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Multiple scattering of thermal waves and non-steady effective thermal conductivity of composites with dense coated fibers

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ABSTRACT

In this study, the multiple scattering of thermal waves by dense coated fibers in composites is considered, and the analytical solution of the non-steady effective thermal conductivity of composites is presented. The Fourier heat conduction law is applied to analyze the propagation of thermal waves in the fibrous composite. The scattered and refracted fields in different material zones are expressed by using wave functions expanded method. The addition theorem for Bessel functions is used to accomplish the translation between different coordinate systems. The theory of quasicrystalline approximation and the conditional probability density function are employed to treat the multiple scattering of thermal waves from the dense fibers in matrix. The effective propagating wave number and non-steady effective thermal conductivity of composites are obtained. As an example, the effects of the material properties of the coating on the effective thermal conductivity of composites are graphically illustrated and analyzed. Analysis shows that the non-steady effective thermal conductivity under higher frequencies is quite different from the steady thermal conductivity. With the increase of the volume fraction of fibers, the effect of the thickness of the coating on the non-steady effective thermal conductivity increases greatly. In different region of wave frequency, the effects of the thickness and properties of the coating and the volume fraction of fibers on the effective thermal conductivity show great difference. Comparisons with the steady thermal conductivity obtained from other methods are also presented.

1. Introduction

The subject of the effective thermal conductivity of composites is one of the classical problems in heterogeneous media which has recently received renewed interest due to the increasing importance of high temperature systems, e.g., car manufacturing, dedicated space structures, etc. These materials usually undergo a complex thermal history. Determining the effective thermal properties of composites is crucial for a successful design and for the manufacture of materials. The development of micromechanical models for accurately predicting the effective thermal conductivity of multiphase composites has been the specific objective to study.

The methods used to measure the thermal conductivity are divided into two groups: the steady state and the non-steady state methods. In the first one, the sample is subjected to a constant heat flow. In the second group, a periodic or transient heat flow is established in the sample [1]. In the past, much attention has been focused on the problems of steady state.

The earliest models for the thermal behavior of composites assumed that the two components are both homogeneous, and are perfectly bounded across a sharp and distinct interface. The Maxwell solution [2] is the starting point to find the effective conductivity of two-phase material systems, but it is valid only for very low concentration of the dispersed phase. Subsequently, many structural models, e.g., Parallel, Maxwell-Eucken [3], and Effective Medium Theory models [4], were proposed. Recently, Samantray et al. applied the unit-cell approach to study the effective thermal conductivity of two-phase materials [5]. The idea of the Generalized Self-Consistent Model was also developed by Hashin [6] to determine the effective thermal conductivity of the two-phase materials.

Recently, coating inclusions have been introduced in the design of composite to enhance the thermal properties of composites. In the modeling, the coating was also introduced for other reasons: first, during the manufacturing process, a chemical reaction between inclusion and matrix can create a third phase: the coating. Second, due to a mismatch between the two phases, the perfect interface assumption is not valid. Thus, the coating contributes to the character of the non-perfect interface. The dramatic effect of interfacial characteristics on thermal conductivities and thermal diffusivities has been experimentally demonstrated by Hasselman and co-workers in particle [7] and fiber reinforced composites [8].

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Based on an equivalent inclusion concept, Hasselman and Johnson extended Maxwell's theory to the systems of spherical and cylindrical inclusions with contact resistance [9]. Benveniste and his co-workers have proposed several analytical models to predict the effective thermal conductivity of composite materials which include the important effects of a thermal contact resistance between the fillers and matrix [10], and the coated cylindrically orthotropic fibers with a prescribed orientation distribution [11]. Lu and Song [12,13] investigated the coated or debonded inclusion and developed a more general model to predict the effective thermal conductivity of composites.

Due to the complexity of non-steady loading, there are few calculations on the effective thermal conductivity in these materials under modulated conditions. Recently, Monde and Mitsutake [14] proposed a method for determining the thermal conductivity of solids by using an analytical inverse solution for unsteady heat conduction. By using modulated photothermal techniques, Salazar et al. [1] studied the effective thermal conductivity of composites made of a matrix filled with aligned circular cylinders of a different material. Most recently, Fang and Hu investigated the distribution of dynamic effective thermal properties along the gradation direction of functionally graded materials by using Fourier heat conduction law [15] and non-Fourier heat conduction law [16].

Nevertheless, little attention has been paid to the non-steady effective thermal conductivity of composites with coated fibers. With the wide application of composites in aerospace, automotive industries, and other high temperature situations, the study on the non-steady effective thermal conductivity of composites with coating fibers plays very important role in the designing and manufacture of materials. Recently, Fang et al. [17] have applied thermal wave method to study the effects of coating on the non-steady effective thermal conductivity of materials. The theory of Waterman and Truell [18] was applied to obtain the non-steady effective thermal conductivity of composites. However, this theory neglects the multiple scattering effects among the fibers, and is only suited to dilute concentrations of fibers.

The main objective of this paper is to extend the work of Fang et al. [17] to the multiple scattering of thermal waves by the dense coated fibers in composites, and the effects of the coating on the non-steady effective thermal conductivity of composites are considered. Thermal wave is often applied with Fourier conduction law. Fourier law underlies "parabolic thermal wave" associated with a nonlinear dependence of thermal conductivity on temperature, and the "thermal wave method" of measuring thermal properties. The same-size fibers of identical properties with same-thickness coating are assumed to be randomly distributed with a statistically uniform distribution. The temperature fields in different regions of composites are expressed by using the wave function expansion method, and the expanded mode coefficients are determined by satisfying the boundary conditions of the coating. Considering that the positions of the fibers are random, the temperature fields in composites are averaged. The averaged equations are solved by using Lax's quasicrystalline approximation [19] to obtain the effective propagating wave number and the non-steady effective thermal conductivity of composites. The variation of effective thermal conductivity under different parameters is graphically illustrated and discussed.

2. Formulation of the problem

Consider a composite material containing a large number N of coated fibers embedded in an infinite matrix. The long, parallel fibers with identical properties are randomly distributed in the matrix. The inner radius of the fibers is a_0, and the outer radius is a_m. The geometry is depicted in Fig. 1, where (x, y, z) is the Cartesian coordinate system with origin at the center of the fiber, and (r, θ, z) is the corresponding cylindrical coordinate system. The fibers are labeled by suffixes $i = 1, 2, \ldots, N$. The position vector of the center of the ith fiber is denoted by $\mathbf{r}_i$, as depicted in Fig. 1. Let λ, c, ρ be the thermal conductivity, specific heat capacity at constant pressure and mass density of the matrix, and λ_f, c_f, ρ_f, those of the fibers. It is assumed that the thickness of the coating is h with material properties λ_c, c_c, ρ_c. Let the boundary of the ith fiber and the coating be denoted by C_{if}, and that of the coating and the matrix by C_{im}.

