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THE SENSITIVITY, LINEARITY AND TEMPERATURE RESOLUTION OF NON-EQUAL ARM THERMISTOR WHEATSTONE BRIDGES NEAR BALANCE

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ABSTRACT

An investigation of the effects of bridge arm configuration, i.e., arm impedance ratios, of a Wheatstone bridge circuit are presented. Although the generalized treatment given can be applied to any resistance change transducer in a Wheatstone bridge, particular emphasis has been placed on the use of thermistors in solution phase calorimetry. The work presented characterizes the various bridge configurations in terms of their sensitivity, linearity, and theoretical temperature resolution.

INTRODUCTION

Much of the instrumentation in analytical calorimetry involves the use of a thermistor in conjunction with some type of Wheatstone bridge circuitry. Although a myriad of special purpose bridge configurations exist, the use of the simple d.c. Wheatstone bridge, in both equal and unequal arm arrangements, is predominant. Such a system, with present day high gain, high input impedance electronic amplifiers has been used extensively in numerous applications of thermometric titrimetry[1,2] and direct injection enthalpimetry[3] as an unbalanced, continuously recording temperature change detector. There are several inherent disadvantages in the equal arm system which manifest themselves when thermal changes are being monitored. One such disadvantage, the non-linear relationship between the bridge output and the actual temperature change, has been discussed by Gunn[4]. The present work is a completely general study of various bridge configurations, i.e., arm ratios, in terms of their sensitivity, signal-to-noise characteristics, and linearity. The results of a recent thermistor study[5] are also used to compare the theoretical temperature resolution of the various configurations.

DISCUSSION

If one considers a Wheatstone bridge circuit such as that shown in Fig. 1, the exact output of the bridge (e_0) can be shown[6] to be:

$$e_0 = E_B \frac{R_1 R_2 - R_3 R_T}{(R_1 + R_3)(R_2 + R_T)} \tag{1}$$

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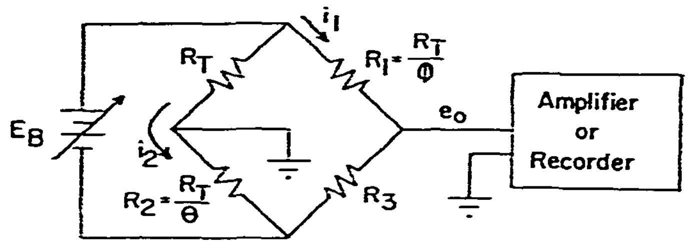

Fig. 1. Wheatstone bridge circuitry commonly used for unbalanced continuous recording enthalpimetric measurements.

under conditions where the voltage source (E_B) has zero equivalent resistance and the voltage measuring device has effectively infinite input impedance. Hence, the output with any resistance values substituted can be calculated. In choosing viable resistance values for the various arms of the bridge, we are limited by the maximum acceptable equivalent resistance of the bridge and by the null condition which requires that:

$$R_1 R_2 = R_3 R_T \tag{2}$$

The equivalent resistance of the bridge with respect to the output is important for impedance matching of the signal source (the bridge) and any subsequent amplifier. This output impedance can be easily computed using the null condition which yields

$$R_{eq} = \frac{R_T(1+\phi)}{\phi(1+\theta)} \tag{3}$$

where ϕ is the ratio R_T/R_1 and θ is the ratio of R_T to R_2. If we assume an equal arm configuration, we find that eqn (3) reduces to $R_{eq} = R_T$ and only the choice of the transducer impedance affects the choice of the following electronics. If we choose

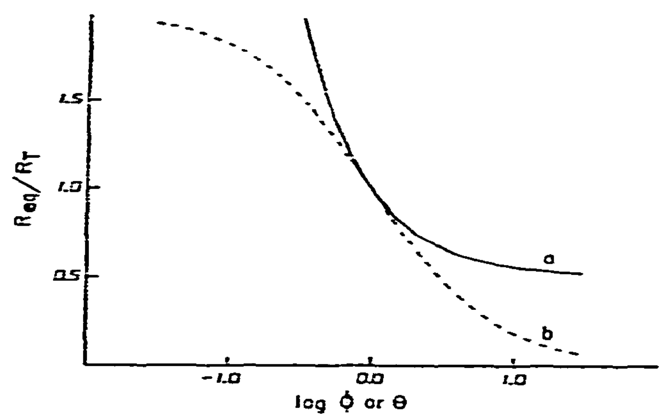

Fig. 2. The normalized equivalent output resistance of the bridge as a function of the bridge configuration. Curve a represents the mathematical form when the independent variable is the ratio of the transducer impedance to the resistance of the parallel element (ϕ). In curve b the independent variable is the ratio of the transducer and the series element (θ). In both cases, the second independent variable is unity.

