Existence of the S Matrix

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We try to develop an analysis of collision experiments according to the method of Wittgenstein. We show that the probability amplitude of a two-particle collision is the matrix element of a unitary operator S. This derivation uses only the superposition principle and the probabilistic interpretation of quantum mechanics. It does not assume any microscopic notion of time nor the validity of the Schrödinger equation.

1. INTRODUCTION

It is a commonly accepted idea that a collision experiment can be described by a unitary scattering matrix S.[1] This notion tends to be more and more central in our description of particles.[2]

Up to now, no epistomological analysis of this notion has really been attempted.[3] It is often accepted as a natural, but cloudy, formulation of experimental results. In the best cases, it is derived by a mathematical analysis starting from the Schrödinger equation.[4] In the most sophisticated instance, it appears as a highly nontrivial consequence of quantum field theory.[5]

Our aim in this paper is to try to remove some of the cloudiness of the direct approach and, in fact, to find a correct framework for its formulation. Accordingly a brief critique of the two other approaches will help to get a better view of the problems involved.

The proof of the existence of the S matrix through quantum mechanics assumes the reliability of the Schrödinger equation. It therefore implies the use of a microscopic time and the existence of an interaction Hamiltonian operator. Both of these notions and the limits of their applicability are not quite clear.

The Haag-Ruelle theory of scattering in quantum field theory, notwithstanding its mathematical beauty, appears as one of the highest peaks of an accumulation of theorems. These theorems stand on a set of axioms whose epistemological value is not clear.[6] However, quantum field theory is the only domain of high-energy physics which has been seriously investigated in that respect. The essential difficulty on which any attempt to achieve a firm foundation has failed is that the very notion of field does not appear as a necessary formulation of our experience, except maybe in electrodynamics. Consequently, there is too much freedom in the properties that one should assume for these fields. They are at best limited by an analogy with electrodynamics, by consistency and by the agreement of their consequences with experiment. Unfortunately, there are very few of these experimental predictions and the lack of any substantial work in the epistemology of quantum electrodynamics since Bohr and Rosenfeld[7] does not make things easier.[8]

To summarize, the customary introductions of the S matrix seem to use steps less general and more questionable than what they are intended to justify.

What else can be done? The answer is obvious: We want to describe collision experiments; therefore let us look at actual collision experiments and let us try to analyze them.

Then comes a more delicate question: how to analyze the experiments? Very often this kind of analysis is obscured beforehand by a set of definitions, like: An experiment consists of a generating apparatus G which produces a state ψ. This state evolves with time and is detected by a measuring apparatus M. This idealistic approach is obviously open to trouble with questions like: What is the difference between G and M? Is ψ always well defined? In a bubble chamber, what is G and what is M? What is the time with which ψ evolves? Etc.

We must therefore first agree on the epistemological approach that we are going to use. For instance, we could take the analysis of the complementary principle of Bohr as a guide.[9] This is the approach which has had the greatest success in showing the logical consistency of quantum mechanics. However, it is essentially Hegelian and remains an approach by "the spirit." It shows that we can choose to adopt quantum mechanics because it is an internally consistent theory and is in agreement with our present experimental knowledge. It does not really give a basis nor a measure to appreciate the degree of necessity of any single concept of quantum mechanics.

Much progress has been made recently in epistemology under the powerful impetus of Wittgenstein, although this new approach does not seem yet to have

[1] J. A. Wheeler, Phys. Rev. **52**, 1107 (1937); W. Heisenberg, Z. Physik **120**, 513 (1943).
[2] See for instance the contribution by G. F. Chew, in *Strong Interaction Physics*, by M. Jacob and G. F. Chew (W. A. Benjamin, Inc., New York, 1964), where a large, although far from extensive, bibliography is given.
[3] Some remarks about these questions have been made by H. P. Stapp (private communication).
[4] See for instance M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1964).
[5] R. Haag, Phys. Rev. **112**, 669 (1958); D. Ruelle, Helv. Phys. Acta **35**, 1 (1962).
[6] See for instance R. F. Streater and A. S. Wightman, *PCT; Spin and Statistics, and All That* (W. A. Benjamin, Inc., New York, 1963).
[7] N. Bohr and A. Rosenfeld, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **12**, No. 8 (1933).
[8] One should, however, mention some lectures by R. Haag which do not seem to be published.
[9] N. Bohr, *Atomic Physics and Human Knowledge* (Pergamon Press, Inc., New York, 1954).

