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GOVERNING EQUATION FOR THE TEMPERATURE DISTRIBUTION IN A TRANSPARENT PLASTIC BAR

JUAN J. MORALES

Departamento de Física, Facultad de Ciencias, Universidad de Extremadura, 06071 Badajoz (Spain)

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ABSTRACT

Three different equations have been used to study the temperature distribution in a plastic poly(methyl methacrylate) bar. The governing equation for the steady state is found to depend on whether the heating is by conduction or radiation. Cooling from the steady state shows a single solution in most of the cases.

INTRODUCTION

The mechanism of heat transfer by conduction and radiation depends on the nature of the material. In solids, there is a clear distinction between metallic and non-metallic materials: in a metallic conductor heat is carried simultaneously by free (valence) electrons and by lattice waves (phonons), whereas in non-metallic materials (dielectrics) heat is carried only by phonons.

In glassy (or amorphous) materials, the atoms or molecules are distributed in a manner lacking both symmetry and periodicity. The atomic arrangement shows short-range order but no long-range order (as in an ideal crystal or pure material), as X-ray diffraction experiments reveal [1]. Other solid materials such as plastics, leather, etc., are heterogeneous in composition, and in a broad sense they are mixtures of solid amorphous materials.

With respect to radiative heat transfer, most solids absorb radiation very strongly through their surfaces, so that practically all the incident radiation is absorbed in a very thin layer below the surface. In metals, the thickness of this layer is a fraction of a micrometre. In electrically non-conducting materials, fractions of a millimetre are usually sufficient. Exceptions are a few solid substances like glass, quartz, rock salt, etc. [2]. When the radiation is absorbed within a certain wavelength range, there is heat production in the solid, and in some cases, such as plastics, the IR is so strongly absorbed that the material may melt.

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Obviously, the flow of heat in opaque solids takes place exclusively by conduction, while in transparent solids conduction and radiation transfer can both occur. In either case, when there is a discontinuity in temperature between the solid and its surroundings, a non-uniform temperature distribution is set up in the system which, together with its variation in time, is usually complex mathematical problem.

In a previous paper [3] a study was made of the best governing equation for the linear flow or heat in a metallic bar. Three different equations were found that described the steady state within statistical error, and two equations that described the cooling. In this paper, a bar made of a transparent poor conductor (a plastic) is studied. Heating is by both conduction and radiation, to see what influence these two agents have on the governing equations for the temperature distribution in the steady state and during cooling.

GENERAL THEORY

The mathematical problem should be treated using the general theory of heat transfer [4] taking into account the nature, the geometry, and the way of heating the solid.

The solid is a transparent plastic with a relatively low melting point. This means that, when the temperature distribution is set up, one cannot be sure about the homogeneity of the solid and the independence of thermophysical properties such as the density, ρ, the specific heat, c, and the thermal conductivity, K, on temperature and position. The plastic is in the form of a cylindrical rod with cross-section, w, diameter, d, and circumference, p, much smaller than its length, L, in order to consider the linear heat flow along a generatrix taken as the X-axis. If the rod is heated at one end by a heater, there is a heat flow from that end into the rest, with no heat production inside the solid, but if it is heated by electromagnetic waves, there should appear internal sources of heat $A(x, t)$ at the points x where a wavelength is being absorbed by the plastic at time t.

The differential equation of heat flow in the bar, which need not be homogeneous or isotropic, with heat sources and immersed in a medium of temperature θ_0, is given by

$$\frac{\partial f_x}{\partial x} + \rho c \frac{\partial \theta}{\partial t} + \frac{Hp}{w}(\theta - \theta_0) = A(x, t) \tag{1}$$

where f_x is the heat flow and H is the surface conductance. This equation holds at any point of the solid, and corresponds to the equation of continuity, with sink and sources, in hydrodynamics.

The treatment of eqn. (1) is very difficult, and one must find some approximation to solve it appropriately for particular cases. Some are briefly described below.