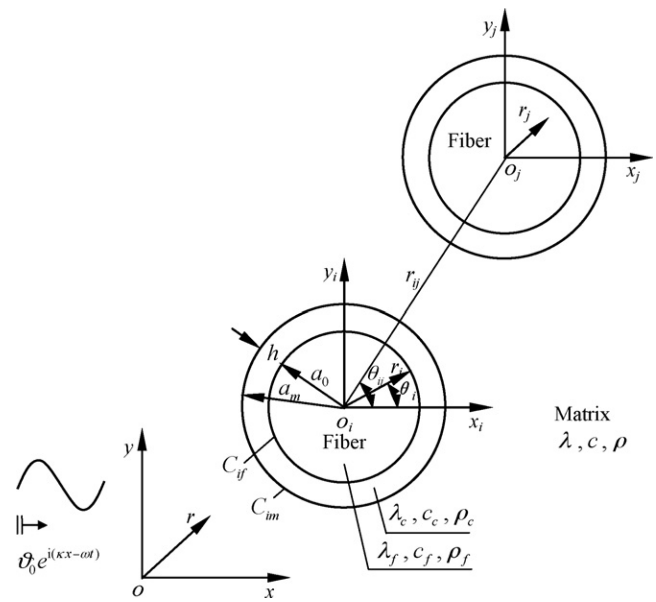

Fig. 1. Coated cylindrical fibers and coordinate systems in composites.

3. Conditional probability density function for fiber distribution

To analyze the correlation of the temperature field among the randomly distributed fibers, the conditional probability density function for fiber distribution must be specified. The probability density of the random variable $(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)$ is denoted by $p(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)$. Then, due to the indistinguishability of the cylindrical fibers, it is symmetric in its arguments, and we have

$$\begin{aligned} &p(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N) \\ &\quad = p(r_i)p(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, ', \ldots, \boldsymbol{r}_N | \boldsymbol{r}_i) \\ &\quad = p(\boldsymbol{r}_i)p(\boldsymbol{r}_j|\boldsymbol{r}_i)p(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, ', \ldots, ' \ldots, \boldsymbol{r}_N|\boldsymbol{r}_j|\boldsymbol{r}_i), p(\boldsymbol{r}_i) \\ &\quad = p(\boldsymbol{r}_1),\ p(\boldsymbol{r}_j|\boldsymbol{r}_i) = p(\boldsymbol{r}_2|\boldsymbol{r}_1),\ i \neq j, \end{aligned} \tag{1}$$

where the probabilities with the vertical bar in their argument denote the customary conditional probabilities. A prime in the first part of Eq. (1) means $\mathbf{r}_i$ is absent, while two primes in the second part of Eq. (1) mean both $\mathbf{r}_i$ and $\mathbf{r}_j$ are absent. For a uniform composite material, the position of a single cylindrical fiber is equally probable within a large region V of the volume of composites, and so its distribution is uniform with density, i.e.,

$$p(\boldsymbol{r}_i) = 1/V, \quad \text{if} \quad \boldsymbol{r}_i \in V; \quad p(\boldsymbol{r}_i) = 0, \quad \text{if} \quad \boldsymbol{r}_i \notin V. \tag{2}$$

If the center of the ith fiber, well within V, is held fixed, the distribution of the cylindrical fibers around it will be cylindrically symmetrical. Thus, the conditional probability density function $p(\mathbf{r}_j|\mathbf{r}_i)$ is usually expressed in term of the pair correlation function

$g(r_{ij})$, i.e.,

$$p(\mathbf{r}_j|\mathbf{r}_i)=\frac{1}{V}[1-g(\mathbf{r}_{ij})], \quad \text{if} \quad \mathbf{r}_j \in V; \quad p(\mathbf{r}_j|\mathbf{r}_i)=0, \quad \text{if} \quad \mathbf{r}_j \notin V. \quad (3)$$

where the pair correlation function $g(r_{ij})$ is a decreasing function of r_{ij}. The normalization condition of $p(\mathbf{r}_j|\mathbf{r}_i)$ gives, in the limit as $V\to\infty$

$$\lim_{R\to\infty}\frac{1}{R^2}\int_0^R g(r_{ij})r_{ij}dr_{ij}=0. \quad (4)$$

Due to the impossibility of interpenetration of the cylindrical fibers and their independence when they are infinitely apart, function $g(r_{ij})$ satisfies the following conditions

$$g(r_{ij})=1 \quad \text{if} \quad r_{ij}<2a_m; \quad \lim_{r_{ij}\to\infty} g(r_{ij})=0. \quad (5)$$

The first of these conditions holds for the non-overlapping sets of cylindrical fibers. The second condition is correct if the correlation in spatial positions of the fibers disappears.

A function satisfying these conditions is expressed as

$$g(r_{ij})=\begin{cases}1, & r_{ij}<2a_m\\ \delta\exp(-r_{ij}/r_0), & r_{ij}>2a_m\end{cases}, \quad (6)$$

where δ $[0<\delta\leq\exp(2a_m/r_0)]$ is the coefficient, and $r_0>0$ is the correlation length.

4. Multiple scattering of thermal waves by fibers and the wave fields in different zones

Based on the Fourier heat conduction law, the heat conduction equation in composites, in the absence of heat sources, is described as

$$\nabla^2 T(r,t)=\frac{1}{D}\frac{\partial T}{\partial t}, \quad (7)$$

where $\nabla^2=\partial^2/\partial x^2+\partial^2/\partial y^2$ represents the two-dimensional Laplace operator, T is the temperature in composite materials, and D is the thermal diffusivity with

$$D=\frac{\lambda}{\rho c}. \quad (8)$$

The solution of periodic steady state is investigated. Suppose that

$$T=T_0+Re[\vartheta\exp(-i\omega t)], \quad (9)$$

where T_0 is the average temperature, and ω is the incident frequency of thermal waves.

Substituting Eq. (9) into Eq. (7), the following equation can be obtained

$$\nabla^2\vartheta+\kappa^2\vartheta=0, \quad (10)$$

where κ is the wave number of complex variables in materials, and

$$\kappa=(1+i)k, \quad (11)$$

with $k=\sqrt{\omega/2D}$ being the incident wave number.

