Evaluation of the $I=0$ Pion-Pion Scattering Length Using a Forward-Scattering Dispersion Relation*

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The $I=0$ $\pi\pi$ scattering length is evaluated with a forward-direction elastic-scattering dispersion relation. The high-energy contribution to the dispersion integral is obtained on the assumption that the high-energy behavior of the forward scattering amplitude is dominated by a few leading crossed-channel Regge poles, while any available experimental information on total $\pi\pi$ cross sections is used to compute the low-energy contribution. The scattering length is found to be negative, with a calculated value of $-1.7_{-0.5}^{+1.3}$ (in pion Compton wavelengths). Evaluation of the $I=1$ amplitude at threshold yields the value -0.4, which is found to be consistent with zero; this indicates that the method used in the evaluation of the scattering length is not unreasonable.

I. INTRODUCTION

EXPERIMENT suggests a large phase shift for the I=0 (I denotes the isotopic spin), S-wave $\pi\pi$ amplitude at low energies[1]; the presence or absence of a resonance, however, remains obscure. In this connection it is evidently of importance to know the value of the scattering length. In this paper we attempt an evaluation, using a forward-direction elastic-scattering dispersion relation. Assuming that the high-energy behavior of the amplitude is adequately represented by a few leading Regge poles, we reach the definite conclusion that the scattering length is negative.

In Sec. 2 we write the scattering length as a sum of two terms, where the first one represents the low-energy contribution to the forward dispersion relation and can, in principle, be evaluated once the total $\pi\pi$ cross sections at low energies are known. It is this term which introduces most of the uncertainties into our calculation. The second term represents the high-energy contribution to the dispersion integral, and is expressed entirely in terms of the parameters for the P, P', and ρ Regge trajectories at zero total center-of-mass energy. The residues are calculated in the Appendix to this paper on the basis of results obtained by Phillips and Rarita on pion-nucleon and nucleon-nucleon scattering. In addition to the scattering length, we evaluate the $I=1$ amplitude at threshold; from Bose statistics we know that it should vanish there since it contains odd partial waves only. Thus the deviation of its value from zero gives us an indication of the reliability of the approximations made in the evaluation of the scattering length.

In Sec. 3 we obtain the numerical results for the scattering length and the $I=1$ amplitude at threshold; their values are found to be -1.7 and -0.4, respectively.

Finally, in Sec. 4 we estimate the errors involved in the calculation, and conclude on the basis of these estimates that the scattering length is definitely negative, and that the value -0.4 of the $I=1$ amplitude at threshold is consistent with zero.

II. FORMULA FOR THE SCATTERING LENGTH

Let $A^I(s,t,u)$ be the amplitude of definite isotopic spin I for the s reaction, where s, t, and u are the usual Mandelstam variables; they are given in terms of the center-of-mass scattering angle θ_s, and the magnitude of the center-of-mass momentum q_s as follows (the subscript s is to remind us that these variables are defined with respect to the s reaction):

$$\begin{aligned} s&=4(q_s^2+\mu^2)\,,\\ t&=-2q_s^2(1-\cos\theta_s)\,,\\ u&=-2q_s^2(1+\cos\theta_s)\,, \end{aligned} \tag{2.1}$$

with

$$s+t+u=4\mu^2.$$

Here μ is the pion mass.[2] We normalize the partial-wave amplitude $A_l^I(s)$ defined by

$$A^I(s,\cos\theta_s)=\textstyle\sum_l (2l+1)A_l^I(s)P_l(\cos\theta_s)\,, \tag{2.2a}$$

so that it is related to the phase shift δ_l^I according to

$$A_l^I(s)=\tfrac{1}{2}(s^{1/2}/q_s)\exp(i\delta_l^I)\sin\delta_l^I. \tag{2.2b}$$

The $I=0$ scattering length λ is then defined as

$$\lambda=\lim_{s\to4} 2A_0^0(s)/s^{1/2}. \tag{2.3}$$

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[1] Evidence for a strong $I=0$, S-wave $\pi\pi$ interaction at low energies has been reported numerous times in the literature. See for example A. Abashian, N. E. Booth, and K. M. Crowe, Phys. Rev. Letters **5**, 258 (1960); **7**, 35 (1961). For a more complete discussion of the ABC enhancement, see N. E. Booth and A. Abashian, Phys. Rev. **132**, 2314 (1963). Other references include N. P. Samios *et al.*, Phys. Rev. Letters **9**, 139 (1962); L. M. Brown and P. Singer, *ibid.* **8**, 460 (1962), and Phys. Rev. **133**, B812 (1964); A. O. Barut and W. S. Au, Phys. Rev. Letters **13**, 165 (1964); Sharashchandra H. Patil, *ibid.* **13**, 261 (1964). Recently, the possible existence of a $I=0$, scalar di-pion near the ρ mass has been suggested; see for example M. Islam and R. Piñon, Phys. Rev. Letters **12**, 310 (1964); V. Hagopian *et al.*, *ibid.* **14**, 1077 (1965); Loyal Durand, III, and Yam Tsi Chiu, *ibid.* **14**, 329 (1965). Further references may be found in the above-mentioned articles.