Homogeneous and isotropic solid

If the solid is homogeneous and isotropic, the thermal conductivity is independent of temperature and the heat flow is given by

$$f_x = -K\frac{\partial \theta}{\partial x} \tag{2}$$

eqn. (1) then becomes

$$\frac{\partial^2 \theta}{\partial x^2} - \frac{1}{\kappa}\frac{\partial \theta}{\partial t} - \frac{\nu}{\kappa}(\theta - \theta_0) = -\frac{A(x,\, t)}{K} \tag{3}$$

where $\kappa = K/c\rho$ and $\nu = Hp/cw\rho$, κ being the diffusivity.

Heat production in the solid

In a first approximation, the rate of heat production is independent of the temperature, and to a better approximation often takes the form

$$A = a + b\theta \tag{4}$$

where a and b are constants which may have either sign [5]. This equation may be used as a very crude first approximation, but the exact solutions may have very different properties from those with linear behaviour.

In some cases it has been found experimentally that an exponential law holds

$$A = A_0\, \mathrm{e}^{b\theta} \tag{5}$$

But analytical solutions do not always exist, and such cases have to be treated numerically.

Isolated solid

If the solid is isolated, there is no radiation into the medium, the surface conductance H is zero and therefore the third term on the left in eqn. (1) and in eqn. (3) is also zero.

Steady state

The case of steady flow, in which $\partial\theta/\partial t = 0$, is of particular importance when there is no radiation into the medium, as eqn. (3) becomes Poisson's equation if A is constant, or Laplace's equation if $A = 0$.

Cooling

In general, cooling from the steady state in a solid does not simplify any of the terms in eqn. (1), but, in the case studied here, when the radiative heater is cut off the internal source of heat produced by the radiation should disappear.

After this brief description of possibilities, the problem now is to select the appropriate solution of eqn. (1) from the experimental data obtained for the steady state and for cooling.

EXPERIMENTAL PROCEDURE AND RESULTS

The experimental procedure was qualitatively the same as that used for an iron bar [3]. The plastic selected was poly(methyl methacrylate) (PMMA), which is well known in solar energy applications [5], and whose physical and geometrical characteristics are given in Table 1. To measure the temperature along the bar, taking into account the sharper temperature gradient than in the iron bar, 18 narrow holes were drilled perpendicular from the generatrix to the axis. Chromel–alumel thermocouples were inserted and connected to a digital thermometer. The first thermocouple, taken as the origin of coordinates, was placed 8 mm from the nearer end of the bar. The bar was heated from this end by a 250 W lamp and with a spectrum similar to that of the sun ('solar lamp') and very rich in IR radiation. In this way the bar was heated by both conduction and radiation. If only conduction was required, an aluminium film was placed on the end, over the cross-section of the bar. The light was collimated by a screen with a hole of the same diameter as the cross-section. The intensity of the radiation was adjusted by altering the distance between lamp and bar. The bar was held on two wooden supports and isolated by means of asbestos cord. The room temperature was taken by a thermocouple inserted from the generatrix to the centre of a cylindrical copper block (5 cm high and 9 cm in diameter).

Optical study of the bar

Before starting to study the temperature distribution in the bar, it was interesting to investigate its optical behaviour. Thus, the study began with the spectral absorption of the radiation.

TABLE 1

Physical and geometrical characteristics of the PMMA bar

$\rho = (1.190 \pm 0.001) \times 10^3$ kg m^{-3} at 20 °C
$c = (1.42 \pm 0.01) \times 10^3$ J kg^{-1} K^{-1}
$K = (0.193 \pm 0.001)$ J s^{-1} m^{-1} K^{-1} (0–50 °C)
$n_D = (1.492 \pm 0.001)$ ($\lambda = 589 \times 10^{-9}$ m)
$L = (135.5 \pm 0.1) \times 10^{-2}$ m
$d = (500.0 \pm 0.5) \times 10^{-4}$ m
$p = (157.08 \pm 0.05) \times 10^{-3}$ m
$w = (196.35 \pm 0.01) \times 10^{-5}$ m^2

