\dot{\beta}}$. Its eigenvalues will be written $p_{\alpha\dot{\beta}}$, and a conjugate space will consist of the Hermitean coordinates $x^{\dot{\alpha}\beta}$. This choice has the following important properties: (a) There is only one invariant, the determinant $\det p_{\alpha\dot{\beta}}$ or $\det x^{\dot{\alpha}\beta}$. (b) The momenta may be classified according to the signs of their matrix eigenvalues. In particular, a momentum with $\det p_{\alpha\dot{\beta}}>0$ and all eigenvalues positive (negative) may be termed "positive (negative) time-like."

One can introduce a complete set of Hermitean matrices $h_A=(1,\sigma_i,\lambda_a,\sigma_i\lambda_a)$, $(A=0, 1, \cdots, 35; i=1, 2, 3; a=1, \cdots, 8)$ where σ_i are the Pauli matrices and λ_a are the $SU(3)$ matrices defined by Gell-Mann.[9] A momentum may then be expanded

$$p_{\alpha\dot{\beta}}=\sum_A p_A(h_A)_{\alpha\dot{\beta}}. \tag{2.1}$$

For a positive time-like momentum with $\det p_{\alpha\dot{\beta}}=M^6$, we shall find it convenient to use the notation

$$p_A=(p_0,p_i,p_{0a},p_{ia})=(p_0,\mathbf{p},p_{\mu a})=(p_0,\mathbf{p},q) \tag{2.2}$$

$(\mu=0, 1, 2, 3)$, where $p_0=p_0(\mathbf{p},q,M)\geq M$. To second order in $\mathbf{p}$ and q, there is the approximation

$$p_0\simeq M+(\tfrac{1}{2}\mathbf{p}^2+\tfrac{1}{3}p_{0a}^2+\tfrac{1}{3}p_{ia}^2)/M. \tag{2.3}$$

Using numerically the same matrices h_A, we shall write

$$x^{\dot{\alpha}\beta}=\sum_A x^A(h_A)^{\dot{\alpha}\beta}, \tag{2.4}$$

$$x^A=(t,x^i,x^{0a},x^{ia}). \tag{2.5}$$

The representations of $SL(6)$ may be designated by a pair of integers (m,n) corresponding to the multiplicities of the representations of two $SU(6)$ groups. In this notation, $p_{\alpha\dot{\beta}}$ is a $(6,6^*)$. Parity (defined, say, as the operation of reversing the sign of all 35 "space" axes) would involve $(m,n)\rightarrow(n,m)$. Since $(6,6^*)\neq(6^*,6)$, there is no parity operator. The situation is different in $SL(2)$, because $2^*=2$ for $SU(2)$. In $SL(6)$ a parity operation would exist if the momentum were chosen as a $(20,20^*)$ (since $\mathbf{20}=\mathbf{20}^*$); but in that case there would

[9] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).

be many invariants formed from a momentum, and it would not be obvious how to formulate quantum mechanics.

We want a Hamiltonian H_0+H', such that H_0 has the complete inhomogeneous $SL(6)$ symmetry. We assume, without detailed verification, that such an H_0 may be constructed by using Lagrangian field theory, the fields being functions of the 36 x^A with $x^0=t$ playing the role of time. There will exist, for instance, a 36×36 stress tensor $\Theta_A{}^B(x)$, such that

$$P_A=\int d^{35}x\ \Theta_{A0}(x)\,, \tag{2.6}$$

where $d^{35}x=dx^1\cdots dx^{35}$. The causality condition, presumably, is that two fields commute unless their separation is positive or negative time-like.

The state vectors may be classified by the invariant M^6 of their momentum and a symbol r denoting a representation of the little group. Since a positive time-like momentum becomes, in its rest frame, a multiple of the unit matrix, the little group is $SU(6)$. Thus a single-particle state vector may be written

$$|M;\mathbf{p},q;r;s\rangle\,, \tag{2.7}$$

where s stands for the quantum numbers of a component of the $SU(6)$ representation r.

$SL(6)$ contains the subgroup $SL(2)\times SU(3)$, where $SL(2)$ can be identified with (the covering group of) the Lorentz group. The notation $(p_0,p_i;\ p_{0a},p_{ia})=(p_\mu,p_{\mu a})$ corresponds to labelling according to the transformation properties under this subgroup. In the "rest frame" ($\mathbf{p}=0$, $p_{\mu a}=0$), one may choose s to include J, J_3 and u, where u is a set of $SU(3)$ quantum numbers, and $\mathbf{J}$ is the ordinary angular momentum defined by the rotation subgroup of $SL(2)$.

