ON REGULAR SINGULAR POINTS OF LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER WHOSE COEFFICIENTS ARE NOT NECESSARILY ANALYTIC*

BY

MAXIME BÔCHER

§ 1. *Introduction.*

Ever since the time of CAUCHY it has been considered of interest to establish the existence of solutions of differential equations whose coefficients are functions of a real variable x, and to do this without requiring these coefficients to be analytic functions of x, but merely continuous functions. It is a natural extension of this point of view to wish to investigate the nature of singular points of such equations, *i. e.*, of points where the coefficients become discontinuous.† It is my object in the present paper to carry through such an investigation in a special, but at the same time important, case.

We shall confine ourselves to the equation:

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0, \tag{1}$$

and shall, as has already been said, regard the independent variable x as real, while p and q are functions of x which are not required to be analytic. In fact, since no further complication is thereby introduced, we shall not even require that p and q be real. A point at which p or q is discontinuous we call a singular point of (1), and we shall restrict ourselves to singular points which, at least on one side, are isolated from all other singular points; *i. e.*, if a is such a point, it must be possible to find a positive constant ϵ so small that *either* in the interval $a < x < a + \epsilon$, *or* in the interval $a > x > a - \epsilon$ there lies no singular point of the differential equation. In order to simplify matters we will take our singular point at the point $x = 0$, and will suppose that throughout the interval $0 < x \leqq b$, b being therefore positive, there lies no singular point of (1). We have, of course, hereby introduced no essential specialization into our problem.

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† Four papers by Kneser (CRELLE, vols. 116, 117, 120 and MATH. ANN., vol. 49) may be regarded as a contribution to this subject, the singular point being at infinity.

Finally we will assume that we can write:

$$p = \frac{\mu}{x} + p_1, \quad q = \frac{\nu}{x^2} + q_1$$

where μ and ν are constants, and where:

$$\int_b^\epsilon |p_1|\, dx \quad \text{and} \quad \int_b^\epsilon x\, |q_1|\, dx$$

approach finite limits as the positive quantity ϵ approaches zero.

When all these conditions are fulfilled we will speak of the point $x = 0$ as a *regular singular point.*

I have already investigated such singular points, to some extent, in a paper presented to the Society in October 1898 and printed in the Bulletin in March 1899. The method which I then used is not sufficiently general to treat more than part of the problem in hand. In the following pages I have adopted a more direct method which enables me to treat all cases of the problem in question. This method also admits of application to linear differential equations of the n^{th} order, as I hope to show on a future occasion. I hope also soon to present to the Society certain applications of the results contained in the present paper.

§ 2. *The Larger Exponent.*

We will now proceed to develop solutions of (1) about the point $x = 0$ into series valid throughout at least that portion of the interval $0\,b$ which lies sufficiently near to $x = 0$. For this purpose we use a method of successive approximations.

We write (1) in the form:

$$\frac{d^2y}{dx^2} + \frac{\mu}{x}\frac{dy}{dx} + \frac{\nu}{x^2}y = -\,p_1\frac{dy}{dx} - q_1 y\,.$$

Starting from the value $y = 0$, we substitute this value in the second member and get the following equation for computing a first approximation Y_0:

$$\frac{d^2y}{dx^2} + \frac{\mu}{x}\frac{dy}{dx} + \frac{\nu}{x^2}y = 0\,. \tag{2}$$

This equation has the solutions:

$$\eta_1 = x^{\kappa'} \quad \eta_2 = x^{\kappa''}$$

where κ' and κ'' are the roots of the quadratic equation in ρ:

$$\rho(\rho - 1) + \mu\rho + \nu = 0\,.$$

This equation we will call the *indicial equation* of the point $x = 0$ and κ', κ'' we will call the *exponents* of this point. Their difference $\kappa' - \kappa''$ we will denote by κ. For the present we exclude the case in which $\kappa' = \kappa''$. We will suppose that our notation has been so chosen that

$$R\kappa' \geqq R\kappa'',$$

where the symbol R stands for the words "the real part of."