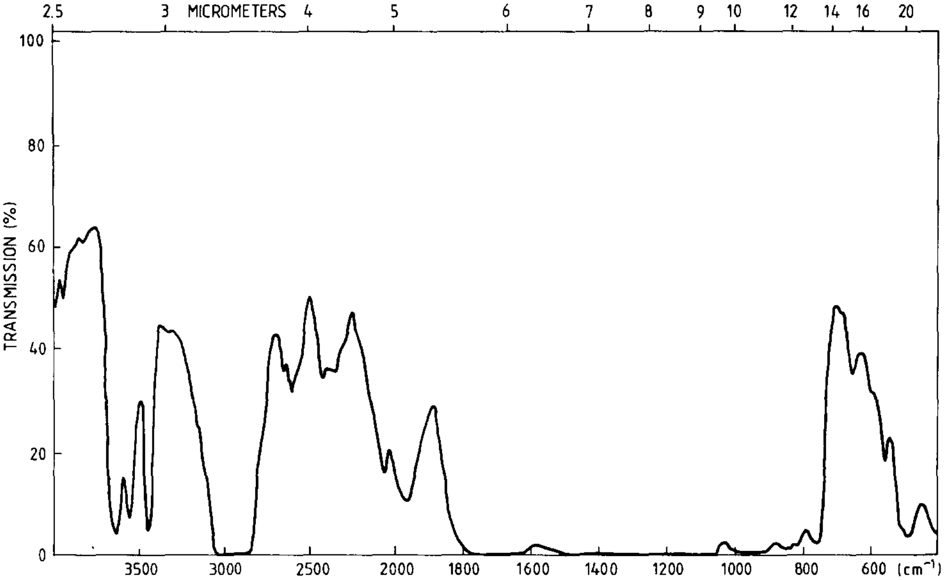

Fig. 1. Transmission of PMMA as a function of wavelength for a sample of thickness 0.38 mm.

The spectrum obtained from the spectrophotometer with a sample of 0.38 mm thickness shows that the UV radiation is absorbed completely, as expected. IR radiation is partially absorbed, as can be seen from Fig. 1, where the wavenumber (in cm^{-1}) at the bottom, or wavelength (in μm) at

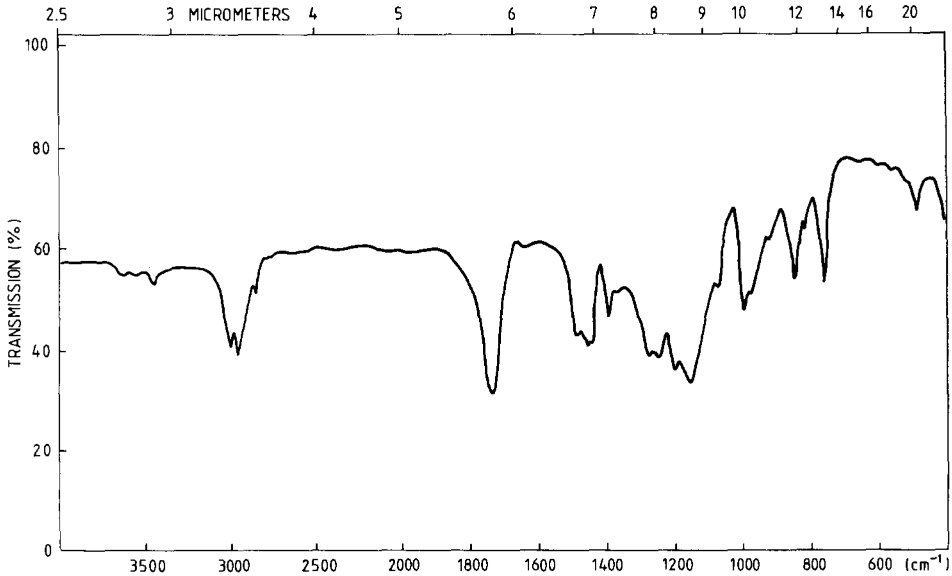

Fig. 2. Absorption for a sample of PMMA and KBr as a function of wavelength.

the top, is plotted against percentage transmission. The transmission is defined as the ratio between the transmitted and the incident intensities. The maximum peak is located at ≈ 2.7 μm with $\approx 70\%$ transmission, whereas the second and third peaks have the same value of about 50% at $\lambda = 4$ and 14 μm, respectively. With a sample of thickness 1 mm, the maximum peak exhibited $\approx 30\%$ transmission.