It is assumed that the thermal waves propagate in the positive x direction. Thus, the incident thermal waves in the matrix are expressed as

$$\vartheta^{(i)}=\vartheta_0 e^{i(\kappa x-\omega t)}, \quad (12)$$

where the superscript (i) stands for the incident waves in the matrix, and ϑ_0 is the temperature amplitude of incident thermal waves in the matrix. It should be noted that all wave fields have the same time variation $e^{-i\omega t}$, which is omitted in all subsequent representations for notational convenience.

When the thermal waves propagate in the fibrous composite material, the waves are scattered by the fibers, and the scattered waves of the fibers are expanded in a series of outgoing Hankel functions. The scattered field around the ith fiber in the matrix is expressed in the form

$$\vartheta_i^{(s)}=\sum_{n=-\infty}^{\infty}A_{in}H_n^{(1)}(\kappa r_i)e^{in\theta_i}, \quad (13)$$

where the superscript(s) stands for the scattered waves, $H_n^{(1)}(\cdot)$ is the nth Hankel function of the first kind, and A_{in} are the mode coefficients that account for the distortion of the scattered cylindrical waves by the ith fiber.

Thus, the total scattered field $\vartheta^{(s)}$ in composites is taken to be a superposition of the scattered fields of every fiber, and is expressed as

$$\vartheta^{(s)}=\sum_{i=1}^{N}\vartheta_i^{(s)}. \quad (14)$$

The total temperature in the matrix $\vartheta^{(m)}$ should be produced by the superposition of the incident field and the total scattered field, i.e.,

$$\vartheta^{(m)}=\vartheta^{(i)}+\vartheta^{(s)}. \quad (15)$$

The refracted waves inside the ith fiber are standing waves, and can be expressed as

$$\vartheta_i^{(r)}=\sum_{n=-\infty}^{\infty}B_{in}J_n(\kappa_f r_i)e^{in\theta_i}, \quad (16)$$

where the superscript (r) stands for the refracted waves, $J_n(\cdot)$ is the nth Bessel function of the first kind, and B_{in} are the mode coefficients of refracted waves.

The temperature $\vartheta_i^{(c)}$ in the coating of the ith fiber may be described by the sum of the two components (outgoing and ingoing), and is expressed in the following form [20,21]

$$\vartheta_i^{(c)}=\left[\sum_{n=-\infty}^{\infty}E_{in}H_n^{(1)}(\kappa_c r_i)e^{in\theta_i}+\sum_{n=-\infty}^{\infty}F_{in}H_n^{(2)}(\kappa_c r_i)e^{in\theta_i}\right], \quad (17)$$

where $H_n^{(2)}$ denoting the ingoing waves is the nth Hankel function of the second kind, and E_{in} and F_{in} are the mode coefficients in the coating.

The wave numbers κ_f in the ith cylindrical fiber and κ_c in the coating are given by

$$\kappa_f=(1+i)\sqrt{\frac{\omega}{2D_f}}, \quad (18)$$

$$\kappa_c=(1+i)\sqrt{\frac{\omega}{2D_c}}, \quad (19)$$

where $D_f=\lambda_f/\rho_f c_f$ and $D_c=\lambda_c/\rho_c c_c$.

5. Boundary conditions and solution of mode coefficients

The boundary conditions on C_{im} and C_{if} of the ith fiber are given by

$$\vartheta_i^{(c)}=\vartheta^{(m)}, \; q_{ri}^{(c)}=q_{ri}^{(m)} \; \text{for} \quad r_i=a_m, \quad (20)$$

$$\vartheta_i^{(r)}=\vartheta_i^{(c)}, \; q_{ri}^{(r)}=q_{ri}^{(c)} \; \text{for} \quad r_i=a_0, \quad (21)$$

where q_{ri} is the heat flow density around the ith fiber in the radial direction, and $q_{ri}=-\lambda(\partial\vartheta)/(\partial r_i)$.

The continuous boundary condition of temperature on C_{in} gives

$$\sum_{n=-\infty}^{\infty}[E_{in}H_n^{(1)}(\kappa_c a_m)e^{in\theta_i}+F_{in}H_n^{(2)}(\kappa_c a_m)e^{in\theta_i}$$

$$=\vartheta_0 e^{i\kappa x}+\sum_{j=1}^{N}\sum_{n=-\infty}^{\infty}A_{jn}H_n^{(1)}(\kappa r_j)e^{in\theta_j}. \tag{22}$$

Multiplying by $e^{-is\theta_i}$ and integrating from 0 to 2π on both sides of Eq. (22), the following equation can be obtained

$$H_s^{(1)}(\kappa_c a_m)E_{is}+H_s^{(2)}(\kappa_c a_m)F_{is}=\vartheta_0 i^s J_s(\kappa a_m)e^{i\kappa r_{io}\cos\theta_{io}}+\sum_{j=1}^{N}\sum_{n=-\infty}^{\infty}A_{jn}K_{jins}, \tag{23}$$

where

$$K_{jins}=\begin{cases}\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}[H_n^{(1)}(\kappa a_m)e^{in\theta_j}e^{-is\theta_i}d\theta_i & j\neq i\\ H_s^{(1)}(\kappa a_m)\delta_{ns} & j=i\end{cases}, \tag{24}$$

in which δ_{ns} is Kronecker delta function. Using the addition theorem for Hankel functions, one can obtain

$$K_{jins}=\frac{1}{2\pi}\int_0^{2\pi}\left\{e^{in\theta_{ij}}(-1)^n\sum_{\nu=-\infty}^{\infty}(-1)^{\nu}J_{\nu}(\kappa a_m)H_{\nu-n}^{(1)}(\kappa r_{ij})e^{i\nu(\theta_i-\theta_{ij})}\right\}\times e^{-is\theta_i}d\theta_i=J_s(\kappa a_m)H_{n-s}^{(1)}(\kappa r_{ij})e^{i(n-s)\theta_{ij}},\qquad (j\neq i), \tag{25}$$

where (r_{ij},θ_{ij}) are the coordinates of o_j referred to o_i as origin.