[2] From here on all values of s, t, and u will be given in units of the pion mass squared.

If the absorptive part of $A^I(s,t,u)$ vanished for large s, then it would satisfy the one-dimensional dispersion relation

$$A^I(s,0)=\frac{1}{\pi}\int_4^\infty ds'\frac{A_s{}^I(s',0)}{s'-s}+\frac{1}{\pi}\int_4^\infty du'\frac{A_u{}^I(0,u')}{u'-(4-s)}\,,\quad(2.4)$$

where $A^I(s,t)\equiv A^I(s,t,4-s-t)$, and where $A_s{}^I(s,0)\equiv(1/2i)\,\mathrm{disc}_s A^I(s,0,4-s)$, and $A_u{}^I(0,u)\equiv(1/2i)\,\mathrm{disc}_u\times A^I(4-u,0,u)$; here disc_x stands for discontinuity in x. Assuming that the asymptotic behavior of the amplitude is determined by the leading Regge poles in the crossed channel,[3] we see that the above dispersion integral is undefined as it stands, since for $s\to\infty$ $A_s{}^I(s,0)\approx s$, which follows from the dominance of the Pomeranchuk pole. We therefore write (2.4) as a sum of two terms, where the first one represents the low-energy contribution (LEC) to $A^I(s,0)$, and where the second term is obtained by approximating the integrand of (2.4) above a certain energy s_1 (which is chosen well beyond the resonance region where Regge behavior presumably sets in) in terms of the leading crossed-channel Regge poles; the latter integral may then be explicitly evaluated by analytic continuation.[4] From crossing symmetry it follows that the u-channel amplitude of definite isotopic spin is the same function of s, t, and u as the s-channel amplitude is of u, t, and s (in that order). Using this fact, we may cast (2.4) in the form

$$\begin{aligned}A^I(s,0)=&\frac{1}{\pi}\int_4^{s_1} ds'\frac{A_s{}^I(s',0)}{s'-s}\\&+\sum_{I'}\alpha_{II'}(-1)^{I+I'}\frac{1}{\pi}\int_4^{s_1} ds'\frac{A_s{}^{I'}(s',0)}{s'-(4-s)}\\&+\sum_{I'}\alpha_{II'}\frac{1}{\pi}\int_{s_1}^\infty ds' R_s{}^{I'}(0,s')\\&\times\left[\frac{1}{s'-s}+\frac{(-1)^{I'}}{s'-(4-s)}\right],\end{aligned}\quad(2.5)$$

where $\alpha_{II'}$ and $\alpha_{II'}(-1)^{I+I'}$ are the t-channel and u-channel crossing matrices, respectively, with

$$\alpha_{II'}=\begin{bmatrix}\frac{1}{3}&1&\frac{5}{3}\\\frac{1}{3}&\frac{1}{2}&-\frac{5}{6}\\\frac{1}{3}&-\frac{1}{2}&\frac{1}{6}\end{bmatrix},\quad(2.6)$$

and where $R_s{}^I(0,s)\equiv(1/2i)\,\mathrm{disc}_s R^I(0,s)$; $R^I(t,s)$ is the contribution to the t-channel amplitude of isotopic spin I coming from the leading Regge poles. There exist a variety of forms for $R^I(t,s)$ in the literature, all of which presumably are good approximations to the true t-channel amplitude at large s and small t. We have chosen the Chew-Jones form[5] (a sum is understood if there are several poles of isospin I)

$$R_s{}^I(t,s)=\tfrac{1}{2}\pi\gamma^I(t)(-q_t{}^2)^{\alpha(t)}P_{\alpha(t)}(-1-s/2q_t{}^2)\,,\quad(2.7)$$

where $q_t{}^2=\frac{1}{4}t-1$, and where $\alpha(t)$ is a particular Regge trajectory in the t channel; $\gamma^I(t)$ is related to the full residue $\beta^I(t)$ associated with the above Regge pole as follows:

$$\gamma^I(t)=[2\alpha(t)+1]\beta^I(t)/(q_t{}^2)^{\alpha(t)}.\quad(2.8)$$

Throughout this paper it is understood that $\alpha(t)$ is a trajectory of definite isotopic spin.