To obtain the minimum transmission, one needs to make a film with plastic filings and KBr as a 'glue', as KBr is transparent to IR radiation. The results are shown in Fig. 2, where there are two minima at $\lambda = 5.8$ and 8.7 μm, with $\approx 30\%$ transmission, and three secondary minima, at $\lambda = 3.4$, 6.9 and 8.0 μm, with $\approx 40\%$ transmission. In this figure, the transmission maxima have disappeared because the distribution of filings in the sample is random, but this affects neither the location nor the intensity of the minima.

Steady state

Table 2 gives the three different steady states (I–III) chosen, the way in which the bar was heated, the distances between the lamp and the end of the bar, and the time before the cooling commenced. In all cases the steady state was reached 4–5 h from the start of heating. The temperature distributions for these cases are given in Table 3, where the first column gives the distances x between thermocouples. Cases I and II were obtained by radiation, having the smoothest and the sharpest temperature gradients, respectively, whereas case III is an intermediate one obtained by heating with the aluminium film placed on the end of the bar.

As a first selection amongst all the possible solutions of eqn. (1), the power, single exponential, and double exponential were chosen for these cases:

$$\theta = a_1 x^{-b_1} \tag{6}$$

$$\theta = a_1 \, e^{-b_1 x} \tag{7}$$

$$\theta = a_1 \, e^{-b_1 x} + a_2 \, e^{-b_2 x} \tag{8}$$

where a_1, b_1, a_2 and b_2 are constants to be determined from the experimental values given in Table 3.

TABLE 2

Initial conditions used to obtain the three steady states I–III

Steady state	Heating source	Source–bar distance (cm)	Time (h)
I	light + heat	20	16
II	light + heat	10	5.5
III	heat	10	6

TABLE 3

Temperature distributions corresponding to the three steady states I–III

$X \times 10^{-3}$ m	I	II	III	$X \times 10^{-3}$ m	I	II	II
0	24.0	71.0	52.0	83	2.6	7.3	2.2
15	12.9	40.0	28.6	91	2.3	6.5	1.7
25	9.1	30.4	20.1	100	2.1	5.9	1.4
33	6.8	22.4	13.8	125	1.8	4.6	0.9
41	5.4	18.1	9.9	150	1.6	4.0	0.6
50	4.5	14.6	7.3	175	1.4	3.4	0.5
58	3.7	11.9	5.3	200	1.2	3.2	0.4
66	3.1	10.0	3.9	250	1.0	2.5	0.4
75	2.8	8.4	2.9	500	0.5	1.2	0.0

TABLE 4

Values of the fit corresponding to the first steady state I

I	$a_1 \pm \sigma$	$-b_1 \pm \sigma$	$a_2 \pm \sigma$	$-b_2 \pm \sigma$	$\sum_{i=1}^{N} (\theta_i - \tilde{\theta}_i)^2$
Eqn. (6)	0.28 ± 0.01	0.923 ± 0.001	–	–	1.23
Eqn. (7)	22.9 ± 0.7	33.14 ± 0.07	–	–	18.09
Eqn. (8)	20.9 ± 0.9	48.47 ± 0.08	3.11 ± 0.08	4.46 ± 0.07	0.11

TABLE 5

Values of the fit corresponding to the second steady state II

II	$a_1 \pm \sigma$	$-b_1 \pm \sigma$	$a_2 \pm \sigma$	$-b_2 \pm \sigma$	$\sum_{i=1}^{N} (\theta_i - \tilde{\theta}_i)^2$
Eqn. (6)	0.82 ± 0.01	0.941 ± 0.001	–	–	40.84
Eqn. (7)	68.4 ± 0.7	30.86 ± 0.07	–	–	92.89
Eqn. (8)	62.9 ± 0.8	41.26 ± 0.08	7.79 ± 0.08	4.54 ± 0.07	3.02

TABLE 6

Values of the fit corresponding to the third steady state III

III	$a_1 \pm \sigma$	$-b_1 \pm \sigma$	$a_2 \pm \sigma$	$-b_2 \pm \sigma$	$\sum_{i=1}^{N} (\theta_i - \tilde{\theta}_i)^2$
Eq. (6)	0.168 ± 0.004	1.2404 ± 0.0004	–	–	40.23
Eq. (7)	51.9 ± 0.8	39.28 ± 0.08	–	–	2.11
Eq. (8)	51.6 ± 0.8	40.39 ± 0.08	0.51 ± 0.08	1.99 ± 0.07	1.07