other values for ϕ and θ, several interesting situations arise. If ϕ is set equal to one and θ is made large, one can reduce the impedance of the source. If we assume that $\theta = 1$ and vary ϕ, we cannot significantly reduce the source impedance even by making ϕ very large. This can be seen graphically in Fig. 2. The source impedance is important since all derivations assume no loading and therefore the value of the equivalent source resistance is required. Secondly, the bridge impedance determines to a large extent the magnitude of the limiting Johnson noise and hence effects both the signal-to-noise performance and the limit of detection of the device. Although the loading phenomenon does merit discussion, the consideration of sensitivity, signal-to-noise ratio characteristics, and limit of detection are more significant factors in the choice of a bridge configuration.

Sensitivity

The use of a thermistor as the transducer–resistance element makes the output of the bridge a function of the solution temperature. The resistance of the device can best be described[6,7] by an equation of the form

$$R_T = R_c \exp\left(\frac{B}{T+K}\right) \tag{4}$$

where R_c, B and K are constants of the thermistor. For small temperature excursions, the functional relationship between resistance and temperature can be approximated by

$$R_T = R_T^\circ + \Delta R_T = R_T^\circ + R_T^\circ \alpha \Delta T \tag{5}$$

where R_T is the impedance of the transducer at any temperature, ΔR_T is the temperature dependent resistance change, R_T° is the thermistor resistance at a convenient reference temperature, α is the temperature coefficient of resistance about ($0.04\,\Omega/\Omega\cdot{}^\circ$C), and ΔT is the solution temperature change. For present considerations, the linear proportionality of eqn (5) will suffice, since we wish to describe the behavior of the

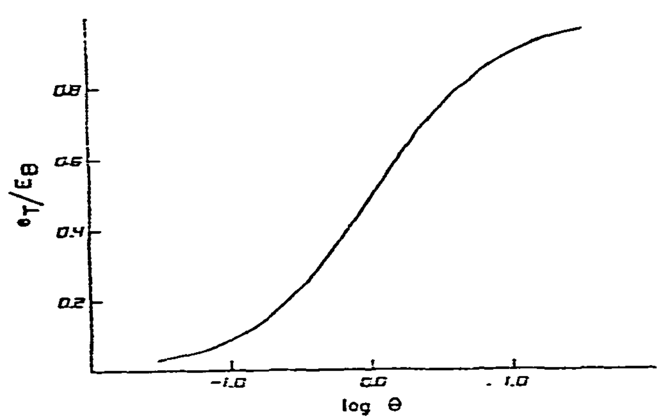

Fig. 3. The voltage drop across the thermistor ($e_T = i_2 R_T$) per Volt applied to the bridge as a function of the ratio θ (see Fig. 1).

bridge configuration. The apparent temperature measured by the thermistor is not only a function of the ambient temperature, but also of the power dissipated in the device[7]. Thus, in order to equitably compare the sensitivity of the various configurations, the current applied to the transducer must be examined. This relationship is given by the following equation which is shown graphically in Fig. 3,

$$i_2 = \frac{\theta}{(1+\theta)} \frac{E_B}{R_T} \tag{6}$$

It is apparent from this equation that the current in the transducer side of the bridge is independent of ϕ.

The sensitivity, defined as the output voltage per unit temperature change, is given by

$$e_0 = \frac{\Delta R}{R_T} \frac{\theta}{(1+\theta)^2} E_B = \alpha \Delta T \frac{\theta}{(1+\theta)^2} E_B \tag{7}$$

provided that the denominator of eqn (1) is held constant, i.e., ΔR is small. Again the output is independent of ϕ while the dependence on θ is very strong. This is shown in Fig. 4. It is readily apparent that an equal arm configuration yields the highest sensitivity. This is not a serious issue since "state of the art" d.c. and a.c. amplifiers are available with input noise less than 10 nV.

