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Time integrals of input signal and output signal in linear measurement systems

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Abstract

The proportionality relation between the time integral of the input signal and that of the output signal is derived in a linear measurement system with lumped parameters and in a system with distributed parameters. The relation is equivalent to that of signals in a most simple case in which the input signal is a time-independent constant and the output signal is proportional to the magnitude of the input signal. In a system with distributed parameters, the relation is valid for signals averaged over surfaces through which they are transmitted in or out. The relation is also valid even when the linear system shows time delay, overshoot or damped vibration.

Keywords: Heat conduction calorimeter; Linear measurement system; Measurement theory; Peak area

1. Introduction

Measurement systems generally have two variable signals, an unknown input signal $x(t)$ and a known output signal $y(t)$, and they are functions of time t. The purpose of measurement is to get the input signal from the observed output. In the most simple and usual case, the input signal is a step function of time

$$x(t) = x_0 u(t) \tag{1}$$

and the observed output signal $y(t) = y_0(t)$ has a limiting value $y_0(\infty)$ as time approaches infinity

$$\lim_{t \to \infty} y_0(t) = y_0(\infty) \tag{2}$$

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Here x_0 is a time-independent value and $u(t)$ is the unit step function

$$\left.\begin{array}{ll} u(t)=0 & t<0 \\ u(t)=1 & t>0 \end{array}\right\} \tag{3}$$

In linear measurement systems, $y_0(\infty)$ is proportional to x_0 and we can get the unknown x_0 from the observed value of $y_0(\infty)$ if the proportionality constant has been determined from calibrating experiments.

When the input signal is time-transient, the output signal does not have a limiting value, $y_0(\infty) \neq 0$, and shows a time-transient behavior. Time-transient signals are found in chromatographic analysis, thermal analysis, etc. In these measurements, the output signal shows a peak on a recording chart and the time integral of the output signal, the "peak area", is used to determine sample amount [1] or energy change due to a phase change of the sample [2].

We show here that the proportionality relation between the time integral of the input signal and that of the output signal is valid in linear measurement systems and the relation is equivalent to that between a constant input signal x_0 and the limiting value of the output signal $y_0(\infty)$.

2. Linear measurement systems with lumped parameters

Let us consider a linear system with lumped parameters described by the following differential equation

$$a_m \frac{\mathrm{d}^m x}{\mathrm{d}t^m} + \ldots + a_1 \frac{\mathrm{d}x}{\mathrm{d}t} + a_0 x = b_n \frac{\mathrm{d}^n y}{\mathrm{d}t^n} + \ldots + b_1 \frac{\mathrm{d}y}{\mathrm{d}t} + b_0 y \quad a_0 \neq 0 \quad b_0 \neq 0 \tag{4}$$

where coefficients $a_m, \ldots, a_1, a_0$ and $b_n, \ldots, b_1, b_0$ are real constants, $x = x(t)$ is the input signal and $y = y(t)$ is the output signal.

When the input and output signals are given by Eqs. (1) and (2) respectively, Eq. (4) yields

$$a_0 x_0 = b_0 y_0(\infty) \tag{5}$$

or

$$y_0(\infty)/x_0 = a_0/b_0 \tag{6}$$

Relations (5) and (6) are the most simple and usual relations encountered in measurement problems.

When all signals are time transient, the signals and their derivatives are zero as $t \to \infty$, and the input and output signals are integral between $0 < t < t_{\mathrm{end}}$, then the time integral of both sides of Eq. (4) gives

$$a_0 \int_0^{t_{\mathrm{end}}} x(t)\,\mathrm{d}t = b_0 \int_0^{t_{\mathrm{end}}} y(t)\,\mathrm{d}t \tag{7}$$