Setting $t=0$ in (2.7) and substituting the result into (2.5), we arrive at the following expression for the high-energy contribution $H^I(\nu)$:

$$H^I(\nu)=\sum_{I'}\alpha_{II'}\gamma^{I'}(0)\frac{1}{2}\int_{\nu_1}^\infty d\nu' P_\alpha(\nu')\times\left[\frac{1}{\nu'-\nu}+\frac{(-1)^{I'}}{\nu'+\nu}\right],\quad(2.9)$$

where $\nu=-1+\frac{1}{2}s$ with corresponding definitions for ν' and ν_1 in terms of s' and s_1, and where $\alpha=\alpha(0)$. Notice that for $\nu=1$, the exchange of the ρ Regge pole in the t channel contributes very little to the integral, since for the ρ, $I'=1$. We shall take advantage of this fact in Sec. 3. For $\mathrm{Re}\,\alpha(0)\geqslant0$, the integral (2.9) is undefined; however, we may use the well-known dispersion relation for the Legendre function of the first kind to rewrite (2.9) as

$$H^I(\nu)=-\sum_{I'}\alpha_{II'}\gamma^{I'}(0)\left[\tfrac{1}{2}\pi\frac{P_\alpha(-\nu)+(-1)^{I'}P_\alpha(\nu)}{\sin\pi\alpha}+\frac{1}{2}\int_1^{\nu_1}d\nu' P_\alpha(\nu')\left(\frac{1}{\nu'-\nu}+\frac{(-1)^{I'}}{\nu'+\nu}\right)\right].\quad(2.10)$$

For our purpose this expression is still inconvenient, since we are interested in the value of $H^I(\nu)$ at $\nu=1$, where $P_\alpha(-\nu)$ has a logarithmic branch point (this branch point is, of course, absent from the full expression). We get around this difficulty by shifting the argument of the Legendre function, using a subtracted dispersion relation for $P_\alpha(\nu)$:

$$P_\alpha(-\nu)=P_\alpha(0)-\frac{\sin\pi\alpha}{\pi}\nu\int_1^\infty d\nu'\frac{P_\alpha(\nu')}{\nu'(\nu'-\nu)}\,.\quad(2.11)$$

Substituting (2.11) into the first term on the right-hand side of (2.10) and combining the various integrals, we

[3] By "leading" we mean those Regge poles lying in the right-half angular-momentum plane.

[4] Our calculation is done in the spirit of Igi's evaluation of the pion-nucleon scattering length; see Keiji Igi, Phys. Rev. Letters **9**, 76 (1962), and Phys. Rev. **130**, 820 (1963).

[5] Geoffrey F. Chew and C. Edward Jones, Phys. Rev. **135**, B208 (1964), Eq. (II-3).

obtain

$$H^I(\nu)=-\sum_{I'}\alpha_{II'}\gamma^{I'}(0)\xi_+^{I'}\left[\tfrac{1}{2}\pi\frac{P_\alpha(0)}{\sin\pi\alpha}+\frac{1}{2}\int_1^{\nu_1}d\nu'\frac{P_\alpha(\nu')}{\nu'}\right]+\sum_{I'}\alpha_{II'}\gamma^{I'}(0)\tfrac{1}{2}\nu$$
$$\times\int_{\nu_1}^{\infty}d\nu' P_\alpha(\nu')[\nu'(\nu'^2-\nu^2)]^{-1}[\nu'\xi_-^{I'}+\nu\xi_+^{I'}], \quad (2.12)$$

where

$$\xi_\pm^I=1\pm(-1)^I. \quad (2.13)$$

Since $\nu_1\gg 1$, and since we will be interested in the value of (2.12) for $\nu=1$, we may neglect ν compared with ν' in the denominator of the second integral, and replace the Legendre function by its asymptotic form,

$$P_\alpha(\nu)\underset{\nu\to\infty}{\longrightarrow}C(\alpha)\nu^\alpha, \quad (2.14a)$$

where

$$C(\alpha)=\frac{2^\alpha}{\pi^{1/2}}\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+1)}; \quad (2.14b)$$

the integral may then be performed explicitly. Our final expression for the forward-scattering amplitude is (recall that $\nu=-1+\frac{1}{2}s$)

$$A^I(s,0)=\frac{1}{\pi}\int_4^{s_1}ds'\frac{A_s^I(s',0)}{s'-s}+\sum_{I'}\alpha_{II'}(-1)^{I+I'}$$
$$\times\frac{1}{\pi}\int_4^{s_1}ds'\frac{A_s^{I'}(s',0)}{s'-(4-s)}+H^I(\nu), \quad (2.15)$$