For certain representations r of $SU(6)$, these quantum numbers (J, J_3 and u) are *sufficient* to label the states. These representations include $r=\mathbf{35}$, **56**, **56***. In their rest frame, such states may be written

$$|M;\mathbf{0},0;r,J,J_3,u\rangle.$$

In a general reference frame the same number of quantum numbers suffices; but J, J_3 must be replaced by J', J_3', which have a slightly more complicated definition. The ensuing discussion is limited to $SU(6)$ multiplets with the above property, otherwise complications concerned with mixing would be encountered. In particular, the representation **405**, which has been proposed for the higher meson resonances, contains two vector octets, and so would require special consideration.

3. THE SYMMETRY-BREAKING HAMILTONIAN

The formalism of Sec. 2 leads to an exact $SL(6)$ symmetry, but it suffers from the presence of 32-momenta (or coordinates) which appear to correspond to nothing physical. Therefore we put a small term H' into the Hamiltonian to break the unwanted translational invariances. It seems to us that, even if H' is quite small, it completely abolishes the problem of physical interpretation.

In more detail, H' must have the following properties:

(i) It must not commute with the unphysical $P_A(A=4,\cdots 35)$.

(ii) It must preserve the subgroup $SL(2)\times SU(3)$. [We do not wish to complicate the discussion by breaking $SU(3)$ at the same time as $SU(6)$, but of course this possibility should be examined.]

(iii) H' must give rise (see Sec. 4) to a "Schrödinger equation" for the physical states which possesses *some* discrete eigenvalues. It is probably physically necessary for *all* the eigenvalues to be discrete. Our model has this property too.

The simplest H' we have been able to construct which meets all these requirements is

$$H'=\lambda^4M^2\int d^{35}x\Theta_\mu{}^\nu(x)V^\mu{}_\nu(x)\,, \tag{3.1}$$

where

$$V^{\mu\nu}(x)=x^{\mu a}x^{\nu a}-\tfrac{1}{2}g^{\mu\nu}x^{\lambda a}x_\lambda{}^a. \tag{3.2}$$

In these equations $\Theta_\mu{}^\nu$ is a subtensor of the symmetric stress tensor $\Theta_A{}^B$ defined in Sec. 2. It is a function of the fields. $V^{\mu\nu}(x)$ on the other hand depends *explicitly* on the unphysical coordinates, and so it breaks the unwanted translational symmetries. A tensor $V^{\mu\nu}$ rather than a scalar V seems to be necessary to satisfy condition (iii) above, by making the "potential" in the "Schrödinger equation" have a definite sign. The identification of $\Theta_\mu{}^\nu$ with the stress tensor serves to restrict the form of its matrix elements. It may not be necessary.

It is hardly to be supposed that (3.1) and (3.2) give the true form of the symmetry-breaking mechanism. We adopt them as a simple model, hoping that some of the conclusions are more general than the model. Presumably the possibility of a spontaneous breakdown of symmetry ought to be explored.

In the ensuing discussion, we shall need matrix elements of H' between the single-particle eigenstates of H_0 (2.7),

$$\begin{aligned}\langle\mathbf{p},q;s|H'|\mathbf{p}',q';s'\rangle\\ &=\lambda^4M^2\langle\mathbf{p},q;s|\Theta_\mu{}^\nu(0)|\mathbf{p}',q';s'\rangle\\ &\quad\times\int d^{35}x\ V^\mu{}_\nu(x)\exp[i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}+i(p_{\lambda a}-p'_{\lambda a})x^{\lambda a}]\\ &=\lambda^4M^2\langle\mathbf{p},q;s|\Theta_\mu{}^\nu(0)|\mathbf{p},q';s'\rangle\\ &\quad\times(2\pi)^{35}\delta^3(\mathbf{p}-\mathbf{p}')V^\mu{}_\nu(i(\partial/\partial q'))\delta^{32}(q-q')\,,\end{aligned} \tag{3.3}$$

where labels M and r have been omitted from the state vectors for brevity. For the matrix elements of $\Theta_\mu{}^\nu$ we have, from Eq. (2.6), for the special case $q=q'$,