Using Eq. (6), Eq. (7) becomes

$$\int_0^{t_{\mathrm{end}}} y(t)\,\mathrm{d}t \Big/ \int_0^{t_{\mathrm{end}}} x(t)\,\mathrm{d}t = a_0/b_0 = y_0(\infty)/x_0 \tag{8}$$

where $\int_0^{t_{\mathrm{end}}} x(t)\,\mathrm{d}t$ is equal to the total amount of sample in chromatographic analysis and to the total amount of energy change of sample in thermal analysis, and the term $\int_0^{t_{\mathrm{end}}} y(t)\,\mathrm{d}t$ is equal to the "peak area" on a recorded chart. Proportionality relation (7) or (8) is used in these analyses to determine the sample amount or the energy change of a sample. Eq. (8) shows that the proportionality relation between the integrals is equivalent to the relation between input signal x_0 and output signal $y_0(\infty)$ in the most usual case of measurement described by Eq. (5) or (6). Under some circumstances, the output signal shows time delay, overshoot or damped vibration. Even in such cases, relations (7) and (8) are valid for the linear system described by Eq. (4).

When the input signal is discontinuous, such as a pulse or step signal, a more sophisticated treatment is given to the linear model of a heat conduction calorimeter with lumped parameters [3].

3. Linear measurement system with distributed parameters

We cannot have a generalized model of a linear system with distributed parameters. However, a linear model of a heat conduction calorimeter with distributed parameters is treated theoretically and the results are given as follows [4]

$$\int_0^{t_{\mathrm{end}}} \bar{y}(t)\,\mathrm{d}t \Big/ \int_0^{t_{\mathrm{end}}} v(t)\,\mathrm{d}t = \bar{y}_0(\infty)/p_0 \tag{9}$$

where

$$\bar{y}(t) = \frac{1}{A}\iint_s y(\boldsymbol{r},t)\,\mathrm{d}A \tag{10}$$

$$y(\boldsymbol{r},t) = T(\boldsymbol{r},t) - T_{\mathrm{B}} \tag{11}$$

where $T(\boldsymbol{r},t)$ is the temperature on surface S of the reaction vessel, $\boldsymbol{r}$ is the position vector and T_{B} is the constant temperature of the surrounding thermal bath of the heat conduction calorimeter; $\bar{y}(t)$ is the surface average of $y(\boldsymbol{r},t)$ over surface S, and A is the surface area; $v(t)$ is the rate of enthalpy change due to physical or chemical change in the reaction vessel and

$$\Delta H = \int_0^{t_{\mathrm{end}}} v(t)\,\mathrm{d}t \tag{12}$$

is the total change of the enthalpy in heat conduction calorimetry; $\bar{y}_0(\infty)$ is the convergence value of $\bar{y}(t)$ when constant power p_0 is generated in the reaction vessel. Eq. (9) has the same form as Eq. (8) in the linear model with lumped parameters if the surface average value of $y(t)$ is chosen as the output signal in the heat conduction calorimeter model with distributed parameters. The detailed mathematical proof is given in Ref. [4].

Next, let us consider another model which consists of three concentric domains D_1, D_2 and D_3, and has boundary surfaces S_1 and S_2 as illustrated in Fig. 1; $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are unit normal vectors on surfaces S_1 and S_2 respectively, and they are directed toward the interior of D_2. The input signal $x(\boldsymbol{r}, t)$ and output signal $y(\boldsymbol{r}, t)$ depend on the position vector $\boldsymbol{r}$ and time t. The input signal in D_1 is transmitted to surface S_2 through surface S_1 and domain D_2. The purpose of measurement here is to find the time integral of the input signal on surface S_1, $X_{S_1}(r, t)$, from the observed output signal on surface S_2, $y_{S_2}(r, t)$.