where

$$H^I(\nu)=-\tfrac{1}{2}\sum_{I'}\alpha_{II'}\gamma^{I'}(0)\xi_+^{I'}\left[\pi\frac{P_\alpha(0)}{\sin\pi\alpha}+\int_1^{\nu_1}d\nu'\frac{P_\alpha(\nu')}{\nu'}\right]-\tfrac{1}{2}\sum_{I'}\alpha_{II'}\gamma^{I'}(0)C(\alpha)\nu_1^\alpha$$
$$\times\left[\frac{\xi_-^{I'}}{\alpha-1}\left(\frac{\nu}{\nu_1}\right)+\frac{\xi_+^{I'}}{\alpha-2}\left(\frac{\nu}{\nu_1}\right)^2\right]. \quad (2.16)$$

Here $\alpha=\alpha^{I'}(0)$. For $s=4$, and $s_1=200$, say, the contribution to $H^I(\nu=1)$ coming from the second summation in (2.16) is small compared with that coming from the first; we, therefore, will ignore it in the subsequent calculations. For $\alpha=1$ (i.e., the Pomeranchuk trajectory), we may of course evaluate (2.10) directly; we find

$$[H^I(1)]_{\mathrm{Pom}}=-\tfrac{1}{3}\gamma_P(0)[\tfrac{1}{2}s_1-1-\tfrac{1}{2}\ln(1-4/s_1)].$$

The factor $\frac{1}{3}$ comes from the crossing-matrix element. Notice that for $\gamma_P(0)>0$, the Pomeranchuk contribution is negative. Superficially it might appear that our result for the scattering length will depend on the choice of s_1. We emphasize that, as long as s_1 is chosen sufficiently large so that the amplitude is well approximated in terms of the leading Regge poles for $s>s_1$, our result will be essentially independent of s_1.

Next, we give the values of the residues $\gamma^I(0)$ as obtained from data furnished by Phillips and Rarita.[6] These authors have fitted the high-energy total pion-nucleon and nucleon-nucleon cross sections, assuming that the forward scattering amplitude is dominated by the leading crossed-channel Regge poles. The π-π residues are then obtained via the factorization theorem.[7] We find

$$\tilde{\gamma}_P=1.05\pm0.02,\qquad \tilde{\gamma}_P'=0.93\pm0.04, \quad (2.17)$$

where the dimensionless quantity $\tilde{\gamma}_i$ is related to the residue defined by (2.7), $\gamma^0(0)$, according to

$$\gamma_i(0)=\frac{2^{\alpha_i+1}}{\pi}\frac{3\tilde{\gamma}_i}{C(\alpha_i)(\bar{s})^{\alpha_i}}, \quad (2.18)$$

where i stands for the P or the P' Regge poles ($I=0$). Here $\bar{s}=2ME_0$, where M is the nucleon mass, and E_0 is a reference energy which Phillips and Rarita chose to be 1 GeV. The factor 3 comes from the crossing matrix element. The reader may consult the Appendix for a detailed discussion. We have not given here the value of the residue associated with the ρ trajectory, since, as is shown in the Appendix, it is very uncertain. Fortunately, as has already been emphasized, ρ exchange in the t channel contributes only very little to $H^I(1)$, so that we shall ignore its contribution here. We shall come back to this point in the following section.

III. EVALUATION OF THE SCATTERING LENGTH

Substituting the values from (2.18) into expression (2.16), we obtain the following results for the P and P' contributions to $H^I(1)$:

$$[H^I(1)]_P=-1.35\pm0.026,\qquad [H^I(1)]_{P'}=-1.67\pm0.072, \quad (3.1)$$

where the errors in (3.1) are those due to the uncertainties in the residues (2.17); the corrections to (3.1) coming from the second summation in formula (2.16) are only of the order of 10^{-4}. To the extent that we are ignoring the contribution of the ρ trajectory, the value of $H^I(1)=[H^I(1)]_P+[H^I(1)]_{P'}$ will be the same for $I=0$, 1, and 2, that is, $H^I(1)\approx-3.0$.

[6] Roger J. N. Phillips and William Rarita, Phys. Rev. Letters 14, 502 (1965).

[7] See, for example, V. N. Gribov and I. Ya. Pomeranchuk, Phys. Rev. Letters 8, 343 (1962). For a more complete treatment see S. C. Frautschi, M. Gell-Mann, and F. Zachariasen, Phys. Rev. 126, 2204 (1962); and M. Gell-Mann, in *Proceedings of the 1962 International Conference on High-Energy Nuclear Physics at CERN*, edited by J. Prentki (CERN, Geneva, 1962), p. 533.