$$(2\pi)^{35}\langle\mathbf{p};q;s|\Theta_\mu{}^0(0)|\mathbf{p},q;s'\rangle\delta^3(\mathbf{p}-\mathbf{p}')\delta^{32}(q-q') = \langle\mathbf{p},q;s|P_\mu|\mathbf{p}',q';s'\rangle,$$

or

$$(2\pi)^{35}\langle\mathbf{p},q;s|\Theta_\mu{}^0(0)|\mathbf{p},q;s'\rangle=2p_0p_\mu\delta_{ss'}, \quad (3.4)$$

using "relativistic" normalization of the state vectors. Since the time component occurring in Eq. (3.4) is in fact the projection along an arbitrary time-like direction, we must have

$$(2\pi)^{35}\langle\mathbf{p},q;s|\Theta_\mu{}^\nu(0)|\mathbf{p},q;s'\rangle=2p_\mu p^\nu\delta_{ss'}. \quad (3.5)$$

4. PHYSICAL STATE VECTORS

Assume that the spectrum of eigenstates of H_0 contains, among others, single-particle states separated by energies of the order of some characteristic mass M. We shall take the same mass M to be the mass of a particular state under consideration. Assume farther that the matrix elements of H' are small compared with M. Then the most important effect of H' will be to mix states which have neighboring values of q but otherwise identical quantum numbers (in so far as the other quantum numbers may be compared when the q's differ). Further, the ground states of H_0+H' will be formed mainly from small values of q. The effect is similar to that of a weak central potential on an otherwise free particle: it breaks the translational symmetry and alters the form of the original continuous spectrum. The way the spectrum is changed depends upon whether the potential is attractive or repulsive, etc. We shall shortly see that the choice (3.1), (3.2) for H' corresponds to an harmonic-oscillator potential, which produces a spectrum of discrete states only.

For these reasons, we seek an approximate eigenstate of H_0+H' of the form (taking $\mathbf{p}=0$).

$$|E;r,JJ_3,u\rangle=\int\sum_{s'}d^{32}q'\,f_{s'}(q')|M;\mathbf{0},q';r,s'\rangle, \quad (4.1)$$

satisfying

$$(H_0+H')|E;r,JJ_3,u\rangle\simeq E|E;r,JJ_3,u\rangle. \quad (4.2)$$

Equation (4.1) is just (1.1) written more completely in the notation of (2.7). Now insert (4.1) into (4.2), take the scalar product with $\langle M;\mathbf{0},q;r,s|$ and use Eq. (3.3). The derivatives of the delta functions in (3.3) are removed by integrating twice by parts. Using (3.5), one finally obtains the equation

$$2p_0(E-p_0)f_s(q)=2\lambda^4M^2p_0{}^2V^{00}(i\partial/\partial q)f_s(q) +2\lambda^4M^3\textstyle\sum_{s'}A_{\mu a}(q;s,s')(\partial f_{s'}(q)/\partial p_{\mu a}) -\lambda^4M^2\sum_{s'}B(q;s,s')f_{s'}(q), \quad (4.3)$$

in which

$$MA_{\mu a}=(\delta_\mu{}^\rho\delta_\nu{}^\sigma-\tfrac{1}{2}g^{\rho\sigma}g_{\mu\nu}) \times[(\partial/\partial p'_{\nu a})\langle q,s|\Theta_{\rho\sigma}(0)|q',s'\rangle]_{q=q'}, \quad (4.4)$$

and

$$B=(\delta_\mu{}^\rho\delta_\nu{}^\sigma-\tfrac{1}{2}g^{\rho\sigma}g_{\mu\nu}) \times[(\partial^2/\partial p'_{\mu a}\partial p_{\nu a}')\langle q,s|\Theta_{\rho\sigma}(0)|q',s'\rangle]_{q=q'}. \quad (4.5)$$