The boundary-initial conditions on the model are assumed to be the following. The rate of change of total amount of $x(\boldsymbol{r}, t)$ in D_2 is assumed to be determined by the gradient of $x(\boldsymbol{r}, t)$ over surface S_1

$$S_1: \quad \iint_{S_1} \left(\frac{\partial x}{\partial n_1}\right)_{S_1} \mathrm{d}S = \frac{\partial}{\partial t} \iiint_{D_2} x(r, t)\, \mathrm{d}\tau \tag{13}$$

The behavior of the transmission of $x(\boldsymbol{r}, t)$ in D_2 is given by

$$D_2: \quad \nabla^2 x = \alpha \frac{\partial^2 x}{\partial t^2} + \beta \frac{\partial x}{\partial t} \tag{14}$$

$x(\boldsymbol{r}, t)$ cannot transmit into D_3 and is reflected by S_2.

$$S_2: \quad \left(\frac{\partial x}{\partial n_2}\right)_{S_2} = 0 \tag{15}$$

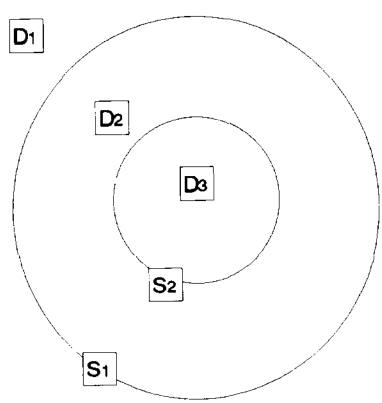

Fig. 1. Model of a linear measurement system with distributed parameters which consists of three concentric domains D_1, D_2, D_3, and boundary surfaces S_1 and S_2; n_1 and n_2 are unit normal vectors on surfaces S_1 and S_2, respectively. The input signal $x(\mathbf{r}, t)$ in domain D_1 is transmitted to surface S_2 through surface S_1 and domain D_2.

The output signal is $x(\boldsymbol{r}, t)$ over surface S_2

$$S_2: \quad y_{S_2}(\boldsymbol{r}, t) = x_{S_2}(\boldsymbol{r}, t) \quad \boldsymbol{r} \in S_2 \tag{16}$$

Here $\mathrm{d}S$ is the area element, $\mathrm{d}\tau$ is the volume element, and α and β are constant coefficients. The input and output signals are assumed to be

$$\lim_{t \to \infty} x(\boldsymbol{r}, t) = 0 \tag{17}$$

$$\lim_{t \to \infty} y(\boldsymbol{r}, t) = 0$$

and they are integrable between $0 < t < \infty$.

We define

$$I(\boldsymbol{r}) = \int_0^\infty x(\boldsymbol{r}, t)\,\mathrm{d}t \tag{18}$$

$$J(\boldsymbol{r}) = \int_0^\infty y(\boldsymbol{r}, t)\,\mathrm{d}t \tag{19}$$

Integrating Eqs. (13)–(16) from $t = 0$ to $t = \infty$, we have

$$S_1: \quad \iint_{S_1} \left(\frac{\partial I}{\partial n_1}\right)_{S_1} \mathrm{d}S = 0 \tag{20}$$

$$D_2: \quad \nabla^2 I = 0 \tag{21}$$

$$S_2: \quad \left(\frac{\partial I}{\partial n_2}\right)_{S_2} = 0 \tag{22}$$

$$S_2: \quad I(\boldsymbol{r}) = J(\boldsymbol{r}) \quad \boldsymbol{r} \in S_2 \tag{23}$$

Applying Green's theorem in the symmetrical form [5] to D_2 enclosed by surfaces S_1 and S_2, we have

$$\iiint_{D_2} (\phi \nabla^2 I - I \nabla^2 \phi)\,\mathrm{d}\tau = \iint_{S_1 + S_2} \left(-\phi \frac{\partial I}{\partial n} + I \frac{\partial \phi}{\partial n}\right) \mathrm{d}S \tag{24}$$

$\phi = \phi(r)$ is defined as

$$D_2: \quad \nabla^2 \phi = 0 \tag{25}$$

and ϕ and $\partial\phi/\partial n$ are uniform on surfaces S_1 and S_2. For example, $\phi = 1/r$ for spherical D_1, D_2, D_3 and $\phi = \ln r$ for cylindrical D_1, D_2, D_3. ϕ is defined as the potential function in the coordinate system.