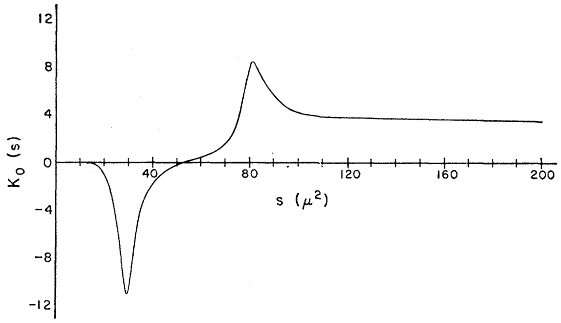

FIG. 1. Plot of $K_0(s)$ in units of $10^{-2}\mu^{-2}$ versus the square of the center-of-mass energy, s, in units of μ^2. (μ=pion mass.) The function has been matched at $s=200$ to the value as obtained from pure P, P', and ρ Regge-pole exchange in the crossed channel: $K_0(200)\approx 3.46\times 10^{-2}\mu^{-2}$.

The next step consists in evaluating the first two integrals in (2.15), using whatever experimental information is available. For this purpose it is convenient to put them into the form ($s=4$)

$$(\mathrm{LEC})_I=\frac{1}{\pi}\int_4^{s_1}\frac{ds'}{s'}\sum_{I'}\left[\delta_{II'}+\alpha_{II'}(-1)^{I+I'}\right]\times A_s{}^{I'}(s',0)+\frac{4}{\pi}\int_4^{s_1}\frac{ds'}{s'(s'-4)}A_s{}^{I}(s',0). \quad (3.2)$$

The contribution coming from the second integral is small compared with that from the first; considering the uncertainties involved in what follows, we shall ignore it. We are interested in the value of $A^I(4,0)$ for $I=0,1$; $A^0(4,0)$ gives us the magnitude of the scattering length, while we expect that $A^1(4,0)\approx 0$, if our approximations are reasonable. We therefore consider expression (3.2) for $I=0$ and 1. Substituting the crossing matrix elements into the integrand of (3.2), and neglecting the second integral, we obtain

$$(\mathrm{LEC})_I=\frac{1}{\pi}\int_4^{s_1} ds K_I(s)\,, \quad (3.3a)$$

where

$$K_0(s)=(1/s)\left[\tfrac{4}{3}A_s{}^0-A_s{}^1+\tfrac{5}{3}A_s{}^2\right] \quad (3.3b)$$

and

$$K_1(s)=(1/s)\left[\tfrac{3}{2}A_s{}^1-\tfrac{1}{3}A_s{}^0+\tfrac{5}{6}A_s{}^2\right]. \quad (3.3c)$$

Here $A_s{}^I$ stands for $A_s{}^I(s,0)$. For $s\to\infty$, $K_I(s)$ approaches twice the contribution coming from the P and P' exchange in the crossed channel,

$$K_I(s)\underset{s\to\infty}{\longrightarrow} 2\alpha_{I0}R_s{}^0(0,s)/s\,, \quad (3.4)$$

where $R_s{}^0(0,s)$ is given by (2.7). We now approximate the amplitude of given isotopic spin I in the resonance region by a Breit-Wigner formula that satisfies elastic unitarity, and has the correct threshold behavior

$$A_s{}^I(s,0)\approx(2l+1)\frac{s^{1/2}}{2q_s}\frac{s_R\Gamma^2(s)}{(s_R-s)^2+s_R\Gamma^2(s)}\,, \quad (3.5a)$$

where we have chosen for $\Gamma(s)$ the form[8]

$$\Gamma^2(s)=\Gamma_0{}^2(s_R/s)(q^2/q_R{}^2)^{2l+1}. \quad (3.5b)$$

For both the ρ and the f_0 we have adjusted the parameters of (3.5a) to give a width of 100 MeV; the positions of the resonances were chosen as 750 and 1250 MeV, respectively. Figures 1 and 2 show a plot of $K_0(s)$ and $K_1(s)$. We have used roughly the following criteria in plotting the curves; (a) near the f_0 and the ρ, $A_s{}^0(s,0)$ and $A_s{}^1(s,0)$ have been approximated by a Breit-Wigner form; (b) for $s\geqslant 200$, we assume that $A_s{}^I(s,0)$ is adequately represented by the exchange of the P, P', and ρ in the crossed channel, that is, we choose $s_1=200$, and use this point as a matching point for the integrand; we have