reached physics.[10] However, it could help us to discriminate which parts of quantum mechanics are necessary consequences of our common understanding of experiments (by which, of course, we do not mean necessary consequences of experiments) and of our common way of communicating our description of them. It could also help us in answering the questions left aside by the idealistic approach.

The present paper is a beginner's exercise in applying Wittgenstein's method to the analysis of collision experiments. It is only meant to try to convey the power of this approach. Any failure or inconsistency is therefore only our own fault; and, should anybody try to do better, that would be the best justification for this attempt.

We have tried to analyze collision experiments and to find the simplest way of describing them. We have accepted without further questioning the basic interpretation of quantum mechanics in terms of probability amplitudes and the superposition principle (without assuming anything about time evolution). Since these axioms have never been discussed along the lines used here, there is a remote chance that some deeper logical inconsistencies could result.

We shall find that the notion of a unitary S matrix is in fact a necessary expression of our common acceptance of the most basic part of quantum mechanics and of the value of experiments. The discussion will unfortunately be rather long, by the very nature of the method, although any one acquainted with Wittgenstein will find it much too sketchy.

Since our approach does not make use of the notion of time, except for the macroscopic time as measured by clocks, it is necessary to give a new derivation of the relation between the matrix elements of the S operator and cross sections. This will be given elsewhere.

In a forthcoming paper we expect to show how simple assumptions about the behavior of collision probabilities as functions of the impact parameter can lead to analyticity properties of the collision amplitudes. The present work can be considered as a foundation for this kind of consideration.

2. DESCRIPTION OF EXPERIMENTS

As we have already mentioned, it is our habit to distinguish in a collision experiment generators and detectors of particles. However, one sometimes meets difficulties in using these notions when dealing with actual experiments. It is not clear, for instance, when a chain of reactions takes place in a bubble chamber, like in the case of

$$\pi^- + p \rightarrow K^+ + \Sigma^-, \tag{1}$$

$$K^+ + p \rightarrow K^+ + p + \pi^0 \tag{2}$$

what is the generator of the K^+ and what its detector. It will therefore be useful to enter further into this important distinction.

The right method, as we learn from Wittgenstein, is not to ask for the exact meaning or the definition of a generator or a detector but to know how these expressions are used by a review of examples. We shall thus learn what "family resemblances" there are between these examples.

Generators of particles are many: The most commonly used are accelerators which deliver particles of relatively well-known energy and targets which deliver them with very small energy. The two together, when supplemented by optical systems, can produce secondary beams of particles so that this larger apparatus acts as a generator. Radioactive sources as well as cosmic radiation can also produce particles.

Of the many types of counters, we shall only mention two: (1) The ordinary ionization counter acts globally in signaling that a particle has been crossing it. Several such counters together with electronic equipment can give correlated information on the effects produced by one or several particles. (2) The bubble chamber acts essentially as a huge set of small ionization counters, each one being the size of a bubble.

It is to be noted that an apparatus is not in itself a generator or a detector. For instance, some accelerators of electrons, if they received a positron, could decelerate it and act as a detector. A counter is often used as a part of the generating optics; or a counter can be used together with a shielding as a filtering part of the generator including an accelerator or cosmic radiation. A target can also be a counter. In the above-mentioned experiment, a part of the bubble chamber has acted as a part of the generator of K^+ while some part of the rest of the chamber acted as a detector of K^+. Therefore, although it would be open to criticism to say that any generator can also be a detector, and vice versa, it seems fair to say that there is a "family resemblance" between generators and detectors and *we can discuss experiments in terms of one where generators and detectors are identical.*

Such a symmetrical experiment, on which we shall base our argument later, is the following one: In order to investigate electron-positron scattering, we build four linear accelerators. Two of them are electron accelerators (and positron decelerators). The two others are positron accelerators (and electron decelerators). They can be moved in order to create different experimental situations. For brevity, we shall call this device the "four-gun" experiment. Although is it not a realistic experiment, it belongs to the family of collision experiments and it will be used mainly to save time in the discussion.