From Eqs. (21) and (25), Eq. (24) becomes

$$0 = \iint_{S_1 + S_2} \left(-\phi \frac{\partial I}{\partial n} + I \frac{\partial \phi}{\partial n}\right) \mathrm{d}S = \iint_{S_1} + \iint_{S_2} \tag{26}$$

where

$$\iint_{S_1} = \iint_{S_1}\left(-\phi\frac{\partial I}{\partial n_1}+I\frac{\partial \phi}{\partial n_1}\right)\mathrm{d}S = \iint_{S_1} I\frac{\partial \phi}{\partial n_1}\mathrm{d}S = \left(\frac{\partial \phi}{\partial n_1}\right)_{S_1}\iint_{S_1} I\,\mathrm{d}S \tag{27}$$

$$\iint_{S_2} = \iint_{S_2}\left(-\phi\frac{\partial I}{\partial n_2}+I\frac{\partial \phi}{\partial n_2}\right)\mathrm{d}S = \iint_{S_2} I\frac{\partial \phi}{\partial n_2}\mathrm{d}S = \left(\frac{\partial \phi}{\partial n_2}\right)_{S_2}\iint_{S_2} I\,\mathrm{d}S \tag{28}$$

Here we have

$$\begin{aligned}\iint_{S_1} I\,\mathrm{d}S &= \iint_{S_1}\left[\int_0^\infty x(\boldsymbol{r},t)\,\mathrm{d}t\right]\mathrm{d}S\\ &= \int_0^\infty\left[\iint_{S_1} x(\boldsymbol{r},t)\,\mathrm{d}S\right]\mathrm{d}t = A_1\int_0^\infty \bar{x}_{S_1}(t)\,\mathrm{d}t\end{aligned} \tag{29}$$

where

$$\bar{x}_{S_1}(t) = \frac{1}{A_1}\iint_{S_1} x(\boldsymbol{r},t)\,\mathrm{d}S \tag{30}$$

is the surface average value of $x(\boldsymbol{r},t)$ on surface S_1, and A_1 is the area of surface S_1. Similarly we have

$$\iint_{S_2} I\,\mathrm{d}S = \iint_{S_2} J\,\mathrm{d}S = A_2\int_0^\infty \bar{y}_{S_2}(t)\,\mathrm{d}t \tag{31}$$

where

$$\bar{y}_{S_2}(t) = \frac{1}{A_2}\iint_{S_2} y(\boldsymbol{r},t)\,\mathrm{d}S \tag{32}$$

is the surface average value of $y(\boldsymbol{r},t)$ on surface S_2, and A_2 is the area of surface S_2. From Eqs. (27)–(29) and (31), Eq. (26) becomes

$$A_1\left(\frac{\partial \phi}{\partial n_1}\right)_{S_1}\int_0^\infty \bar{x}_{S1}(t)\,\mathrm{d}t + A_2\left(\frac{\partial \phi}{\partial n_2}\right)_{S_2}\int_0^\infty \bar{y}_{S_2}(t)\,\mathrm{d}t = 0 \tag{33}$$

and

$$\int_0^\infty \bar{y}_{S_2}(t)\mathrm{d}t\bigg/\int_0^\infty \bar{x}_{S_1}(t)\,\mathrm{d}t = -A_1\left(\frac{\partial \phi}{\partial n_1}\right)_{S_1}\bigg/A_2\left(\frac{\partial \phi}{\partial n_2}\right)_{S_2} \tag{34}$$

Eqs. (33) and (34) are the proportionality relations between the time integral of the input signal and that of the output signal in a linear measurement system with distributed parameters and correspond to proportionality relations (7) and (8) in a linear measurement system with lumped parameters. In a system with distributed parameters, the same proportionality relation is obtained if the signals are averaged over surfaces through which the signals are transmitted in or out.