$$K_I(s=200)\approx 3.46\times 10^{-2}\mu^{-2}; \quad (3.6)$$

(c) we assume that the various absorptive parts of the amplitudes approach their asymptotic limit "smoothly" from above; (d) we have seen that ρ exchange makes no contribution to $K_I(s)$; it does, of course, make substantial contributions to the individual absorptive parts that make up $K_I(s)$. In view of the fact that the ρ residue is poorly known, we are led to make the additional assumption that an approximate curve for $K_I(s)$ may be obtained by consistently ignoring the effect of ρ exchange on the individual absorptive parts that make up $K_I(s)$; this assumption does not seem unreasonable. Figures 1 and 2 show what we think is a reasonable curve for $K_0(s)$ and $K_1(s)$. In both cases the functions have been matched at $s_1=200$ to their full asymptotic value (3.6). The reader may wonder what happened to the f_0 resonance in Fig. 2. The reason for its absence is the following: Let us assume, for simplicity, that $A_s{}^0(s,0)$ and $A_s{}^1(s,0)$ are given near the f_0 and the ρ

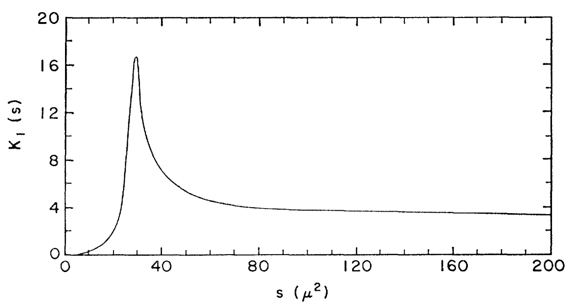

FIG. 2. Plot of $K_1(s)$ in units of $10^{-2}\mu^{-2}$ versus the square of the center-of-mass energy s in units of μ^2. The matching point and the value of $K_1(s)$ at that point are identical to that given in Fig. 1.

[8] See J. D. Jackson, Nuovo Cimento 34, 1644 (1964), for a discussion on the phenomenological analysis of resonances.

by their unitarity bounds, $A_s^I(s,0)=(2l+1)(s^{1/2}/2q)$; it then follows that

$$\frac{\frac{3}{2}A_s^1/m_1^2}{\frac{1}{3}A_s^0/m_0^2}\approx 6,$$

where m_1 and m_0 are the masses of the ρ and the f_0, respectively. A more detailed analysis shows that, as a first approximation, we may omit the "bump" in the curve for $K_1(s)$ arising from the f_0 resonance, since the uncertainties involved in the plot are already substantial. We obtained for the two LEC's the values

$$\frac{1}{\pi}\int_4^{200} ds\, K_0(s)\approx 1.3,$$

$$\frac{1}{\pi}\int_4^{200} ds\, K_1(s)\approx 2.6.$$

Combining these results with the high-energy contribution $H^I(1)\approx -3.0$ we arrive at the values for $A^0(4,0)$ and $A^1(4,0)$ of

$$A^0(4,0)\approx -1.7, \tag{3.7a}$$

$$A^1(4,0)\approx -0.4. \tag{3.7b}$$

IV. CONCLUSION

The main source of error in the value of the scattering length λ comes from the low-energy contribution; the deviation of $A^1(4,0)$ from zero gives us an indication of the reliability of the approximations made in the evaluation of λ since we have used the same criteria for obtaining $A^0(4,0)$ and $A^1(4,0)$. We shall presently give an estimate of the error involved; in view of the lack of information on total $\pi\pi$ cross sections, it is clear that this is only a rough estimate.