Then, Eq. (23) is rewritten as

$$H_s^{(1)}(\kappa_c a_m)E_{is}+H_s^{(2)}(\kappa_c a_m)F_{is}=A_{is}H_s^{(1)}(\kappa a_m)+J_s(\kappa a_m)\times\left[\vartheta_0 i^s e^{i\kappa r_{io}\cos\theta_{io}}+\sum_{j=1,j\neq i}^{N}\sum_{n=-\infty}^{\infty}A_{j,n+s}H_n^{(1)}(\kappa r_{ij})e^{in\theta_{ij}}\right]. \tag{26}$$

Similarly, the continuous boundary conditions of temperature on C_{if} give

$$B_sJ_s(\kappa_f a_0)=E_sH_s^{(1)}(\kappa_c a_0)+F_sH_s^{(2)}(\kappa_c a_0). \tag{27}$$

According to the continuous boundary conditions of heat flux density on C_{im} and C_{if}, one can obtain

$$\lambda_c\left[E_s\frac{\partial}{\partial a_m}H_s^{(1)}(\kappa_c a_m)+F_s\frac{\partial}{\partial a_m}H_s^{(2)}(\kappa_c a_m)\right]=\lambda\left\{\frac{\partial}{\partial a_m}H_s^{(1)}(\kappa a_m)A_{is}+\frac{\partial}{\partial a_m}J_s(\kappa a_m)\left[\vartheta_0 i^s e^{i\kappa r_{io}\cos\theta_{io}}+\sum_{j=1,j\neq i}^{N}\sum_{n=-\infty}^{\infty}A_{j,n+s}H_n^{(1)}(\kappa r_{ij})e^{in\theta_{ij}}\right]\right\}, \tag{28}$$

$$\lambda_f\left[B_s\frac{\partial}{\partial a_0}J_s(\kappa_f a_0)\right]=\lambda_c\left[E_s\frac{\partial}{\partial a_0}H_s^{(1)}(\kappa_c a_0)+F_s\frac{\partial}{\partial a_0}H_s^{(2)}(\kappa_c a_0)\right]. \tag{29}$$

According to Eqs. (17–22), the expanded coefficient of scattered waves A_s can be expressed as

$$A_{is}=A'_sT_i^s, \tag{30}$$

where

$$A'_s=X_s\frac{H_s^{(1)}(\kappa_c a_m)}{H_s^{(1)}(\kappa a_m)}+Y_s\frac{H_s^{(2)}(\kappa_c a_m)}{H_s^{(1)}(\kappa a_m)}-\frac{J_s^{(1)}(\kappa a_m)}{H_s^{(1)}(\kappa a_m)}, \tag{31}$$

$$T_i^s=\vartheta_0 i^s e^{i\kappa r_{io}\cos\theta_{io}}+\sum_{j=1,j\neq i}^{N}\sum_{n=-\infty}^{\infty}A'_{n+s}T_j^{n+s}H_n^{(1)}(\kappa r_{ij})e^{in\theta_{ij}}. \tag{32}$$

in which X_s and Y_s are shown in Appendix A. It should be noted that T_i^s is the temperature field in any point of composites.

When either o_i or o_i and o_j together are held fixed, to determine the mean temperature field in composites $\langle T_i^s\rangle_i$, the conditional expectation of the fiber distribution is used. From Eq. (32), one can obtain

$$\langle T_i^s\rangle_i=\vartheta_0 i^s e^{i\kappa r_{io}\cos\theta_{io}}+n_0\left(1-\frac{1}{N}\right)\sum_{n=-\infty}^{\infty}A'_{n+s}\times\int_{r_{io},r_{jo}\in S}[1-g(r_{ij})\langle T_j^{n+s}\rangle_{ij}H_n^{(1)}(\kappa r_{ij})e^{in\theta_{ij}}d\tau_jH_n^{(1)}(\kappa r_{ij})e^{in\theta_{ij}}d\tau \tag{33}$$

where $n_0=N/S=c/(\pi a_0^2)$ is the number of cylindrical fibers per unit area, c is the volume fraction of fibers in the matrix, and $g(r_{ij})$ shown in Eq. (6) is the pair correlation function of fiber distribution. Eq. (33) involves the conditional expectation with two cylindrical fibers held fixed. If we take the conditional expectation of Eq. (33) with two cylindrical inclusions held fixed, the resulting equation will contain the conditional expectation with three cylindrical inclusions held fixed, and so on. To eliminate this hierarchy, Lax's quasicrystalline approximation theory [19] is applied. In Lax's quasicrystalline approximation theory [19], the two-inclusion correlation function is involved, and the mean temperature field is expressed as

$$\langle T_i^s\rangle_{ij}=\langle T_i^s\rangle_i,\quad i\neq j, \tag{34}$$

According to the extinction theorem, when S and N become infinitely large, the incident wave is extinguished on entering the composite, so that the corresponding term in Eq. (33) can be dropped. Thus, this equation is simplified to

$$\langle T_i^n\rangle_i=n_0\sum_{s=-\infty}^{\infty}A'_{n+s}\int_{|r_{io}-r_{jo}|>2a_n}[1-g(r_{ij})]\langle T_j^{n+s}\rangle_jH_s^{(1)}(\kappa r_{ij})e^{is\theta_{ij}}d\tau_j. \tag{35}$$

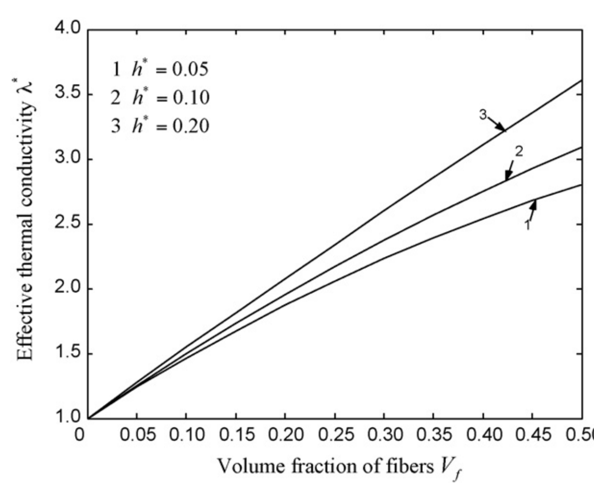

Fig. 2. Non-steady effective thermal conductivity as a function of volume fraction of fibers ($k^*=1.0$, $\lambda_f^*=4.0$, $c_f^*=\rho_f^*=2.0$, $\lambda_c^*=2.5$, $c_c^*=\rho_c^*=1.5$).