Since generators and detectors are not distinguishable by themselves, what makes us distinguish between them?

[10] L. Wittgenstein, *Philosophical Investigations*, English transl. by G. E. M. Anscombe (The Macmillan Company, New York, 1953).

A distinction which is checked practically in only a few cases is that the detector acts *after* the generator. By that we mean that a macroscopic (i.e., detectable) amount of time separates the action of a part of the accelerator (for instance an entrance counter) and of a part of the detector (for instance an exit counter). In most cases, we ascertain that the particle has a certain direction of motion dictated by our understanding of how the generator works and we keep track of the direction of this motion. In this way, we can make sure that some part of the apparatus was crossed after another part. By this we mean that the track of the particle (or particles) gives an order in which to arrange macroscopic parts of the experimental apparatus. It is important to realize that contrary to the direct time ordering, this space ordering depends strongly on some assumptions about the motion of particles and heavily in practice on the laws of relativity. We shall therefore assume these laws to be correct without further analysis.[11]

Even when one takes into account this ordering of the experimental apparatus, some freedom often remains in what exactly we shall call generator or detector. In many cases, this can only be done once the experiment is made. For instance, in the case of reactions (1) and (2), we can choose to concentrate on reaction (1), and consider a part of the bubble chamber containing the track of the pion as belonging to the generator, the rest of it as belonging to the detector. Or we may be interested in reaction (2), in which case we shall consider the part of the chamber containing the pion and the initial kaon tracks as the generator, the part containing the K^+, p, π^0 tracks as detector (of course, we speak only for brevity of the track of a π^0). If we want to look at the process as a reaction

$$\pi^- + p + p \rightarrow \Sigma^- + K^+ + p + \pi^0,$$

the overlap of generator and detector is more intricate.

To summarize: *We can restrict the discussion to cases where generators and detectors are clearly defined beforehand*, but we should be aware that it is not necessarily so in all cases.

A last and fundamental property of generators and detectors is that *two of them are always macroscopically separated*. This separation is most often spatial: two counters are at different locations. In extreme cases, this spatial separation can be quite small, like the radius of an emulsion grain or a chamber bubble.[12] The separation can also be in time: for most purposes, two successive pulses of an accelerator can be viewed as the action of two different generators.

[11] E. P. Wigner, in Proceedings of the International School of Physics "Enrico Fermi," Varenna, 1963 (to be published).

[12] In that sense, a good definition of macroscopic is "whatever can at least be seen with a microscope."

3. ASSUMPTION OF COMPLETENESS

When we are planning an experiment, we expect to get some information from it. In fact, we behave as if we expected much more than that and as if, at least in principle, this information could be complete. The question this time is not "What is the information given by experiments?" but "How do we use the readings of an experiment?"

Any experiment furnishes us with a finite number of events that we classify according to their characteristics. There is much prejudice in this classification. For instance, in the four-gun experiment, we enter as important parameters only the total energy and the scattering angle, because of our acceptance of special relativity. The results are a set of numbers that we consider, except for slight random fluctuations, as measuring probabilities (this is true in experiments on classical physics as well as in quantum physics).

An important assumption is that, by making an experiment with enough statistics and by enough refining of the precision of measurements, we shall get practically complete information. This amounts to assuming that there are no hidden regularities, for instance, in the behavior of a cross section as a function of energy, on a scale systematically unattainable by experiment. Furthermore, as soon as the precision gives a smooth and stable variation of the data, we agree that this is good enough information, at least qualitatively.

In more technical but more useful terms, we should say that it is our common understanding that, at least for some experiments in practice, and for all experiments in principle we are able: (1) to make a complete set of measurements, (2) to prepare any experimental situation, i.e., any set of initial data. This last point is of course only true for finite energies but we can without trouble restrict it to a finite domain of energies, since we have accepted the conservation of energy when granting relativity.