Since $A_{\mu a}$ and B are dimensionless we assume them to be of order unity. Then the orders of magnitude of the three terms on the right-hand side of (4.3) are in the ratio

$$M^2:M\bar{q}:\bar{q}^2,$$

where $\bar{q}$ is a measure of the range of values of q over which $f_s(q)$ varies appreciably. If we assume that $\bar{q}\ll M$, the second and third terms on the right-hand side of Eq. (4.3) may be neglected, as a zeroth-order approximation. This is the neglect of the "spin-orbit" coupling referred to in the Introduction. The approximated equation may then be solved, and the consistency of the assumption $\bar{q}\ll M$ verified. For the same reasons, we insert the approximation (2.3) for p_0. Using Eq. (3.2) to substitute for V^{00}, the zeroth approximation to (4.3) is

$$(E-M)f_s(q)=[(p_{0a}{}^2+p_{ia}{}^2)/(3M) -\tfrac{1}{2}\lambda^4M^3(\partial^2/\partial p_{0a}{}^2+\partial^2/\partial p_{ia}{}^2)]f_s(q). \quad (4.6)$$

Equation (4.3), with its approximate form (4.6), is the "Schrödinger equation" mentioned in the Introduction. Since it is written in momentum space, the differentials constitute the "potential." It is now clear how the forms of (3.1) and (3.2) were chosen to give an attractive potential in (4.6). A scalar $V(x)$ in H' would inevitably have given a potential of indefinite sign because of the Lorentz metric.

Eq. (4.6) is just the Schrödinger equation for an harmonic oscillator in 32-dimensions. The ground-state solution was given in (1.2), and its energy is

$$E=M+32\lambda^2M/\sqrt{6}. \quad (4.7)$$

Thus to zeroth approximation, we have an unsplit $SU(6)$ multiplet, whose spin and $SU(3)$ quantum numbers are given by the original quantum numbers s, with no "orbital" admixture (it is consistent with the approximations already made to disregard the distinction between J, J_3 and J', J_3' made at the end of Sec. 2).

Expression (1.2) gives the wave function in the Lorentz frame $\mathbf{p}=0$. In a general frame, (1.2) becomes

$$\exp[-(p'_a{}^2-p'_a{}^\mu p'_{\mu a})/(6^{1/2}\lambda^2M^2)], \quad (4.8)$$

where

$$p_a'=p^\mu p_{\mu a}(p^\lambda p_\lambda)^{-1/2},$$
$$p_{\mu a}'=p_{\mu a}-p_\mu p_a'(p^\lambda p_\lambda)^{-1/2}.$$

The excited-state energies are separated from (4.7) by multiples of $(\frac{2}{3})^{1/2}\lambda^2M$. The first excited wave functions may be written

$$p_{\mu a}f_s(q). \quad (4.9)$$

In order to discover the spin and $SU(3)$ content of these states, observe that $p_{\mu a}$ transforms like an $SU(6)$ 35 with its vector singlet (p_i) missing. Thus, if $f_s(q)$ represents a **35**, the excited states (4.9) correspond to the content of **35**×**35**, but with the following spin-$SU(3)$ states missing, (1,1), (3,1), (5,1), (3,8), (1,8), (3,8), (5,8)—a total of $3\times35=105$ missing states. [Here (n,m) denotes states of spin multiplicity n and $SU(3)$ multiplicity m.] Similarly, the first excited state corresponding to a **56** consists of the content of **35**×**56** but with the following 168 states missing (2,8), (4,8), (2,10), (4,10), (6,10). In the zeroth approximation, *all* the first excited states are degenerate. Apart from this degeneracy, the excited states are not expected to exhibit any $SU(6)$ properties, both because the multiplets are not complete and because there are "orbital" contributions to the spin and $SU(3)$ quantum numbers.

We now mention briefly the sort of corrections to be expected from the "spin-orbit" terms $A_{\mu a}$ and B in Eq. (4.3). If these terms are treated by perturbation theory, it is consistent at the same time to expand (4.4) and (4.5) in powers of q/M. Thus the "spin-orbit" potential effective to first order is proportional to q/M, and so the first-order correction to the wave function contains the same factor. Take the example of a **35**. In the nonrelativistic limit, the $SU(6)$ spin state may be represented by a traceless 6×6 Hermitian matrix

$$S=\sum_{A=1}^{35} h_A S_A\,,$$

where $S_A=(S_i,S_a,S_{ia})$, and the h_A were defined before Eq. (2.1). The zeroth-order wave function may be written $F(q^2)S$, and then the first-order correction must have the structure

$$G(q^2)[\{q,S\}-\mathrm{Tr}\{q,S\}]/M\,, \qquad (4.10)$$

where

$$q=\sum_{1}^{35} h_A q_A \quad \text{and} \quad q_A=(0,p_a,p_{ia})\,.$$

The expression (4.10) exemplifies the "orbital" contributions to spin and $SU(3)$ which are generated by the "spin-orbit" parts of the potential.