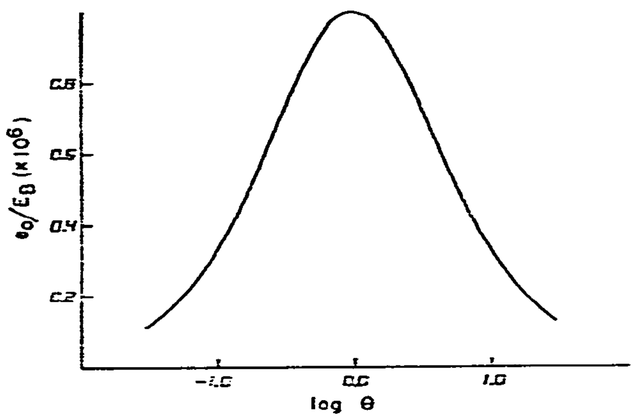

Fig. 4. The bridge response to a 100 μ°C change per Volt applied to the bridge as a function of the ratio θ (see Fig. 1).

Signal-to-noise considerations

In a recent publication[5], the rms temperature noise of thermistors under the conditions of solution phase enthalpimetry has been characterized by an equation of the form

$$\Delta T_{rms} = \sqrt{\frac{A}{E_B^2} + B + CE_B^4} \tag{8}$$

where A is a Johnson noise dependent term, B is a solution inhomogeneity coefficient, and C is a power dissipative term. The Johnson noise[8] is given by

$$e_{JN}^2 = 4kT\,\Delta f\,R_{eq} \tag{9}$$

where k is the Boltzmann constant, T is the ambient temperature, Δf is the system bandwidth, and R_{eq} is the system impedance. Combining eqns (3), (7) and (9), a temperature noise equivalent to the electrical variations can be given by the equation

$$\Delta T_{JN}^2 = \frac{4kT\,\Delta f}{\alpha^2 E_B^2}\left(\frac{R_T(1+\phi)(1+\theta)^3}{\phi\theta^2}\right) \tag{10}$$

In a similar manner, the noise due to the power dissipated in the sensor can be derived by combining eqn (6) and the relation between the temperature and the applied power[7], resulting in the equation,

$$\Delta T_\delta^2 = \frac{i_2^4 R_T^2}{f^2(\delta)} = \frac{\theta^4}{f^2(\delta) R_T^2 (1+\theta)^4} E_B^4 \tag{11}$$

where $f(\delta)$ is a function of the dissipation constant[5]. If these noise sources are summed as statistical deviations and put in the form of eqn (8), it is found that

$$A = \frac{4kT\,\Delta f}{\alpha^2}\,\frac{(1+\phi)(1+\theta)^3}{\phi\theta^2}\,R_T \tag{12}$$

and

$$C = \frac{\theta^4}{f^2(\delta) R_T^2 (1+\theta)^4} \tag{13}$$

It has been shown[5] that if B is assumed to be zero, the minimum temperature resolution is given by the relation

$$\Delta T_{\text{min.,rms}} \cong 2.2\,A^{1/3} C^{1/6} \tag{14}$$

Substituting for A and C,

$$\Delta T_{\text{min.}} = \frac{2.2(4kT\,\Delta f)^{1/3}}{\alpha^{2/3} f^{1/3}(\delta)}\,\frac{(1+\phi)^{1/3}(1+\theta)^{1/3}}{\phi^{1/3}} = K\,\frac{(1+\phi)^{1/3}(1+\theta)^{1/3}}{\phi^{1/3}} \tag{15}$$

The optimum bridge voltage is given by

$$E_{\text{B,opt.}} = \sqrt{\frac{A}{2C}} = K'\,\frac{(1+\phi)^{1/2}}{\phi^{1/2}}\,\frac{(1+\theta)^{7/2}}{\theta^3} \tag{16}$$

Inspection of eqn (8) indicates that $\Delta T_{\text{min.}}$ will merely be increased by an increase in B without changing its dependence upon A or C (see eqn (11)); further, B has no effect on $E_B^{\text{opt.}}$. This function is shown graphically in Fig. 5. It can be easily seen from this representation that for a given value of θ, the temperature resolution improves as ϕ is made larger. Conversely, for a specified value of ϕ, a smaller value for θ gives better temperature resolution. To carry this extrapolation to a logical conclusion, the ultimate temperature resolution will be obtained for an infinite value of ϕ and a zero θ value. Substituting these values into eqn (15) the maximum improvement in resolution, one can obtain over the equal arm case, is a factor of 1.59.