Assuming the existence of an average plane wave, the solution of Eq. (35) is proposed as

$$\left\langle T_i^n \right\rangle_i = i^n T_n e^{iKr_{io}\cos\theta_{io}}, \tag{36}$$

where T_n is a constant, and K is the wave number of the effective thermal waves. Making use of the Green's theorem and wave function expansion method, the following can be obtained

$$e^{iKr_{jo}\cos\theta_{jo}} = e^{iKr_{io}\cos\theta_{io}} \sum_{m=-\infty}^{\infty} i^{-m} J_m(Kr_{ji}) e^{-im\theta_{ji}}. \tag{37}$$

The first integral appearing in Eq. (35) can be simplified as

$$\begin{aligned} &\int_{|r_{io}-r_{jo}|>2a_n} e^{iKx_{jo}} H_s^{(1)}(\kappa r_{ij}) e^{is\theta_{ij}} d\tau_j \\ &= \frac{1}{\kappa^2-K^2} \int_{|r_{io}-r_{jo}|>2a_n} [(\nabla^2 e^{iKx_{jo}}) H_s^{(1)}(\kappa r_{ij}) e^{is\theta_{ij}} \\ &\quad - e^{iKx_{jo}} \nabla^2 H_s^{(1)}(\kappa r_{ij}) e^{is\theta_{ij}}] d\tau_j = e^{iKx_{io}} \frac{2\pi a_m i^{-s}}{\kappa^2-K^2} \\ &\quad \times \left[J_s(2Ka_m) \frac{\partial}{\partial a_m} H_s^{(1)}(2\kappa a_m) - H_s^{(1)}(2\kappa a_m) \frac{\partial}{\partial a_m} J_s(2Ka_m) \right]. \end{aligned} \tag{38}$$

In the same way, the second integral in Eq. (35) can be also simplified. Then, Eq. (35) reduces to the system of equations

$$\begin{aligned} T_n = 2\pi n_0 \sum_{s=-\infty}^{\infty} A'_{n+s} T_{n+s} \Bigg\{ &\frac{a_m}{\kappa^2-K^2} \left[J_s(2Ka_m) \frac{\partial}{\partial a_m} H_s^{(1)}(2\kappa a_m) \right. \\ &\left. - H_s^{(1)}(2\kappa a_m) \frac{\partial}{\partial a_m} J_s(2Ka_m) \right] - \int_{2a_n}^{\infty} g(r_{ij}) J_s(Kr_{ij}) H_s^{(1)}(\kappa r_{ij}) r_{ij} dr_{ij} \Bigg\}. \end{aligned} \tag{39}$$

The set of Eq. (39) consists of an infinite number of homogeneous linear equations determining the coefficients T_n. For a nontrivial solution of T_n, the determinant must vanish, and this leads to the equation for the effective wave number K. It is noted that the effects of multiple scattering on the coherent waves are of great practical importance for the volume fraction of fibers ($V_f = 0.1$–0.5). At very low volume fraction of fibers ($V_f < 0.1$), the multiple scattering effect can be neglected and each fiber can be treated as independent.

The expressions of $J_s(x)$ and $H_s^{(1)}(x)$ are written as

$$J_s(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(s+k)!} \left(\frac{x}{2}\right)^{s+k}, \tag{40}$$

$$H_s^{(1)}(x) = \frac{i}{\sin(s\pi)} [J_s(x) e^{-is\pi} - J_{-s}(x)]. \tag{41}$$

When r_0 is sufficiently small compared to the wavelength, by using Eqs. (40–41) and retaining the lowest order terms for $h/a_0 = 0$, one can obtain

$$T_n \simeq -\pi c \sum_{s=-\infty}^{\infty} A''_{n+s} T_{n+s} \left[\frac{1}{\pi} \left(\frac{K}{\kappa}\right)^s \frac{1}{1-(K/\kappa)^2} + \frac{i}{2} \kappa^2 P_s \right], \tag{42}$$

where

$$A''_0 = \frac{\rho_f}{\rho} - 1, \tag{43}$$

$$A''_{\pm 1} = \frac{\lambda - \lambda_f}{\lambda + \lambda_f}, \tag{44}$$

$$A''_q = 0, \quad (|q| \geq 2), \tag{45}$$

$$P_0 \simeq \frac{2i}{\pi} V r_0^2 \left(1 + \log \frac{\kappa r_0}{2} - \frac{i\pi}{2} \right), \tag{46}$$

$$P_{\pm 1} \simeq \frac{i}{\pi} V r_0^2 \frac{K}{\kappa}, \tag{47}$$

$$P_{\pm 2} \simeq \frac{i}{2\pi} V r_0^2 \left(\frac{K}{\kappa}\right)^2, \tag{48}$$

$$P_\nu = 0, \quad |\nu| \geq 3. \tag{49}$$

6. Non-steady effective properties of the fiber reinforced composites

According to Eq. (11), the non-steady effective thermal conductivity λ^{eff} can be easily obtained from the effective propagating wave number as follows:

$$\lambda^{eff} = \frac{\rho^{eff} c^{eff} \lambda}{\rho c} [Re(k/K)]^2, \tag{50}$$

where $\mathrm{Re}(\cdot)$ denotes the real part, and ρ^{eff} and c^{eff} are the effective mass density and effective heat capacity of composites. From Ref. [1], it is known that ρ^{eff} and c^{eff} always follow the mixture rule, and $\rho^{eff} c^{eff}$ is given by

$$\rho^{eff} c^{eff} = \rho c \left\{ 1 - V_f \left(1 + \frac{h}{a_0}\right)^2 \right\} + \rho_f c_f V_f + \frac{hV_f}{a_0} \rho_c c_c \left(2 + \frac{h}{a_0}\right). \tag{51}$$

7. Numerical examples and discussion

To examine the effect of material properties on the non-steady effective thermal conductivity of composites, for a given value of k, A'_s is computed. Next, the complex coefficient matrix M corresponding to T_n in Eq. (39) is formed. The complex determinant of the coefficient matrix is computed using standard Gauss elimination techniques. For a given value of k, the root of the equation $\mathrm{del}(M) = 0$ is searched in the complex plane using Muller's method. Good initial guesses are provided by Eq. (42) at low values of k and these can be used systematically to obtain quick convergence of roots at increasingly higher values of k.

In the following analysis, it is convenient to make the variables dimensionless. To accomplish this step, a representative length scale a_0, where a_0 is the radius of fibers, is introduced. The following dimensionless variables and quantities have been chosen for computation: the incident wave number $k^* = ka_0 = 0.1$–2.0, $h^* = h/a_0 = 0.05$–0.20, $\lambda_f^* = \lambda_f/\lambda = 2.0 - 8.0$, $c_f^* = c_f/c = 2.0 - 4.0$, $\rho_f^* = \rho_f/\rho = 2.0 - 4.0$, $\lambda_c^* = \lambda_c/\lambda = 0.5 - 8.0$, $c_c^* = c_c/c = 1.0 - 4.0$, and $\rho_c^* = \rho_c/\rho = 1.0 - 4.0$. The dimensionless effective thermal conductivity is $\lambda^* = \lambda^{eff}/\lambda$.