Tables 4–6 list the results for these constants and their standard deviation σ for the steady states I–III. The last column shows the sum of the squares of the differences between the experimental value θ_i and the expected value $\tilde{\theta}_i$ as a measure of the goodness of the fit [7]. From Tables 4–6, the best equation for the steady states in the plastic bar is seen to be the double exponential with a first (second) exponential driving the sharper (smoother) temperature gradient; the second exponential is much smaller and could be considered as a correction to the first. The accuracy of the fit is better when the temperature is lower, and for case I, the fit is practically perfect.

The other two fits are affected by the manner of heating. The power equation yields a value of b_1 very close to unity, being below 1 for cases I and II and above 1 for case III. The differences between the experimental data an theoretical values are close to those of the double exponential in case I, but higher for the other two cases, each giving a value of ≈ 40. The single exponential equation tends to give a value of a_1 which is θ_1, the temperature at the first point of the bar. For cases I and II these values of a_1 are lower than θ_1, but for case III it is exactly the same in consequence of the marked decrease in the uncertainty shown in the last column.

Cooling

Once the steady state had been reached cooling was started by switching off the lamp and removing it from the vicinity of the end of the bar (because of its thermal inertia). The first temperatures were taken after 30 s and the other measurements were done every 5 min up to 190 min, when the temperatures in the bar were seen to be near room temperature.

Case III, with temperature intermediate between that for cases I and II, was chosen for cooling, and the exponential decay was studied after discarding other kinds of solutions with far from reasonable behaviour. Tables 7 and 8 show the results of the fit for the simple and double exponential, respectively, where θ_1, θ_2, $\theta_3, \ldots, \theta_i$ are the temperatures at the points $x = 0.0$, 15, 25, …, x_i (see Table 3). The cooling obeys the general form for the single exponential, as the double exponential has the same coefficient for

TABLE 7

Fit for the cooling starting from steady state III, using eqn. (7)

Cool.	$a_1 \pm \sigma$	$-b_1 \pm \sigma$	$\sum_{i=1}^{N} (\theta_i - \tilde{\theta}_i)^2$
θ_1	52.2 ± 0.8	0.57 ± 0.01	13.24
θ_2	32.2 ± 0.7	0.36 ± 0.01	31.52
θ_3	23.9 ± 0.7	0.31 ± 0.01	31.60
θ_4	17.1 ± 0.8	0.26 ± 0.02	25.13

TABLE 8

Fit for the cooling starting from steady state III, using eqn. (8)

Cool.	$a_1 \pm \sigma$	$-b_1 \pm \sigma$	$a_2 \pm \sigma$	$-b_2 \pm \sigma$	$\sum_{i=1}^{N}(\theta_i - \tilde{\theta}_i)^2$
θ_1	31.6 ±1.2	0.81±0.06	21.83±0.03	0.39±0.02	4.08
θ_2	20.7 ±1.0	0.36±0.03	11.45±0.01	0.360±0.009	31.52
θ_3	12.5 ±1.1	0.31±0.04	11.38±0.01	0.308±0.009	31.60
θ_4	11.04±0.03	0.26±0.01	6.0 ±0.3	0.26 ±0.02	25.13

the exponent ($b_1 = b_2$) and the total coefficient a is the sum of a_1 and a_2. Only for the first point does the double exponential clearly predominate over the single exponential, but the differences between the experimental and expected values are not large for either case. For points beyond $i = 4$ the temperature excess over room temperature is not so large and the curve tend to straight lines.

Figure 3 presents the cooling for the first ten points in the bar. The first point presents the sharpest decay, the temperature after 25, 40, 55,... min being lower than for the points $i = 2, 3, 4,...$ From the second point on, the thermal inertia is evident and increases as the temperature difference between the points and the laboratory diminishes.

