first-forbidden unique decay with an endpoint energy of only 2.6 keV and a half-life of about $4\times10^{10}$ years.[13] This decay has extreme astrophysical importance, moreover, since it provides a galactic clock.[14] The decay $Rb^{87}(\beta^-)Sr^{87}$ with a half-life of $4.7\times10^{10}$ yr may be enhanced by the allowed photobeta transition to the first excited state of $Sr^{87}$. The same statement applies to the decay $Pd^{107}(\beta^-)Ag^{107}$, with a half-life of $7\times10^6$ yr.

The above examples are but a few of the astrophysically interesting possibilities suggested from a preliminary examination of the Nuclear Data Sheets.[12] The photobeta process (together with excited-state beta decay) provides a mechanism for transforming a stable $(Z,A)$ nucleus to an unstable $(Z+1,A)$ nucleus and thence to another stable $(Z+2,A)$ nucleus (e.g., $Ca^{48}\rightarrow Sc^{48}\rightarrow Ti^{48}$). On the other hand the cycle illustrated by $\gamma+Te^{125}\rightarrow I^{125}+e^-+\bar{\nu}$, $e^-+I^{125}\rightarrow Te^{125}+\nu$ is essentially the catalytic conversion of a photon into a neutrino-antineutrino pair.

It is not possible to make any general statement concerning the relative importance of the photobeta process and excited-state beta decay: Such evaluation requires a knowledge of the level structure and *ft* values for decay *from* excited states (which only rarely can be determined experimentally). On the other hand the photobeta process is largely independent of such considerations (except for photobeta to excited states as in $Ca^{48}$) and, since it does not interfere with excited-state beta decay, one can always obtain a *lower limit* to rates for the processes $Z\rightarrow Z+2$ and $\gamma\rightarrow\nu+\bar{\nu}$ as discussed above. For example, from Fig. 10 we see that at $12\times10^8$ °K every $Te^{125}$ nucleus produces about 0.3 MeV in neutrinos *at least* as rapidly as once every $10^4$ years, regardless of the excited-state beta-decay rate.

[13] R. L. Brodzinski and D. C. Conway, Phys. Rev. **138**, B1368 (1965).

[14] Donald D. Clayton, Astrophys. J. **139**, 637 (1964).

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# Equations of Motion without Infinite Self-Action Terms in General Relativity*

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The role of the Bianchi identity in obtaining the equations of motion in general relativity is further discussed in connection with the problem of the infinite self-action terms. It is shown that because of this identity and under certain assumptions (concerning the type of singularity of the Christoffel symbols near the particle), there is a possibility of obtaining an equation of motion, free of infinite self-action terms, without referring to any renormalization procedure. This is the Infeld equation of motion which describes the motion of a particle of finite mass and in which the time coordinate is taken to be the independent parameter. Besides that, however, the Bianchi identity imposes certain constraints that the field functions have to satisfy in addition to the equation of motion.

## INTRODUCTION

ONE of the most important achievements, and one of the most attractive features, of the general theory of relativity since its publication is the discovery by Einstein and Grommer[1] in 1927 that the equations of motion need not be postulated separately in addition to the gravitational field equations; rather, they follow from them.

Indeed, 11 years later, Einstein, Infeld, and Hoffmann[2] succeeded in developing an approximation method by means of which they found the equations of motion of finite-mass particles represented as singularities of the gravitational field. The equation of motion for each particle obtained in this way (Einstein-Infeld-Hoffmann equation) includes a relativistic correction of order $1/c^2$ to the well-known Newton equation and tends to the latter for velocities $v\ll c$, where $c$ is the speed of light in vacuum. This additional force term is the one associated with the advance of the perihelion of the planetary motion, a phenomenon to which only general relativity can give a satisfactory answer.