The non-steady effective thermal conductivity of composites as a function of volume fraction of fibers with parameters: $k^* = 1.0$, $\lambda_f^* = 4.0$, $c_f^* = \rho_f^* = 2.0$, $\lambda_c^* = 2.5$, $c_c^* = \rho_c^* = 1.5$ is presented in Fig. 2. Because the thermal conductivity of the fiber is greater than that of the matrix, the non-steady effective thermal conductivity increases with the volume fraction of fibers. It can be seen that the non-steady effective thermal conductivity increases with the increase of the thickness of the coating. When the volume fraction of the fiber is small, the effect of the thickness of the coating is little. When the volume fraction of the fiber is great, the effect of the thickness of the coating becomes more distinct. When the thickness of the coating is great, the non-steady effective thermal conductivity of composites increases greatly with the increase of the volume fraction of fibers.

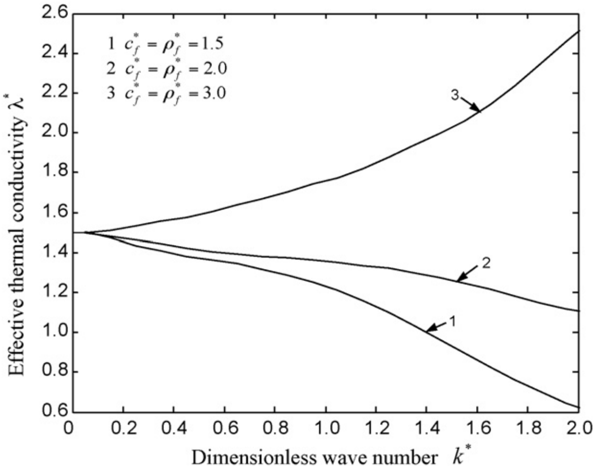

Fig. 3. Non-steady effective thermal conductivity as a function of dimensionless wave number ($h^* = 0.1,\ V_f = 0.1,\ \lambda_f^* = 4.0,\ \lambda_c^* = 2.5,\ c_c^* = \rho_c^* = 1.5$).

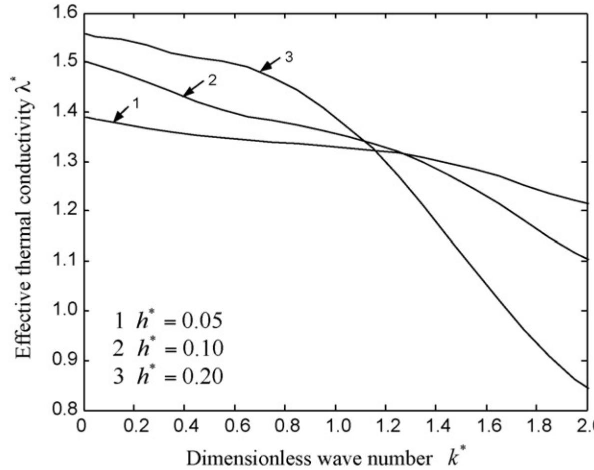

Fig. 5. Non-steady effective thermal conductivity as a function of dimensionless wave number ($V_f = 0.1,\ \lambda_f^* = 4.0,\ c_f^* = \rho_f^* = 2.0,\ \lambda_c^* = 2.5,\ c_c^* = \rho_c^* = 1.5$).

Fig. 3 illustrates the non-steady effective thermal conductivity of composites as a function of the incident wave number with parameters: $h^* = 0.1, V_f = 0.1, \lambda_f^* = 4.0, \lambda_c^* = 2.5, c_c^* = \rho_c^* = 1.5$. As expected, the steady effective thermal conductivity is not dependent on the specific heat and density of the two phases. So, in the region of very low frequency, the variation the specific heat and density of the two phases nearly expresses no effect on the effective thermal conductivity. With the increase of the incident wave number, the effect of the specific heat and density of the two phases on the non-steady effective thermal conductivity increases greatly. The non-steady effective thermal conductivity increases with the specific heat and density ratio of the fibers and matrix.

Fig. 4 illustrates the non-steady effective thermal conductivity of composites as a function of the incident wave number with parameters: $h^* = 0.1,\ V_f = 0.4,\ \lambda_f^* = 4.0,\ \lambda_c^* = 2.5,\ c_c^* = \rho_c^* = 1.5$. Comparing the results with those in Fig. 3, it is clear that with the increase of the volume of fibers, the effects of the values of c_f^* and ρ_f^* on the non-steady effective thermal conductivity of composites increase greatly. The effect of the volume fraction of fibers on the non-steady effective thermal conductivity increases with the increase of the wave frequency. When the volume fraction of fibers is great, in the region of low frequencies, the specific heat and density of the two phases also show great effect on the non-steady effective thermal conductivity of composites.

Fig. 5 illustrates the non-steady effective thermal conductivity of composites as a function of dimensionless wave number with parameters: $V_f = 0.1,\ \lambda_f^* = 4.0,\ c_f^* = \rho_f^* = 2.0,\ \lambda_c^* = 2.5,\ c_c^* = \rho_c^* = 1.5$. It can be seen that in the region of low frequency the non-steady effective thermal conductivity increases with the increase of the value of h^*. However, in the region of high frequency, the non-steady effective thermal conductivity decreases with the increase of the value of h^*.

Fig. 6 illustrates the non-steady effective thermal conductivity of composites as a function of dimensionless wave number with parameters: $V_f = 0.4,\ \lambda_f^* = 4.0,\ c_f^* = \rho_f^* = 2.0,\ \lambda_c^* = 2.5,\ c_c^* = \rho_c^* = 1.5$. Comparing the results with those in Fig. 5, it can be seen that with the increase of the volume fraction of fibers, in the region of low and high frequencies, the effect of the value of h^* on the non-steady effective thermal conductivity expresses little variation. However, in the region of intermediate frequencies, the effect of the value of h^* on the non-steady effective thermal conductivity shows great difference. The effect of the value of λ_c^* on the non-steady effective thermal conductivity increases with the volume fraction of fibers.