4. THE QUANTUM DESCRIPTION OF STATES

As has been said in Sec. 2, we can limit the discussion to cases where detectors and generators are clearly defined beforehand, as in the four-gun experiment.

It is by now a part of common language that such a generator, as for instance the electron accelerator, prepares a state of the electron which we shall call ψ_1. This state is defined when the accelerator works alone and the beam does not suffer any interference.

Once we know what state ψ_1 is produced by the accelerator when it works at a given time in a given position in space, it is easy to specify the state produced by the same accelerator at another time or in another position. In fact, if the connection between the two accelerators is defined by an element $\{a,\Lambda\}$ of the inhomogeneous Lorentz group, where a is a space-time translation and Λ a Lorentz transformation, then it has been shown by Bargmann, Wightman, and Wigner[13] that the new state is $U\{a,\Lambda\}\psi_1$, where $U\{a,\Lambda\}$ is a unitary operator representing the inhomogeneous Lorentz group. This result depends only upon the assumption of relativistic invariance and the basic hypothesis of quantum mechanics, which we have assumed. It is worth emphasizing at this stage that a is a macroscopic translation and that its definition does not imply the use of microscopic space or time.

In the same way, the positron accelerator produces a state ψ_2.

When both accelerators are working in concert, we shall consider them as together constituting the generator and we shall call ψ the state they produce. Let us suppose, to be more definite, that the two accelerators produce two intersecting beams during pulses of duration T and that we can vary the (macroscopic) time t at which the pulse of accelerator 2 begins while the pulse of accelerator 1 always begins at time zero. Let us therefore write $\psi(\psi_1,\psi_2,t)$ to emphasize the dependence of ψ on t, ψ_1, and ψ_2.

In the same way we shall denote by ϕ_1 and ϕ_2 the states that decelerators 1 and 2 are able to measure when acting independently. We shall restrict ourselves for simplicity to two-body collisions which can give rise only to two-body final states.

5. THE ASSUMPTIONS OF USEFULNESS

Let us now see what bearing our use of experiments has upon the properties of ψ. More precisely, let us investigate how our common agreement upon the interest of making experiments allows us to state properties of $\psi(\psi_1,\psi_2,t)$ as a function of t.

Let us first consider the case where $t\gg T$. We agree that in that case the two beams do not affect each other. In fact, one of them has already vanished far away when the other starts. In the language of quantum mechanics, it means that

$$\psi\approx\psi_1\otimes U\{t\}\psi_2. \tag{3}$$

In the same way, two detectors act independently only when their distance is macroscopic so that the state measured by the two decelerators in the four-gun experiment will be

$$\phi\approx\phi_1\otimes\phi_2. \tag{4}$$

What can be said when $t\ll T$? Here, it should be stressed that we agree that our experiment will give us practically pure information about the properties of the particles and not about the functioning of our apparatus. That such is our understanding is borne out by the very fact that we build accelerators and make experiments. That this convention is reasonable comes from the fact that similar experiments done in a different place or time give comparable results. In the language of quantum mechanics it means that for $t\ll T$, the dependence of $\psi(\psi_1,\psi_2,t)$ is trivial, i.e.,

$$\psi(\psi_1,\psi_2,t)\approx U\{t\}\psi(\psi_1,\psi_2,0). \tag{5}$$

Equation (5) could be called the assumption of the usefulness of experiment.

6. INTERPRETATION OF MEASUREMENTS

The results of the experiments are usually interpreted according to the basic axioms of quantum mechanics, namely the probability that the detectors register the outgoing particles should be equal to $|\langle\phi|\psi(t)\rangle|^2$. Now, according to Eqs. (4) and (5), this is nothing but

$$|\langle\phi_1\otimes\phi_2,U\{t\}\psi(0)\rangle|^2.$$

In the experiment under consideration, ϕ_1 and ϕ_2 correspond to values of the momenta p_1, p_2 of the particles which are well-defined, up to a small error. According to Bargmann, Wightman, and Wigner,[13] this means that

$$U\{t\}\phi_1\otimes\phi_2\simeq e^{i(p_1^0+p_2^0)t}\phi_1\otimes\phi_2,$$

so that the probability is given by

$$|\langle\phi_1\otimes\phi_2,\psi(0)\rangle|^2, \tag{6}$$

and the scalar product in this expression is antilinear in ϕ_1 and ϕ_2.