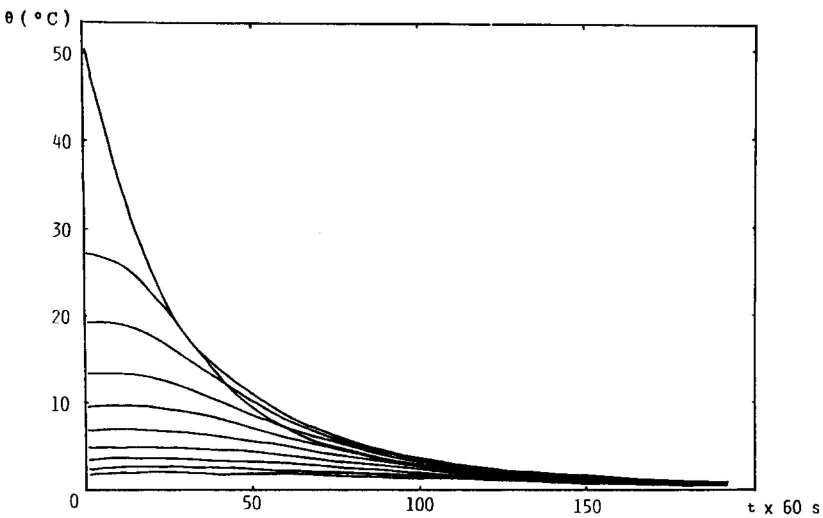

Fig. 3. Cooling curves for several points in the PMMA bar, with the cooling started from steady state III.

DISCUSSION AND CONCLUSIONS

As can be seen from the results, three different equations explain the steady state temperature distribution in the bar, but one equation predominates depending on which agency produces the heating. If the bar is heated by the lamp, with thermal and radiative effects, the power solution of eqn. (6) is better than the single exponential solution of eqn. (7), but when heating is without radiation the opposite is the case. The explanation could be in the nature of the bar (Table 1) and in the role of the internal source of heat in the plastic (Figs. 1 and 2). The power law for the steady state is one of the limited number of special analytical solutions for the differential equation of linear flow, eqn. (1), when the thermal properties of a solid vary with position. The internal sources of heat created in the transparent medium as the radiation passes along the bar make the bar less homogeneous, and the power solution predominates over the single exponential solution (Tables 4 and 5). When radiation is prevented, the internal sources disappear ($A = 0$), lessening the inhomogeneity of the bar and permitting eqn. (3) to be used instead of eqn. (1). The resulting differential equation has a straightforward solution, a single exponential as in eqn. (7) with coefficient $a_1 = \theta_1$, which agrees with the results given in Table 6. The double exponential, which is a general solution of eqn. (1), gives the best fit to the steady state as the number of parameters available to fit the data is double (a_1, b_1, a_2 and b_2). The first exponential describes the sharpest decay of the temperature distribution, whereas the second describes the smoother decay of temperature (Tables 4–6). In some practical cases it is easier to handle a single exponential than a double exponential when $a_1 \gg a_2$ and $b_1 \gg b_2$, as in the case under consideration, despite the slight loss of accuracy in the results.

The general behaviour for the cooling is a single exponential law, except for the first point in the bar where the double exponential is clearly seen (Fig. 3). The faster temperature decay at this point is a consequence of the closeness of the end of the bar (only 8 mm), permitting heat loss through its cross-section. Apart from this case, the double exponential solutions obtained reduce to single exponentials, as the exponents are the same ($b_1 = b_2$) and the resulting coefficient a is the sum of each coefficient a_1 and a_2 (Tables 7 and 8).

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REFERENCES

1 C. Kittel, Introduction to Solid State Physics, Wiley, New York, 1986, Chaps. 5 and 17.
2 E.R.G. Eckert and R.M. Drake, Analysis of Heat and Mass Transfer, McGraw-Hill, Tokyo, 1972, Chaps. 2 and 15.
3 J.J. Morales, Thermochim. Acta, 143 (1989) 211.
4 H.S. Carslaw and J.C. Jaeger, Conduction of Heat in Solids, Oxford University Press, Oxford, 1978, Chaps. 1 and 17.
5 J.C. Jaeger, Br. J. Appl. Phys., 28 (1949) 107.
6 A. Blaga, Solar Energy, 21 (1978) 331.
7 V.P. Spiridonov and A.A. Lopatkin, Mathematical Treatment of Physico-chemical Data, Mir, Moscow, 1973.