Following Einstein, Infeld, and Hoffmann's original paper, other works on the subject were devoted to improving the mathematical methods towards better understanding of the problem.[3] Nowadays the Einstein-Infeld-Hoffmann equation has been extended to include

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[1] A. Einstein and J. Grommer, Sitzber. Preuss. Akad. Wiss. Physik. Math. Kl. **2**, 235 (1927).

[2] A. Einstein, L. Infeld, and B. Hoffmann, Ann. Math. **39**, 65 (1938); L. Infeld, Phys. Rev. **53**, 836 (1938); A. Einstein and L. Infeld, Ann. Math. **41**, 455 (1940); A. Einstein and L. Infeld, Can. J. Math. **1**, 209 (1949).

[3] See, for example, J. N. Goldberg, in *Gravitation, An Introduction to Current Research*, edited by L. Witten (John Wiley & Sons, Inc., New York, 1962), Chap. 3.

corrections due to possible gravitational radiation,[4] and other approximation methods have been developed.[5]

## EQUATIONS OF MOTION AS A CONSEQUENCE OF FIELD EQUATIONS

In order to establish the relation between the Einstein field equations and the equations of motion one proceeds as follows. We notice that because of the contracted Bianchi identity[6]

$$\mathcal{G}^{\mu\nu}{}_{;\nu}\equiv 0\,, \tag{1}$$

it follows that the energy-momentum tensor density appearing in the Einstein gravitational field equations,

$$\mathcal{G}^{\mu\nu}=8\pi\mathcal{T}^{\mu\nu}\,, \tag{2}$$

satisfies a generally covariant conservation law of the form

$$\mathcal{T}^{\mu\nu}{}_{;\nu}\equiv \mathcal{T}^{\mu\nu}{}_{;\nu}+\Gamma^{\mu}{}_{\alpha\beta}\mathcal{T}^{\alpha\beta}=0\,. \tag{3}$$

In Eqs. (1)–(3) $\mathcal{G}^{\mu\nu}\equiv(-g)^{1/2}G^{\mu\nu}$, where $G^{\mu\nu}$ is the Einstein tensor,

$$G^{\mu\nu}\equiv R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R\,,$$

$R^{\alpha\beta}$ is the Ricci tensor, $R\equiv R^{\alpha\beta}g_{\alpha\beta}$, $\mathcal{T}^{\mu\nu}\equiv(-g)^{1/2}T^{\mu\nu}$ ($T^{\mu\nu}$ being the energy-momentum tensor), $g\equiv\det g_{\alpha\beta}$, and $\Gamma^{\mu}{}_{\alpha\beta}$ is the Christoffel symbol of the second kind. Greek indices run from 0 to 3 ($x^0\equiv t$), ordinary partial differentiation is denoted by a comma, whereas covariant differentiation is denoted by a semicolon, and we use units in which the velocity of light $c$ and Newton's gravitational constant $G$ are equal to unity.

For a system of $N$ particles of any *finite* mass, represented as singularities of the gravitational field, $\mathcal{T}^{\mu\nu}$ may be taken in the form

$$\mathcal{T}^{\mu\nu}=\sum_{A=1}^{N} m_A v_A{}^{\mu} v_A{}^{\nu}\delta_A(\mathbf{x}-\mathbf{z}_A)\,. \tag{4}$$

Here $z_A{}^{\mu}$ are the coordinates of the $A$th particle. (Capital Latin indices $A$, $B$, $\cdots$, run from 1 to $N$. For these indices the summation convention will be suspended.) $v^{\mu}=\dot{z}^{\mu}$, where the dot denotes time differentiation ($v_A{}^0=\dot{z}_A{}^0=1$), and $\delta$ is the three-dimensional Dirac delta function satisfying the following conditions:

$$\delta(\mathbf{x})=0 \quad\text{for}\quad \mathbf{x}\neq 0\,, \tag{5a}$$

$$\int\delta(\mathbf{x}-\mathbf{z})d^3x=1\,, \tag{5b}$$

$$\int f(\mathbf{x})\delta(\mathbf{x}-\mathbf{z})d^3x=f(\mathbf{z})\,, \tag{5c}$$

for every continuous function $f(\mathbf{x})$ in the neighborhood of $\mathbf{z}$. In Eq. (4), $m_A$ is a function of time which may be called the *inertial mass* of the $A$th particle.