In our discussion so far we have ignored the possibility that the absorptive parts $A_s^0(s,0)$ and $A_s^2(s,0)$ might make substantial contributions to the integrals in (2.15) at energies roughly below the f_0 region (because of the threshold behavior of the $I=1$ amplitude, we do not expect $A_s^1(s,0)$ to make an important contribution below the ρ mass). Without these low-energy contributions we have estimated the error in λ to lie between $-0.5\mu^{-1}$ and $+0.3\mu^{-1}$ [this estimate was made essentially on intuitive grounds; we have however considered the effects arising from the uncertainties in the residues (2.17)]. As we have mentioned at the beginning of the paper, however, there seems to exist a strong $I=0$, S-wave $\pi\pi$ interaction at low energies. To estimate its effect on λ we have assumed a constant S-wave phase shift of 45 deg over an energy range extending from $s=5$ to $s=40$ (this phase shift is suggested by the value of the scattering length given by ABC in Ref. 1). We find that the additional contribution to λ is roughly $+0.9\mu^{-1}$; we have included in this error the contribution coming from the second integral in (3.2) which is no longer negligible if there exists a strong low-energy enhancement in the $I=0$ amplitude. If this enhancement can be associated with a new particle lying on a vacuum trajectory, then the exchange of such a trajectory in the crossed channel would give rise to additional contributions. From the analysis of total pion-nucleon cross sections at high energies, we expect the residue associated with this Regge pole to be small, so that its effect on the scattering length will probably not be significant. Concerning the $I=2$ amplitude, experiment seems to indicate that the total $\pi^-\pi^-$ cross section is of the order of 3 mb over an energy range extending roughly from 400 to 1200 MeV.[9] We estimated its effect on the scattering length to be less than $+0.1$. Our conclusion that the scattering length is negative does not come as a complete surprise[10]; we know that at the symmetry point,[11] that is, $s=t=u=\frac{4}{3}$, $A^0/A^2=\frac{5}{2}$; furthermore, one expects that the $I=0$ and $I=2$ amplitudes are dominated at that point by their respective S-wave components, so that $A_0^0(\frac{4}{3})\approx(\frac{5}{2})A_0^2(\frac{4}{3})$. Analysis of the angular distribution in charged ρ decay however indicates that the $I=2$ amplitude is negative in the ρ region; now, we do not expect the above-mentioned ratio to change sign between $s=\frac{4}{3}$ and $s=4$, nor do we expect a change in sign of the $I=2$ amplitude between the ρ region and threshold; such reasoning leads to the conclusion that the scattering length is negative.[12]

The corrections to the value of $A^1(4,0)$ introduced by the low-energy $I=0$ enhancement, and the low-energy $I=2$ cross section, have been estimated in a similar manner and were found to be about $\frac{1}{6}$ and $\frac{1}{2}$ of those for the $I=0$ amplitude.

Finally we wish to point out that in our calculation we have taken Γ_0 [see definition (3.5a, 3.5b)] to be 100 MeV. An increase in the width of the ρ would increase the value of $A^1(4,0)$ while decreasing the value of the scattering length; the modifications that need to be made in Fig. 2 to ensure the vanishing of $A^1(4,0)$ are rather modest, and certainly within the limits of uncertainty of our plot. In conclusion, then, we find that the $\pi\pi$ scattering length is negative, with a calculated value of $\lambda=(-1.7_{-0.5}^{+1.3})\mu^{-1}$, and that the value of $A^1(4,0)=-0.4$ is consistent with zero.

ACKNOWLEDGMENT

The author wishes to express his appreciation to Professor Geoffrey F. Chew for suggesting the problem,

[9] Saclay-Orsay-Bari-Bologna Collaboration, Nuovo Cimento **37**, 361 (1965).

[10] Our sign for the scattering length is different from that obtained by most other authors; it agrees, however, with the sign found by R. E. Kreps, L. F. Cook, J. J. Brehm, and R. Blankenbecler, Phys. Rev. **133**, B1526 (1964); they obtain a scattering length of -1.72.

[11] Geoffrey F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1962), p. 65.

[12] I am indebted to Professor Geoffrey F. Chew for this remark.

and for the many helpful suggestions given throughout the course of this work.

APPENDIX

We obtain the values of the residues for the P and P' trajectories, using the data of Phillips and Rarita.[6] These authors have fitted the total pion-nucleon and nucleon-nucleon cross sections, assuming that the forward-scattering amplitude is dominated at high energies by a few leading crossed-channel Regge poles. They normalize the spin-averaged forward elastic scattering amplitude so that the optical theorem reads

$$\mathrm{Im}\hat{A}_{ab}(s,0)=\sigma_{ab}^{\mathrm{tot}}(s)\,, \tag{A1}$$

where the subscripts a and b denote the particles involved in the elastic reaction, and write the contribution to $\hat{A}_{ab}(s,0)$ coming from the ith crossed-channel Regge pole (for large s) as

$$[\hat{A}_{ab}(s,0)]_i=B_i(ab)\left(\frac{1\pm\exp(-i\pi\alpha_i)}{\sin\pi\alpha_i}\right)\left(\frac{E}{E_0}\right)^{\alpha_i-1}, \tag{A2}$$

where the $+$ and $-$ signs correspond to the even- and odd-signature trajectories, respectively, and where $B_i(ab)$ is a coefficient with dimensions of millibarns; E_0 is a scale factor which they chose as 1 GeV, and E is the energy of the bombarding particle in the laboratory system. As usual we have written $\alpha_i\equiv\alpha_i(0)$. For large s, $E\approx s/2M$, where M is the mass of the particle at rest (the nucleon in our case). The relation between $\hat{A}_{ab}(s,0)$ and our amplitude $A_{ab}(s,0)$, normalized according to Eqs. (2.2a), (2.2b) is

$$A_{ab}(s,0)=(q_s s^{1/2}/8\pi)\hat{A}_{ab}(s,0)\underset{s\to\infty}{\longrightarrow}(s/16\pi)\hat{A}_{ab}(s,0)\,. \tag{A3}$$