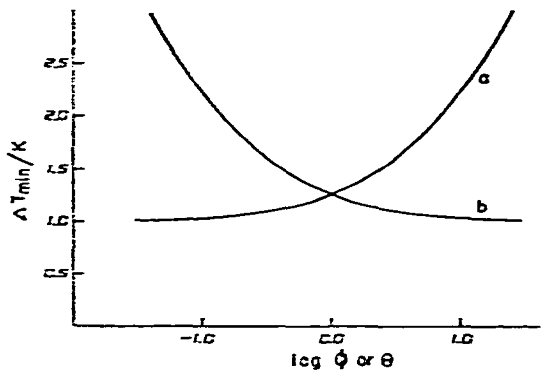

Fig. 5. The normalized temperature minimum as a function of the ratio θ in curve a and ϕ in curve b. (See Fig. 1 for the definition of ϕ and θ.) For both cases, the second independent variable was assumed to be one.

Linearity

As was mentioned previously, for sufficiently small temperature excursions the resistance response of the thermistor is linear. For the present purposes, a linear response will be assumed to simplify the relationship between the normalized bridge output and the temperature change. As stated above, modern linear d.c. amplifiers preclude the necessity of obtaining maximum bridge sensitivity. In order to compare the non-linearity inherent in the various configurations with a minimum of distortion due to changes in sensitivity, the solution obtained from eqns (1), (2), and (5) was normalized with respect to the sensitivity and the temperature change to yield

$$\frac{e_G(1+\theta)^2}{E_B \Delta T \theta} = \frac{\alpha}{1 + \dfrac{\phi\theta}{(1+\phi)(1+\theta)} \alpha \Delta T} \tag{17}$$

The graphical representation of the normalized function is shown in Fig. 6 for several values of ϕ and θ. The absolute deviation of the response of the various bridge arrangements from the horizontal line (— — — in Fig. 6) is an indication of the absolute non-linearity of the bridge output as a function of temperature change.

It is quite interesting that a decrease in θ, which results in temperature resolution improvements, also advantageously affects the bridge linearity. It should be noted, however, that in Fig. 6 the maximum temperature excursion is 5 °C from balance. Thus the error in even the least favorable case (shown in Fig. 6d) results in only a 2.0% deviation from linearity in the course of a 1.0 °C temperature change. It is therefore apparent that for small thermal excursions, the bridge configuration is not important, at least in terms of the linearity. For large magnitude temperature monitoring, an unequal arm bridge would seem more appropriate if a high degree of linearity were required. A thorough treatment of these relationships is given by Gunn[4].

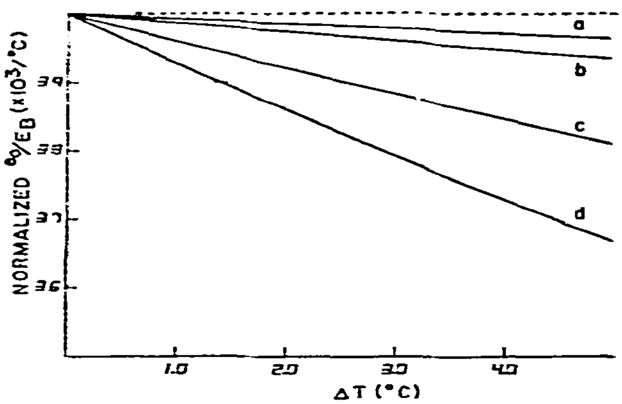

Fig. 6. The bridge sensitivity normalized with respect to the bridge driving voltage and the bridge resistances as a function of the solution temperature. For curve a, $R_1 = R_T$ and $R_2 = 10\,R_T$; curve b, $R_1 = 0.1\ R_T$ and $R_2 = 10\,R_T$; for curve c, $R_T = R_1 = R_2$; and for curve d, $R_1 = 0.1\,R_T$ and $R_2 = R_T$.