Fig. 7 shows the non-steady effective thermal conductivity of composites as a function of dimensionless wave number with

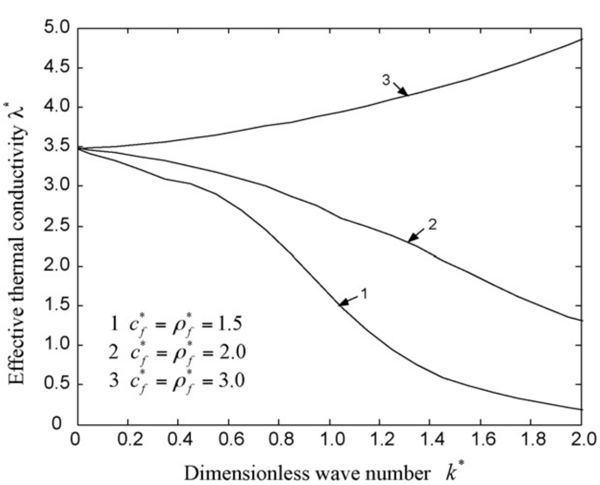

Fig. 4. Non-steady effective thermal conductivity as a function of dimensionless wave number ($h^* = 0.1,\ V_f = 0.4,\ \lambda_f^* = 4.0,\ \lambda_c^* = 2.5,\ c_c^* = \rho_c^* = 1.5$).

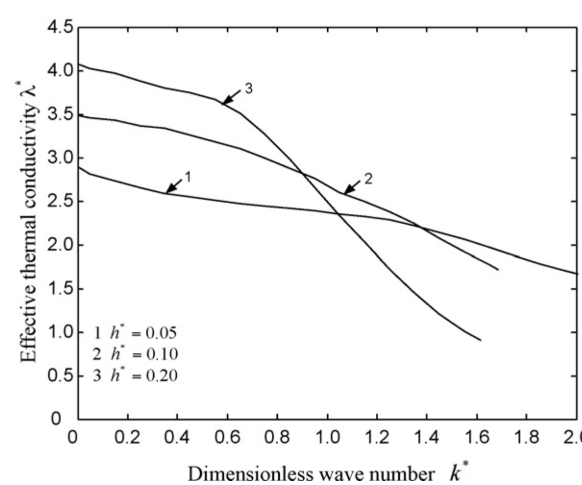

Fig. 6. Non-steady effective thermal conductivity as a function of dimensionless wave number ($V_f = 0.4,\ \lambda_f^* = 4.0,\ c_f^* = \rho_f^* = 2.0,\ \lambda_c^* = 2.5,\ c_c^* = \rho_c^* = 1.5$).

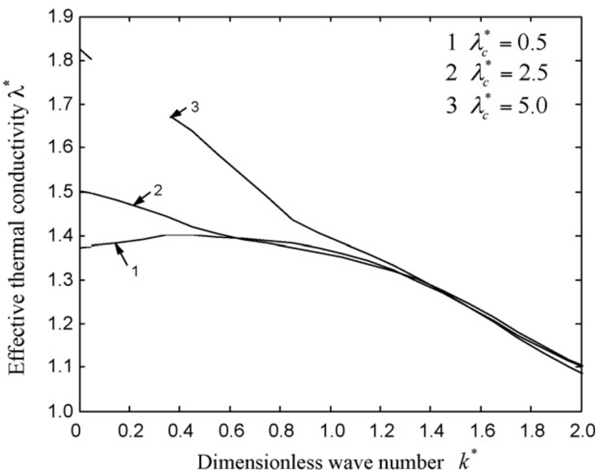

Fig. 7. Non-steady effective thermal conductivity as a function of dimensionless wave number ($V_f = 0.1,\ h^* = 0.1,\ \lambda_f^* = 4.0,\ c_f^* = \rho_f^* = 2.0,\ c_c^* = \rho_c^* = 1.5$).

parameters: $V_f = 0.1,\ h^* = 0.1,\ \lambda_f^* = 4.0,\ c_f^* = \rho_f^* = 2.0,\ c_c^* = \rho_c^* = 1.5$. It can be seen that in the region of low frequency, the non-steady effective thermal conductivity increases with the increase of the value of λ_c^*. However, in the region of high frequency, the non-steady effective thermal conductivity nearly expresses no variation with the value of λ_c^*.

Fig. 8 shows the non-steady effective thermal conductivity of composites as a function of dimensionless wave number with parameters: $V_f = 0.4,\ h^* = 0.1,\ \lambda_f^* = 4.0,\ c_f^* = \rho_f^* = 2.0,\ c_c^* = \rho_c^* = 1.5$. Comparing the results with those in Fig. 7, it can be seen that with the increase of the volume fraction of fibers, only in the region of low frequencies, the effect of the value of λ_c^* on the non-steady effective thermal conductivity is great. However, in the region of high frequencies, the effect of the value of λ_c^* on the non-steady effective thermal conductivity expresses little variation. The effect of the value of λ_c^* on the non-steady effective thermal conductivity increases with the decrease of the volume fraction of fibers.

Finally, to demonstrate the validity of this dynamic thermal model, the steady effective thermal conductivity of two-phase composites without coating is given. As $k^* \to 0$, the dynamic effective thermal conductivity tends to the steady solutions. In Fig. 9, the results obtained from the present model, Effective Medium Theory [4] and Hasselman and Johnson [9] are plotted. It is noted that the steady effective thermal conductivity equations obtained from Effective Medium Theory [4] and Hasselman and Johnson [9] are listed in Appendix B. Close agreement is seen to exist between the models; however, the present model predicts a lower value of effective thermal conductivity than the Effective Medium Theory. This is consistent with regards to criticism of the conventional Effective Medium Theory for overestimating the effective thermal conductivity of two-phase composites when $\lambda_f > \lambda$. This is attributed to the assumption that the fibers are regarded as the effective medium even at close range.

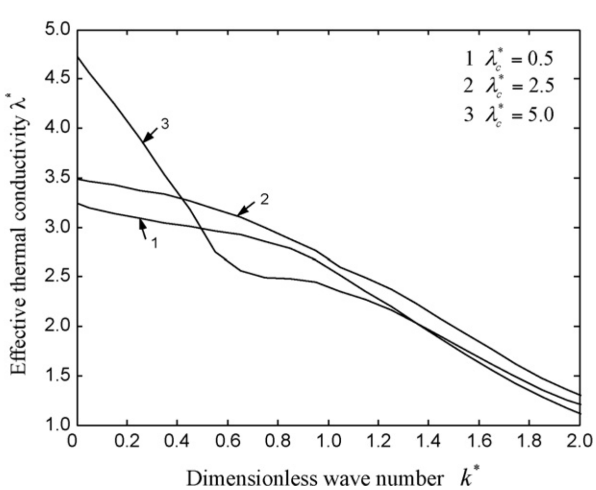

Fig. 8. Non-steady effective thermal conductivity as a function of dimensionless wave number ($V_f = 0.4,\ h^* = 0.1,\ \lambda_f^* = 4.0,\ c_f^* = \rho_f^* = 2.0,\ c_c^* = \rho_c^* = 1.5$).