Using (A.3) and (A.2), with a-b taken as the pion-nucleon, and nucleon-antinucleon, respectively, we have

$$[A_{\pi N}(s,0)]_i=\frac{\bar{s}}{16\pi}B_i(\pi N)\left(\frac{1\pm\exp(-i\pi\alpha_i)}{\sin\pi\alpha_i}\right)\times\left(\frac{s}{\bar{s}}\right)^{\alpha_i}, \tag{A4a}$$

and

$$[A_{N\bar{N}}(s,0)]_i=\frac{\bar{s}}{16\pi}B_i(N\bar{N})\left(\frac{1\pm\exp(-i\pi\alpha_i)}{\sin\pi\alpha_i}\right)\times\left(\frac{s}{\bar{s}}\right)^{\alpha_i}, \tag{A4b}$$

where we have used the asymptotic form for s, $s\approx 2ME$, and have written $\bar{s}=2ME_0$ (≈ 98.5). Phillips and Rarita provide us with the coefficients $B_i(\pi N)$ and $B_i(N\bar{N})$ for the case where N is the proton, and π is the negatively charged pion. They obtain the following values[6]: $B_P(\pi N)=-19.9\pm0.1$ mb, $B_{P'}(\pi N)=-18.1\pm0.2$ mb, $B_\rho(\pi N)=2.4\pm0.4$ mb, $B_P(N\bar{N})=-36.2\pm0.2$ mb, $B_{P'}(N\bar{N})=-33.8\pm0.6$ mb, and $B_\rho(N\bar{N})=1.0\pm1.2$ mb, (the sign of B_P and $B_{P'}$ has been misprinted in Ref. 6). Notice that the value of $B_\rho(N\bar{N})$ is consistent with zero. In a similar way to (A4a), (A4b), we write the contribution of the ith crossed-channel Regge pole to the $\pi\pi$ amplitude, at large s, as

$$[A_{\pi\pi}(s,0)]_i=\frac{\bar{s}}{16\pi}B_i(\pi\pi)\left(\frac{1\pm\exp(-i\pi\alpha_i)}{\sin\pi\alpha_i}\right)\times\left(\frac{s}{\bar{s}}\right)^{\alpha_i}. \tag{A5}$$

The coefficient $B_i(\pi\pi)$ is given by the factorization theorem for residues[7]:

$$B_i(\pi\pi)=B_i^2(\pi N)/B_i(N\bar{N})\,. \tag{A6}$$

Since the value of $B_\rho(N\bar{N})$ is consistent with zero, and since it appears in the denominator of formula (A6), we shall limit ourselves to the evaluation of the P and P' residues; fortunately, as we have pointed out before, a knowledge of the ρ residue is not critical for our problem, since the ρ contribution to $H^I(1)$ is small [see Eq. (2.16)]. Substituting the values for the coefficients $B_i(\pi N)$ and $B_i(N\bar{N})$ into (A6), we obtain

$$\tilde{\gamma}_P=1.05\pm0.02\,,\quad\text{and}\quad \tilde{\gamma}_{P'}=0.93\pm0.04\,, \tag{A7}$$

where ($i=P$, P')

$$\tilde{\gamma}_i\equiv-\bar{s}B_i(\pi\pi)/16\pi\,. \tag{A8}$$

We have taken the value $\alpha_{P'}(0)=0.5$ from Ref. 6 [$\alpha_P(0)=1$, of course].

To get the relation between the residue $\gamma(0)$ defined in (2.7) and the quantities $\tilde{\gamma}_i$, we notice that the P and P' contributions to $\mathrm{Im}A_{\pi\pi}(s,0)$ are also given by

$$[\mathrm{Im}A_{\pi\pi}(s,0)]_i=\tfrac{1}{3}[R_s^0(0,s)]_i\,, \tag{A9}$$

where $R_s^0(0,s)$ is given by (2.7). Approximating the right-hand side of (A9) by its leading term for $s\to\infty$, and comparing the resultant expression with the imaginary part of (A5), we obtain

$$\gamma_i(0)=(2^{\alpha_i+1}/\pi)[3\tilde{\gamma}_i/C(\alpha_i)(\bar{s})^{\alpha_i}]\,, \tag{A10}$$

where $C(\alpha)$ is defined by (2.14b), and where we have written $\gamma_i(0)$ rather than $\gamma^I(0)$ for notational consistency; the subscript i stands for the P and P' Regge poles.