If we put the energy-momentum tensor density, given by Eq. (4), into Eq. (3) and integrate over the three-dimensional region surrounding the first singularity we obtain, using Eqs. (5a)–(5c),

$$dp^{\mu}/dt=\int F^{\mu}\delta(\mathbf{x}-\mathbf{z})d^3x\,, \tag{6}$$

where

$$p^{\mu}=mv^{\mu}\,,$$
$$F^{\mu}=-m\Gamma^{\mu}{}_{\alpha\beta}v^{\alpha}v^{\beta}\,,$$

and we have put, for simplicity, $m\equiv m_1$, $z^{\mu}\equiv z_1{}^{\mu}$, $v^{\mu}\equiv v_1{}^{\mu}$, and $\delta(\mathbf{x}-\mathbf{z})\equiv\delta_1(\mathbf{x}-\mathbf{z}_1)$.

## INFELD'S "EXACT EQUATION OF MOTION"

Equation (6) may be interpreted as an "exact equation of motion" of the first particle. However, since the Christoffel symbol is singular at the location of the first particle, Eq. (6) contains infinite self-action terms.

In order to overcome this serious difficulty, so that Eq. (6) will have a definite meaning, Infeld[7,8] introduced a new delta function. In addition to the requirements given by Eqs. (5a)–(5c), the Infeld delta satisfies

$$\int|x|^{-p}\delta(\mathbf{x})d^3x=0\,, \tag{7}$$

for any finite positive integer $p$.

Repeating now the procedure adopted above for deriving Eq. (6), but using the Infeld delta function in the energy-momentum tensor density rather than the Dirac delta, we obtain

$$dp^{\mu}/dt=\bar{F}^{\mu}\,, \tag{8}$$

where the bar above a function $F$ means

$$\bar{F}=\int F\delta(\mathbf{x}-\mathbf{z})d^3x\,, \tag{9}$$

the delta function now being that of Infeld.

Contrary to Eq. (6), Eq. (8) has a definite meaning and resembles the familiar Newton law of motion. It is especially suitable for slow motion.[4] The bar on $F$ means two things[7]: Singularities are ignored, and $\mathbf{z}$ replaces $\mathbf{x}$.[9–13]

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[4] M. Carmeli, Phys. Letters **9**, 132 (1964); Nuovo Cimento **37**, 842 (1965).

[5] P. Havas and J. N. Goldberg, Phys. Rev. **128**, 398 (1962); M. Carmeli, Ann. Phys. (N. Y.) (to be published).

[6] See, for example, J. Weber, *General Relativity and Gravitational Waves* (Interscience Publishers, Inc., New York, 1962), p. 33.

[7] L. Infeld, Rev. Mod. Phys. **29**, 398 (1957).

[8] L. Infeld and J. Plebański, *Motion and Relativity* (Pergamon Press, Inc., New York, 1961).

[9] In fact the bars on a function do *not* mean simply putting $z^k$ for $x^k$ (in addition to neglecting singular terms) as has been stated by Infeld in Ref. 7. Consider, for example, the function $F^{rs}=(x^r-z^r)(x^s-z^s)/r^2$. $F^{rs}$ is finite at $x^s=z^s$. However, it has no definite value at $x^s=z^s$, because the limit depends on the angle of approach to $\mathbf{z}$. If we evaluate $\bar{F}^{rs}$ by means of Eq. (9), using the Infeld—or the Dirac—delta, however, we obtain $\bar{F}^{rs}=\frac{1}{3}\delta^{rs}$. Indeed, functions like $F^{rs}$ do appear in the equations of motion of singularities (see Refs. 4, 10).