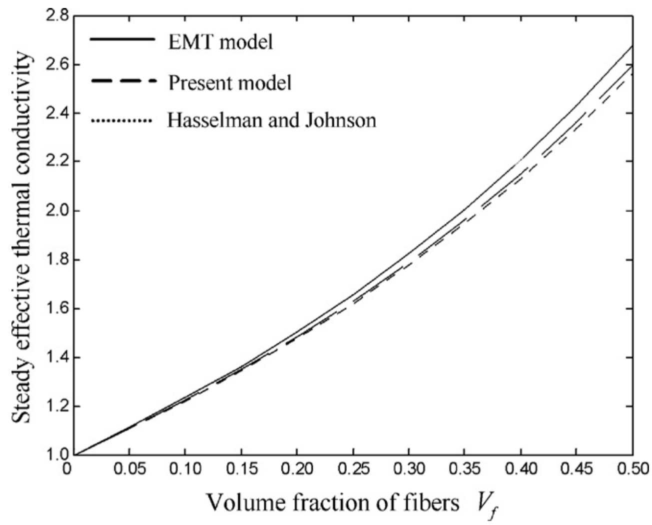

Fig. 9. Comparison of the steady effective thermal conductivity with EMT model and Hasselman and Johnson (Ref. [9]) ($\lambda_f^* = 10.0,\ c_f^* = 2.0,\ \rho_f^* = 2.0,\ h^* = 0,\ k^* = 0$).

8. Conclusions

The multiple scattering of thermal waves in composites reinforced by dense coated fibers is investigated theoretically by employing wave functions expansion method. The interactions of temperature field between the fibers in the matrix are considered. The Lax's quasicrystalline approximation is applied to obtain the effective propagating wave number of thermal waves. The analytical solution of the non-steady effective thermal conductivity of composites is presented. Comparison with the steady effective thermal conductivity demonstrates the validity of the dynamical thermal model.

It has been found that the non-steady effective thermal conductivity of composites is dependent on the incident wave number, the volume fraction of fibers, the material properties ratio of the fiber and matrix and the properties of the coating. The non-steady effective thermal conductivity of the composites increases with an increase of the thickness of the coating, the volume fraction of fibers, and the thermal conductivity ratio of the fiber and matrix. The effects of the thickness of the coating and the thermal conductivity ratio of the fiber and matrix on the non-steady effective thermal conductivity increase with the increase of the volume fraction of fibers. In contrast to the steady case, the frequency of the thermal waves has great influence on the effective thermal conductivity. In the region of low frequency, the variation the specific heat and density of the two phases nearly expresses no effect on the effective thermal conductivity. With the increase of the incident wave number, the effects of the specific heat and density of the two phases on the non-steady effective thermal conductivity increase greatly. In different region of frequency, the effect of the thickness of the coating also shows great difference. Therefore, to gain a higher effective thermal conductivity of composites, when the frequency of thermal loading is low, the greater thickness and thermal conductivity of the coating and the greater thermal conductivity ratio of the fibers and matrix should be chosen. However, in the region of high frequency, the smaller thickness of the coating is preferable. In addition, in the region of lower frequencies, a greater volume fraction of fibers can also help us obtain a higher effective thermal conductivity of composites. However, in the region of very high frequency, a smaller volume fraction of fibers is preferable.

The results of this paper can provide guidelines for the design of fiber reinforced composites in the presence of coating and would be helpful in understanding the thermal behavior of composites.

Appendix A

The expressions of X_s and Y_s are given by

$$X_s = \frac{P_s K_s}{M_s K_s - N_s L_s}, \tag{A1}$$

$$Y_s = \frac{-P_s L_s}{M_s K_s - N_s L_s}, \tag{A2}$$

where

$$M_s = H_s^{(1)}(\kappa_c a_m)\frac{\partial}{\partial a_m}H_s^{(1)}(\kappa a_m) - \frac{\lambda_c}{\lambda}H_s^{(1)}(\kappa a_m)\frac{\partial}{\partial a_m}H_s^{(1)}(\kappa_c a_m), \tag{A3}$$

$$N_s = H_s^{(2)}(\kappa_c a_m)\frac{\partial}{\partial a_m}H_s^{(1)}(\kappa a_m) - \frac{\lambda_c}{\lambda}H_s^{(1)}(\kappa a_m)\frac{\partial}{\partial a_m}H_s^{(2)}(\kappa_c a_m), \tag{A4}$$

$$P_s = J_s(\kappa a_m)\frac{\partial}{\partial a_m}H_s^{(1)}(\kappa a_m) - H_s^{(1)}(\kappa a_m)\frac{\partial}{\partial a_m}J_s(\kappa_c a_m), \tag{A5}$$

$$L_s = \frac{\lambda_f}{\lambda_c}H_s^{(1)}(\kappa_c a_0)\frac{\partial}{\partial a_0}J_s(\kappa_f a_0) - J_s(\kappa_f a_0)\frac{\partial}{\partial a_0}H_s^{(1)}(\kappa_c a_0), \tag{A6}$$

$$K_s = \frac{\lambda_f}{\lambda_c}H_s^{(2)}(\kappa_c a_0)\frac{\partial}{\partial a_0}J_s(\kappa_f a_0) - J_s(\kappa_f a_0)\frac{\partial}{\partial a_0}H_s^{(2)}(\kappa_c a_0). \tag{A7}$$

Appendix B

The effective thermal conductivity equation obtained from Effective Medium Method model [4] is expressed as

$$V_f \frac{\lambda_f - \lambda^{eff}}{\lambda_f + 2\lambda^{eff}} + (1 - V_f)\frac{\lambda - \lambda^{eff}}{\lambda + 2\lambda^{eff}} = 0. \tag{B1}$$

The effective thermal conductivity equation obtained from Hasselman and Johnson [9] is expressed as

$$\lambda^{eff} = \lambda \frac{\left(\frac{\lambda_f}{\lambda - 1}\right) V_f + \left(1 + \frac{\lambda_f}{\lambda}\right)}{\left(1 - \frac{\lambda_f}{\lambda}\right) V_f + \left(1 + \frac{\lambda_f}{\lambda}\right)}. \tag{B2}$$

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