Corrections to the energy-levels of the Schrödinger equation (4.3) appear only to second order in q/M, and are therefore of order $\lambda^4 M$.

5. MATRIX ELEMENTS IN $SL(6)$

In this section we argue that our model will impose $SL(6)$ rules on scattering and decay matrix elements, in the zeroth approximation and under certain circumstances. The rules however allow "irregular" terms[5] (spurions or kinetons), even in the zeroth approximation.

There is an S matrix corresponding to the unbroken $SL(6)$ symmetry of H_0 (see Sec. 2). Call it $\mathcal{S}$. It is a function of the 35-momentum $(\mathbf{p},q)$ and the $SU(6)$ quantum numbers (r,s) for each of the scattering particles. For certain ranges of the momenta (to be discussed below), the main effects of the symmetry breaking H' should be (i) to smear out slightly the $\delta^{32}(q_{\mathrm{final}}-q_{\mathrm{initial}})$ which appears in $\mathcal{S}$, and (ii) to alter the interpretation of the initial and final states (see Sec. 4). Denote the S matrix with the smeared-out δ^{32} function by $\mathcal{S}'$. We assume that the physical S matrix, S (between the lower particle or resonance states), is to be obtained approximately by multiplying $\mathcal{S}'$ by a zeroth-order wave function $f_s(q)$ (see Sec. 4) and integrating $d^{32}q$ for each scattering particle. The wave function will pick out values of $q\ll M$.

We must now discuss unitarity. Since we start from an Hermitian Hamiltonian H_0+H', the exact S matrix must be unitary; the question is whether the above procedure leads to a unitary approximation to it. Again, since H_0 is Hermitian, $\mathcal{S}$ must be unitary. At first sight we seem to be simply picking out the submatrix of $\mathcal{S}$ in which all the q's are zero, and it would be remarkable if this submatrix were also unitary. However, it must be remembered that Eq. (4.3) has a series of excited solutions, which presumably correspond to higher physical states. The true unitarity equation will have contributions from these higher intermediate states; and their combined effect may be approximated by an integral with respect to q over a continuum, which is manifestly $SL(6)$-covariant.

Thus there is no conflict with the gross effects of unitarity. However, in regions which are sensitive to the exact spectrum of intermediate states, the replacement of the discrete spectrum by the continuum will have a bigger effect, and we cannot expect the $SL(6)$ rules to be well obeyed. This will be the case near a branch point, for example.[10] Thus we arrive at a condition similar to that proposed by Salam *et al.*[3] Note, however, the crucial role in our model played by the discrete spectrum of excited states in simulating the effect of an $SL(6)$-invariant continuum.

Let us now examine some of the effects of the above procedure. Suppose that there are terms in $\mathcal{S}$ which depend on the momenta $(p_\mu,p_{\mu a})$ only through their mutual $SL(6)$ invariants, that is, terms without spin flip. The effect of the wave functions $f_s(q)$ is to make the q's effectively zero (compared with the p_μ's), so the resulting S-matrix term is a function of the mutual Lorentz invariants of the p_μ's, together with an $SL(6)$-invariant spin matrix element involving the $SU(6)$ quantum number s. Such a term has been called "regular."[5]

On the other hand, there may be terms in $\mathcal{S}$ which involve the momenta as 6×6 matrices $p_{\alpha\beta}$, not all their indices being saturated on other momenta. The wave

[10] This condition is not as stringent as it might appear. For example, *poles* due to $SU(6)$ multiplets in intermediate states seem *not* to violate the $SL(6)$ rules (with irregular terms). This is because the unphysical momenta q in the propagator at the pole are forced to be nearly zero by momentum conservation (or near conservation) if the external q's are close to zero. It is therefore no further restriction to enforce the condition that the pole correspond to a physical intermediate state. Where integrals over intermediate states are concerned (at branch points), however, this argument does not apply.

functions $f_s(q)$ remove the unphysical components and leave the "incomplete" matrix

$$p=\sum_{A=0}^{3} p_A h_A = p_\mu \sigma^\mu .$$

These terms therefore finally become $SL(6)$ invariants made out of the $SU(6)$ quantum numbers s and the "incomplete" momentum matrices $p_\mu\sigma^\mu$, and so are just the "irregular" terms proposed by Rühl.[5] Contrary to what might have been supposed, these terms are not expected to be smaller than the "regular" ones, but just part of the zeroth approximation in the theory.