[10] M. Carmeli, Physics Letters **11**, 24 (1964).

[11] One has to distinguish between Eq. (6), or Eq. (8), and the usual geodesic equation. See Refs. 10 and 12. Compare, however, Ref. 13.

[12] H. Bondi, *Lectures on General Relativity* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1964), p. 411; A. Peres, Phys. Rev. **137**, B1126 (1965).

[13] P. Havas, J. Math. Phys. **5**, 373 (1964).

## EQUATIONS OF MOTION WITHOUT THE INFELD DELTA

We now show how infinite self-action terms can be removed from Eq. (6) in an *exact* way without using the Infeld delta.[14]

Putting Eq. (4) into Eq. (3) we obtain

$$[\sum_{A=1}^{N} m_A v_A{}^\mu \delta_A]_{,0}+[\sum_{A=1}^{N} m_A v_A{}^\mu v_A{}^n \delta_A]_{,n}+\sum_{A=1}^{N} m_A \Gamma^\mu{}_{\alpha\beta} v_A{}^\alpha v_A{}^\beta \delta_A=0, \quad (10)$$

where Latin indices run from 1 to 3 and the delta function now is that of Dirac. The first term on the left-hand side of Eq. (10) can be written as

$$[\sum_{A=1}^{N} m_A v_A{}^\mu \delta_A]_{,0}=\sum_{A=1}^{N} (m_A v_A{}^\mu)_{,0}\delta_A+\sum_{A=1}^{N} (m_A v_A{}^\mu)(\delta_A)_{,0}. \quad (11)$$

Now

$$\begin{aligned}(\delta_A)_{,0}&=[\delta_A(x^s-z_A{}^s)]_{,0}\\ &=(\delta_A)_{,z_A{}^n} v_A{}^n\\ &=-(\delta_A)_{,n} v_A{}^n.\end{aligned} \quad (12)$$

Using Eqs. (11) and (12) in Eq. (10), we obtain

$$\sum_{A=1}^{N} \{(m_A v_A{}^\mu)_{,0}+m_A \Gamma^\mu{}_{\alpha\beta} v_A{}^\alpha v_A{}^\beta\}\delta_A=0. \quad (13)$$

Equation (13), which is identical with Eq. (3), is supposed to be satisfied for any space-time point, since otherwise Eq. (1) or Eq. (2) will not be satisfied.

We now examine the behavior of Eq. (13) in the infinitesimal neighborhood of the first singularity, which we assume not to contain any other singularity. In this region

$$\delta_B(\mathbf{x}-\mathbf{z}_B)=0;\quad B=2, 3, \cdots, N.$$

Hence, Eq. (13) gives for the conservation law near the first singularity:

$$\{(mv^\mu)_{,0}+m\Gamma^\mu{}_{\alpha\beta} v^\alpha v^\beta\}\delta(\mathbf{x}-\mathbf{z})=0. \quad (14)$$

Following Infeld,[7] let us further assume that the Christoffel symbols in Eq. (14) can be expanded near the first singularity into a power series in the infinitesimal distance $r$, defined by

$$r^2=(x^s-z^s)(x^s-z^s);\quad z^s\equiv z_1{}^s,$$

in the vicinity of the first particle. Then we have[15]

$$\Gamma^\mu{}_{\alpha\beta}={}_{-k}\Gamma^\mu{}_{\alpha\beta}+{}_{-k+1}\Gamma^\mu{}_{\alpha\beta}+\cdots+{}_0\Gamma^\mu{}_{\alpha\beta}+\cdots, \quad (15)$$

where the indices written as subscripts on the left of a function indicate its behavior with respect to $r$, and $k$ is a positive integer. For example, ${}_0\Gamma^\mu{}_{\alpha\beta}$ is that part of the Christoffel symbol which varies as $r^0$, i.e., is finite at the location of the first particle.