We begin by taking as our first approximation :

$$Y_0 = \eta_1 = x^{\kappa'}.$$

Then we compute in succession the quantities Y_1, Y_2, $\cdots$ from the equations :

$$(3) \qquad \frac{d^2 Y_m}{dx^2} + \frac{\mu}{x}\frac{dY_m}{dx} + \frac{\nu}{x^2} Y_m = -p_1 \frac{dY_{m-1}}{dx} - q_1 Y_{m-1} \qquad (m = 1, 2, \cdots),$$

the constants that come in through the integration of these differential equations being so determined that

$$Y_m = x^{\kappa'}\varphi_m(x), \text{ where } \lim_{x=0} \varphi_m(x) = 0.$$

That such a determination of these constants of integration is possible will appear in the course of our work.

The equation (3) is a non-homogeneous linear differential equation for determining Y_m. We solve it by the method of variation of constants, *i. e.*, we let:

$$Y_m = f_m^{(1)}\eta_1 + f_m^{(2)}\eta_2$$

and determine $f_m^{(1)}$ and $f_m^{(2)}$ so that not only the above expression satisfies (3) but also:

$$Y_m' = f_m^{(1)}\eta_1' + f_m^{(2)}\eta_2',$$

(accents here and in what follows denoting differentiation). This gives for $f_m^{(1)}$ and $f_m^{(2)}$ the values:

$$(4) \qquad f_m^{(1)} = -\int \frac{\eta_2 P_m}{\eta_1\eta_2' - \eta_2\eta_1'}\,dx, \qquad f_m^{(2)} = \int \frac{\eta_1 P_m}{\eta_1\eta_2' - \eta_2\eta_1'}\,dx,$$

where, for the sake of abbreviation, we have denoted the second member of (3) by P_m. The constants involved in these indefinite integrals we determine by writing them as definite integrals with upper limit x and lower limit 0. This must be shown to be allowable by proving that these definite integrals converge.

As we have already said, we shall not consider the whole interval $0b$ but only a portion $0c$ of it, which is presently to be chosen so short that certain conditions are satisfied. We will introduce the positive constant C by the following inequalities:

$$C > |\kappa'|\,, \quad C > |\kappa''|\,, \quad C > 1\,.$$

Further let:

$$M(x) = \int_0^x \{\, C|p_1| + x|q_1|\} dx\,.$$

It should be noticed that $M(x) \leqq M(c)$ when $0 \leqq x \leqq c$.

We will now establish the convergence of the integrals (4), which may at once be reduced to:

$$(4') \qquad \begin{aligned} f^{(1)}_m &= \frac{1}{\kappa}\int_0^x x^{1-\kappa'} P_m dx\,, \\ f^{(2)}_m &= \frac{-1}{\kappa}\int_0^x x^{1-\kappa''} P_m dx\,, \end{aligned} \qquad (m = 1,\ 2,\ 3,\ \cdots)$$

and at the same time prove that Y_m satisfies the inequalities:

$$(5) \qquad \begin{aligned} |Y_m(x)| &\leqq \frac{C^m\{2M(x)\}^m}{|\kappa|^m}\, x^{R\kappa'}\,, \\ |Y'_m(x)| &\leqq \frac{C^{m+1}\{2M(x)\}^m}{|\kappa|^m}\, x^{R\kappa'-1}\,. \end{aligned} \qquad (m = 0,\ 1,\ 2,\ \cdots)$$