In summary, the rules for finding the effect of the symmetry in the zeroth approximation to the model are:

(1) Write down all the formally $SL(6)$-invariant structures, regular and irregular.

(2) Check whether there are any nearby branch points to spoil the symmetry.

6. PARITY AND $\tilde{U}(12)$

The formalism described thus far contains no parity operation and must therefore be regarded as a model interesting, if at all, for its qualitative features.

To include parity one must go to a theory such as Rühl's[4] or $\tilde{U}(12)$.[3] In either case the inhomogeneous group must have more than 36 generators (72 and 144, respectively), and a momentum vector will possess more than one invariant (compare Sec. 2). In such a space, it is not obvious how to formulate quantum mechanics, what the causality condition is, and so forth. Which types of momenta describe physical particles may even be uncertain.

Nevertheless, we inquire briefly what one would expect to find *if* a model of the kind proposed here could be formulated in a $\tilde{U}(12)$ framework.

We assume that a momentum in the inhomogeneous $\tilde{U}(12)$ transforms as a **144**,

$$p=\sum_{A=0}^{143} p_A \Gamma_A ,$$

with $\gamma_0 p$ Hermitian, in the notation of Ref. 3. We further assume that the momentum of a particle (in the unbroken symmetry) is such that the equation

$$p\psi = M\psi \qquad (6.1)$$

(where ψ is a 12-component "quark") has 6 independent solutions $\psi(p)$. These solutions may then be used to represent the particle states in interactions. When the symmetry is broken, the physical states will involve an integral over a wave function $f(q)$ (where the q are the *independent* unphysical momenta) analogous to Eq. (4.1); and we conjecture that this wave function will force *all* the unphysical momenta to be small. The effect is, to the zeroth approximation, that the p in (6.1) is replaced by $p_\mu\gamma^\mu$, where p_μ is the physical 4-momentum. Thus (6.1) becomes, in this approximation, the Wigner-Bargmann equation proposed by Salam *et al.*,[3] which we therefore believe may be a natural consequence of a model of the kind proposed here. We have given the argument for a quark, but it clearly generalizes to higher representations of $\tilde{U}(12)$.

A similar argument shows that "irregular" terms are to be expected in zeroth approximation in $\tilde{U}(12)$, just as in the $SL(6)$ theory (see Sec. 5).

One would also like to find the structure of the excited states (see Sec. 4) in $\tilde{U}(12)$; but to do this would require a more detailed model. It is clear however, that the qualitative features (degenerate sets of incomplete multiplets) would reappear.

7. SUMMARY

The model described above has the following properties:

(1) It is Lorentz-invariant.

(2) It seems not to conflict with unitarity.

(3) The ground-state wave functions, to a zeroth approximation, correspond to pure $SU(6)$ multiplets.

(4) S-matrix elements between these ground states obey simple $SL(6)$ rules, in regions which are insensitive to the detailed spectrum of intermediate states.

(5) These rules allow for the presence of "irregular" terms, even in zeroth approximation.

(6) There are excited states which correspond to approximately degenerate groups of incomplete $SU(6)$ multiplets. These play an important rôle as intermediate states in the unitarity condition.

(7) The $SL(6)$ model does not contain parity; but, *if* it can be generalized to $\tilde{U}(12)$, the Wigner-Bargmann equations will probably appear naturally, and properties (1) to (6) will probably be qualitatively unchanged.

We conclude that a broken translational symmetry scheme of this kind is capable of accounting for the known successes of $SU(6)$ and its relativistic generalisations.

Note added in proof. In Sec. 4, the states of the Schrödinger-like harmonic-oscillator equation (4.6) may be classified according to $SU(32)$. But to find their spin and $SU(3)$ content—which is the problem—the $SU(32)$ orbital wave function has to be combined with the $SU(6)$ "intrinsic spin." $SU(32)$, by itself, is *not* a symmetry of the theory. We therefore have not been able to use $SU(32)$ to find a general rule for the decomposition of the higher excited states.

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