CONCLUSION

The generalized treatment given above can be applied to any resistance change transducer in a Wheatstone bridge configuration with the exception of the signal-to-noise considerations which apply only to transducers in which the major noise is a result of the *Johnson noise and dissipative phenomena*. The presentation of the data in the previous section illustrated the mathematical form of variations in the bridge configuration. From this, several important trends arise. First, the resistance element in the same arm as the transducer is important in the determination of the sensitivity and linearity. In the latter case, the impedance should be as large as possible for optimal results. Another possible benefit is an increase in the voltage applied to the bridge with an accompanying increase in the output voltage. It has been shown[5] that a voltage optimum exists corresponding to the temperature minimum alluded to in eqn (8) and is given by eqn (16). Hence, the smaller θ (the ratio of the resistance elements in the transducer side of the circuit) the larger the optimal bridge voltage. The sensitivity derived in eqn (7) and shown in Fig. 4 is normalized with respect to E_B, so that by increasing the applied voltage, an enhanced output can be achieved. This is a direct result of the considerations as discussed in the section on sensitivity. Secondly, the impedance of the arm parallel to the transducer has a lesser effect on the bridge system. Its influence on both the equivalent resistance and the temperature minima are somewhat secondary in importance. In spite of this, the impedance of this parallel resistance should be made as small as possible for optimum bridge signal-to-noise behavior.

The performance of several specific configurations is summarized in Table 1. These specific cases were chosen to illustrate the trends discussed above. Making R_1 greater than R_T and R_2 less than R_T constitutes a reversal of these trends and is characterized by decreases in the sensitivity, resolution, and linearity as compared to the equal arm case as illustrated in the last row. Another interesting facet

TABEL 1

SUMMARY OF THE PERFORMANCE OF SEVERAL SPECIFIC CONFIGURATIONS

Bridge configuration	R_{eq}	*Sensitivity* ($\times 10^6$)[a]	*Resolution*[b]	*Deviation*[c] (%)
$R_1 = R_T$; $R_2 = 100\ R_T$	$1.98\ R_T$	0.039	1.26	1.94
$R_1 = R_T$; $R_2 = 10\ R_T$	$1.82\ R_T$	0.33	1.22	1.78
$R_1 = R_2 = R_T$	R_T	1.0	1.00	0.990
$R_1 = 0.1\ R_T$; $R_2 = R_T$	$0.55\ R_T$	1.0	1.22	0.182
$R_1 = 0.01\ R_T$; $R_2 = R_T$	$0.50\ R_T$	1.0	1.26	0.020
$R_1 = 0.1\ R_T$; $R_2 = 10\ R_T$	R_T	0.33	1.50	0.330
$R_1 = 0.01\ R_T$; $R_2 = 100\ R_T$	R_T	0.039	1.57	0.040
$R_1 = 0$; $R_2 = \infty$	R_T	0	1.59	0
$R_1 = 10\ R_T$; $R_2 = 0.1\ R_T$	R_T	0.024	0.11	0.330

[a] Sensitivity was calculated from eqn (6) as the dimensionless quantity e_0/E_B, assuming a 100 μ°C temperature change. [b] Resolution here is defined as the ratio of ΔT_{min} as calculated from eqn (15) for the equal arm case to the ΔT_{min} for the given bridge configuration. [c] The deviation is expressed as the % difference between the linear case and the result of eqn (17) for a 1 °C temperature change.

is the linearity exhibited by all of the arrangements. Even in the least favorable case shown, a 1 °C change results in less than 2.5% error. For high sensitivity work, this deviation would be negligible for a millidegree temperature excursion. Of particular interest is the improvement in resolution seen in rows 6–8. Choosing $R_1 = 0.1\,R_T$, $R_2 = 10\,R_T$ and $R_3 = R_T$ (see Fig. 1) yields a bridge configuration which realistically approximates the theoretical limits of signal-to-noise behavior, linearity, and sensitivity. The improvement over the equal arm bridge arrangement, however, is at best modest. The ultimate choise of a bridge configuration, then, should be based on the application and a thorough investigation of the bridge behavior such as that presented here.

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