Next we examine the output signal $y_0(\boldsymbol{r},t)$ when the input signal in D_1 is given by

$$x_0(\boldsymbol{r},t) = x_0(\boldsymbol{r})\,u(t) \tag{35}$$

The boundary conditions of $x_0(\boldsymbol{r}, t)$ are described by eqs. (13)–(16). When time t approaches infinity, we assume in D_2 and D_3

$$\lim_{t\to\infty} x(\boldsymbol{r}, t) = x(\boldsymbol{r}, \infty) \tag{36}$$

$$\lim_{t\to\infty} y_0(\boldsymbol{r}, t) = y_0(\boldsymbol{r}, \infty) \tag{37}$$

and the boundary conditions of $x_0(\boldsymbol{r}, \infty)$ become

$$S_1\colon \quad \iint_{S_1} \left(\frac{\partial x_0}{\partial n_1}\right)_{S_1} \mathrm{d}S = 0 \tag{38}$$

$$D_2\colon \quad \nabla^2 x_0(\boldsymbol{r}, \infty) = 0 \tag{39}$$

$$S_2\colon \quad \left(\frac{\partial x_0}{\partial n_2}\right)_{S_2} = 0 \tag{40}$$

$$S_2\colon \quad y_{0,S_2}(\boldsymbol{r}, \infty) = x_{0,S_2}(\boldsymbol{r}, \infty) \quad \boldsymbol{r} \in S_2 \tag{41}$$

Application of Green's theorem to D_2 enclosed by surface S_1 and S_2 gives

$$\iiint_{D_2} (\phi \nabla^2 x_0 - x_0 \nabla^2 \phi) \mathrm{d}\tau = \iint_{S_1+S_2} \left(-\phi \frac{\partial x_0}{\partial n} + x_0 \frac{\partial \phi}{\partial n}\right) \mathrm{d}S \tag{42}$$

where ϕ is defined as before. Considering Eqs. (38)–(42), we have

$$\left(\frac{\partial \phi}{\partial n_1}\right)_{S_1} \iint_{S_1} x_0 \, \mathrm{d}S + \left(\frac{\partial \phi}{\partial n_2}\right)_{S_2} \iint_{S_2} y_0 \, \mathrm{d}S = 0 \tag{43}$$

Then we have

$$A_1 \left(\frac{\partial \phi}{\partial n_1}\right)_{S_1} \bar{x}_{0,S_1} + A_2 \left(\frac{\partial \phi}{\partial n_2}\right)_{S_2} \bar{y}_{0,S_2}(\infty) = 0 \tag{44}$$

where

$$\bar{x}_{0,S_1} = \frac{1}{A_1} \iint_{S_1} x_0(\boldsymbol{r}) \, \mathrm{d}S \tag{45}$$

$$\bar{y}_{0,S_2}(\infty) = \frac{1}{A_2} \iint_{S_2} y_0(\boldsymbol{r}, \infty) \, \mathrm{d}S \tag{46}$$

From Eqs. (35) and (44), we have

$$\int_0^\infty \bar{y}_{S_2}(t) \, \mathrm{d}t \Big/ \int_0^\infty \bar{x}(t) \, \mathrm{d}t = \bar{y}_{0,S_2}(\infty)/\bar{x}_{0,S_1} = -A_1 \left(\frac{\partial \phi}{\partial n_1}\right)_{S_1} \Big/ A_2 \left(\frac{\partial \phi}{\partial n_2}\right)_{S_2} \tag{47}$$

Eq. (47) shows equivalence between the most simple relation (44) and the proportionality relation of the time integrals of the signals in a linear measurement system with distributed parameters.

It should be noticed that the relation is also valid when the system shows time delay, overshoot or damped vibration. When distribution of the input and output signals is not uniform in the domains, for example, the temperature distribution in the reaction vessel of a heat conduction calorimeter is not uniform, the relation is also valid between surface average signals or surface average temperatures over the reaction vessel.

References

[1] H. Purnel, Gas Chromatograpy, John Wiley & Sons, New York, 1962, p. 397.
[2] W.W. Wendlandt, Thermal Methods of Analysis, Wiley-Interscience, New York, 3rd edn., 1976, p. 266.
[3] S. Tanaka, Thermochim. Acta, 141 (1989) 181.
[4] S. Tanaka, Thermochim. Acta, 204 (1992) 261.
[5] M. Schwartz, S. Green and W.A. Rutledge, Vector Analysis, Harper and Row, New York, 1964, p. 309.