The expansion given by Eq. (15) means, when one uses spherical coordinates $r$, $\theta$, and $\varphi$, that

$$\begin{aligned}{}_{-k}\Gamma^\mu{}_{\alpha\beta}&=r^{-k}A^\mu{}_{\alpha\beta}(\theta,\varphi),\\ {}_{-k+1}\Gamma^\mu{}_{\alpha\beta}&=r^{-k+1}B^\mu{}_{\alpha\beta}(\theta,\varphi),\\ &\cdots\\ {}_{-1}\Gamma^\mu{}_{\alpha\beta}&=r^{-1}C^\mu{}_{\alpha\beta}(\theta,\varphi),\\ {}_0\Gamma^\mu{}_{\alpha\beta}&=r^0 D^\mu{}_{\alpha\beta}(\theta,\varphi),\\ {}_1\Gamma^\mu{}_{\alpha\beta}&=rE^\mu{}_{\alpha\beta}(\theta,\varphi),\ \text{etc.}\end{aligned} \quad (16)$$

Terms like ${}_1\Gamma^\mu{}_{\alpha\beta}$, ${}_2\Gamma^\mu{}_{\alpha\beta}$, $\cdots$, however, need not be taken into account when one puts the expansion (15) into Eq. (14), since

$$r^j\delta(\mathbf{x}-\mathbf{z})=0 \quad (17)$$

for any positive integer $j$. If we denote $mA^\mu{}_{\alpha\beta}v^\alpha v^\beta$, $\cdots$ by $A^\mu$, $\cdots$ we can write Eq. (14) in the form

$$\{r^{-k}A^\mu+r^{-k+1}B^\mu+\cdots+r^{-1}C^\mu+D_1{}^\mu\}\delta(\mathbf{x}-\mathbf{z})=0, \quad (18)$$

where we have used the notation

$$D_1{}^\mu=(mv^\mu)_{,0}+D^\mu. \quad (19)$$

In order to get rid of terms proportional to negative powers of $r$ in Eq. (18) we proceed as follows. Multiplying Eq. (18) by $r^k$ and using Eq. (17), we obtain

$$A^\mu(\theta,\varphi)\delta(\mathbf{r})=0, \quad (20)$$

the integration of which over the three-dimensional region yields, using spherical coordinates,

$$\int\int A^\mu(\theta,\varphi)\sin\theta\, d\theta\, d\varphi\int r^2\delta(\mathbf{r})dr=0. \quad (21)$$

To evaluate the last integral of Eq. (21) we write Eq. (5b) in spherical coordinates, thus getting

$$\int\int \sin\theta\, d\theta\, d\varphi\int \delta(\mathbf{r})r^2 dr=1. \quad (22)$$

Hence

$$\int \delta(\mathbf{r})r^2\, dr=(4\pi)^{-1}. \quad (23)$$

We thus obtain, from Eq. (21),

$$\int\int A^\mu(\theta,\varphi)\sin\theta\, d\theta\, d\varphi=0, \quad (24)$$

[14] Other methods for removal of infinite self-action terms are also known; they are sometimes based on renormalization procedures valid only for the equations of motion obtained up to a definite accuracy. Sometimes the self-action terms are even left in the equation of motion. These equations, then, can hardly be given any meaning. See, for example, B. Bertotti and J. Plebański, Ann. Phys. (N. Y.) **11**, 169 (1960).