7. EXISTENCE OF THE S MATRIX

As we have seen in Sec. 2 and as is particularly clear in the four-gun experiment, there are cases where no distinction can be made between generator and detector except in their time ordering. However, the scalar product in Eq. (6) is antilinear in ϕ_1 and ϕ_2 and does not depend upon time. Therefore, since it is linear in ψ, it must also be linear in ψ_1 and ψ_2. Since ψ_1 and ψ_2 belong to a Hilbert space, this property, according to the Riesz theorem,[14] can be satisfied only if

$$\langle\phi_1\otimes\phi_2,\psi(0)\rangle=\langle\phi_1,\otimes\phi_2,S\psi_1\otimes\psi_2\rangle, \tag{7}$$

where S is a certain linear operator acting between the Hilbert spaces of (ψ_1,ψ_2) and (ϕ_1,ϕ_2).

8. UNITARITY OF S

Let us recall that, according to the assumption of completeness, we consider that our states ϕ_1, ϕ_2, ψ_1, ψ_2, which are in practice limited, can be used as well as a complete set of states.

Applied to the measuring states, this means that a complete set of measurements in a practical sense should

[13] V. Bargmann, A. S. Wightman, and E. P. Wigner (unpublished).

[14] See, for instance, F. Riesz and B. Sz. Nagy, *Functional Analysis* (Hungarian Academy of Sciences, Budapest, 1951).

give the same result as the projection on a complete set of states in the mathematical sense. Since the total probability is 1, we have

$$\sum_{\{\phi_0\}} |\langle\phi_0,S\psi_0\rangle|^2=1\,, \tag{8}$$

where $\phi_0=\phi_1\otimes\phi_2$, and $\{\phi_0\}$ is a complete set.

Applied to $\psi_0=\psi_1\otimes\psi_2$, the assumption of completeness means that Eq. (8) is true for any normed vector ψ_0 in the Hilbert space; i.e., Eq. (8) is the very definition of a unitary operator.

9. CONCLUSIONS

We can state our conclusions as the following proposition.

Proposition: It is a necessary consequence of our common understanding of what an experiment is as well as of our use of it that a collision between particles can always be described by a unitary operator S, the probability amplitude for a particular measurement being given by $|\langle\phi_0 S\psi_0\rangle|^2$.

The very formulation of this proposition is rather unusual. In fact such a result has not the compelling strength of a mathematical theorem. It is open to a re-evaluation of our behavior with respect to experiments, as could happen from a new and unexpected result. It would seem interesting, although long and difficult, to re-examine some of our concepts, in classical physics as well as in the foundations of quantum mechanics, along similar lines.

Finally let us state a few important points of the present approach:

The introduction of the S matrix does not need any consideration of a microscopic time. In that sense, the S matrix is on a higher level of simplicity than the Schrödinger equation.

Our considerations have been relativistic. Of course, the same results could be obtained in a nonrelativistic approach. However, the role played here by the unitary representations of the inhomogeneous Lorentz group would have to be played by the representations of the Galilei group, which are somewhat more complicated. This was our main reason for sticking to the relativistic case which has the advantage of being both more general and simpler.

It is not trivial to extend the present considerations to three-body collisions, the problem in that case being to extend Eq. (4). We expect to go back to this point later.

The connection between S-matrix elements and cross sections is customarily made by using explicitly the Schrödinger equation or some of its equivalent forms. It is an easy, though somewhat lengthy, exercise to give a derivation which does not use time and characterizes positions by the Newton-Wigner operator. This derivation will be given elsewhere. It is to be emphasized that Planck's constant enters only at that stage.

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