[15] The assumption that field functions can be expanded in power series in $r$ near the world-line of the particle is inherent in this approach to the problem of motion. See Ref. 7, Sec. 2; also Ref. 4.

independent of the value of the variable $r$. Thus the angular distribution of $A^\mu(\theta,\varphi)$ is such that its average equals zero. However, not only does Eq. (24) hold, but also ($s$ is any finite positive integer)

$$a(r)\equiv r^{-s}\int\int A^\mu(\theta,\varphi)\sin\theta\,d\theta\,d\varphi=0 \tag{25}$$

for small values of $r$ as well as when $r$ tends to zero, as can be verified by using L'Hospital's theorem, for example. It follows then that $a(r)$, as defined by Eq. (25), is a function of $r$ whose value is zero for any small $r$, including $r=0$. We now use the property given by Eq. (5c) for the $\delta$ function, and take for $f(\mathbf{x})$ a continuous function of $r$ only. We obtain, using spherical coordinates,

$$\int r^2\delta(\mathbf{r})f(r)dr=(4\pi)^{-1}f(0). \tag{26}$$

Since $a(r)$ is certainly continuous, one obtains

$$\int r^2\delta(\mathbf{r})a(r)dr=0. \tag{27}$$

Hence when one integrates Eq. (18) over the three-dimensional space, there will be no contribution from the first term.

In order to show that the second term of Eq. (18) will also not contribute to the three-dimensional integration of Eq. (18), we multiply the latter by $r^{k-1}$. We obtain, using Eq. (17),

$$\{r^{-1}A^\mu(\theta,\varphi)+B^\mu(\theta,\varphi)\}\delta(\mathbf{r})=0. \tag{28}$$

Integration of this equation, again using spherical coordinates, shows that the first term of Eq. (28) will not contribute anything because of Eqs. (25) and (27), and we are left with

$$\int\int B^\mu(\theta,\varphi)\sin\theta\,d\theta\,d\varphi\int r^2\delta(\mathbf{r})dr=0. \tag{29}$$

Using Eq. (23), we obtain

$$\int\int B^\mu(\theta,\varphi)\sin\theta\,d\theta\,d\varphi=0, \tag{30}$$

independent of $r$. From this equation one obtains two others, analogous to Eqs. (25) and (27) but with $B^\mu$ instead of $A^\mu$:

$$b(r)\equiv r^{-s}\int\int B^\mu(\theta,\varphi)\sin\theta\,d\theta\,d\varphi=0, \tag{31}$$

$$\int r^2 b(r)\delta(\mathbf{r})dr=0. \tag{32}$$

Proceeding in this way, one verifies that the angular distribution of all functions $A^\mu(\theta,\varphi)$, $B^\mu(\theta,\varphi)$, etc., is such that they all satisfy equations like Eqs. (24), (30), etc. From this it is clear that one obtains

$$\int D_1{}^\mu(\theta,\varphi)\delta(\mathbf{r})d^3x=0, \tag{33}$$

which gives

$$dp^\mu/dt+mv^\alpha v^\beta\int {}_0\Gamma^\mu{}_{\alpha\beta}\delta(\mathbf{r})d^3x=0. \tag{34}$$

Equation (34) is the "exact equation of motion" and is essentially the same as that of Infeld, Eq. (8), though the Dirac delta function has been used rather than the Infeld delta.

## CONCLUDING REMARKS

We thus come to the conclusion that one need not assume the existence of a new delta function, like that of Infeld, in order to obtain the Infeld equation of motion. Rather, it follows from the Einstein field equations alone if one assumes, as Infeld indeed does, the expansion (15).

Terms which are singular at the particle's location are grouped according to their behavior; their angular distribution is such that the average of each group is equal to zero. This is a consequence of the Bianchi identity which is an integral part of Einstein's formulation of the theory of gravitation. Equations (24), (30), etc., may be interpreted as constraints which the field functions have to satisfy, in addition to Eq. (34).

One may be encouraged by this property of Einstein's theory and ask whether general relativity will be incorporated into the theory of microscopic phenomena of nature, where singularities play a major role.[16]

## ACKNOWLEDGMENTS

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[16] L. D. Landau, in *Niels Bohr and the Development of Physics*, edited by W. Pauli (Pergamon Press, Inc., New York, 1962), 2nd ed.