The inequalities (5) obviously hold when $m = 0$. Let us then assume that when $m = 0\,,\ 1\,,\ \cdots\ (m_1 - 1)$, the inequalities (5) hold and the integrals (4′) converge. Then, using the ordinary method of mathematical induction, we have merely to prove that when $m = m_1$ the integrals (4′) converge and formulæ (5) hold. Now we have:

$$P_{m_1} = -p_1 Y'_{m_1-1} - q_1 Y_{m_1-1}\,;$$

therefore:

$$|P_{m_1}| \leqq |p_1|\,|Y'_{m_1-1}| + |q_1|\,|Y_{\ \ 1-1}| \leqq \frac{C^{m_1-1}\{2M(x)\}^{m_1-1}}{|\kappa|^{m_1-1}}\, x^{R\kappa'-1}\{\, C|p_1| + x|q_1|\}\,,$$

from which we see not merely that the integrals (4′) converge when $m = m_1$, but that

$$\begin{aligned} |f^{(1)}_{m_1}| &\leqq \frac{1}{|\kappa|}\int_0^x x^{1-R\kappa'}|P_{m_1}|dx \leqq \frac{C^{m_1-1}\{2M(x)\}^{m_1-1}M(x)}{|\kappa|^{m_1}} \\ |f^{(2}_{m_1}| &\leqq \frac{1}{|\kappa|}\int_0^x x^{1-R\kappa''}|P_{m_1}|dx \leqq \frac{C^{m_1-1}\{2M(x)\}^{m_1-1}M(x)}{|\kappa|^{m_1}}\, x^{R\kappa}. \end{aligned}$$

Therefore:

$$|Y_{m_1}| \leqq |f^{(1)}_{m_1}|\,|\eta_1| + |f^{(2)}_{m_1}|\,|\eta_2| \leqq \frac{C^{m_1-1}\{2M(x)\}^{m_1}}{|\kappa|^{m_1}} x^{R\kappa'},$$

$$|Y'_{m_1}| \leqq |f^{(1)}_{m_1}|\,|\eta'_1| + |f^{(2)}_{m_1}|\,|\eta'_2| \leqq \frac{C^{m_1}\{2M(x)\}^{m_1}}{|\kappa|^{m_1}} x^{R\kappa'-1}.$$

From these inequalities formulæ (5), for the case $m = m_1$, follow at once when we remember that $C > 1$.

By means of (5) we not only verify a statement made above, but deduce another fact not yet stated. Let:

$$Y_m = x^{\kappa'}\varphi_m(x), \qquad Y'_m = x^{\kappa'-1}\bar{\varphi}_m(x) \qquad (m = 0, 1, 2, \cdots).$$

From (5) we get immediately:

$$(\bar{5}) \qquad |\varphi_m(x)| \leqq \frac{C^m\{2M(x)\}^m}{|\kappa|^m}, \qquad |\bar{\varphi}_m(x)| \leqq \frac{C^{m+1}\{2M(x)\}^m}{|\kappa|^m},$$

these inequalities holding throughout the interval $0 < x \leqq c$. Since $M(x)$ approaches zero with x the same will be true of $\varphi_m(x)$ and $\bar{\varphi}_m(x)$ when $m > 0$. We will therefore define: $\varphi_m(0) = \bar{\varphi}_m(0) = 0$ $(m = 1, 2, 3, \cdots)$.

The absolute and uniform convergence of the two series:

$$(6) \qquad \sum_{m=0}^{m=\infty}\varphi_m(x), \qquad \sum_{m=0}^{m=\infty}\bar{\varphi}_m(x),$$

throughout the interval $0 \leqq x \leqq c$ now follows at once from $(\bar{5})$, by means of the fundamental Weierstrassian criterion, * provided that:

$$\frac{2CM(c)}{|\kappa|} < 1.$$

We will, from now on, suppose that c is so small that this inequality is true.

By multiplying the series (6) by $x^{\kappa'}$ and $x^{\kappa'-1}$ respectively we get the series:

$$\sum_{m=0}^{m=\infty} Y_m, \qquad \sum_{m=0}^{m=\infty} Y'_m.$$

From the uniform convergence of the series (6) throughout the interval $0 \leqq x \leqq c$ we cannot infer the uniform convergence of the last written series throughout this interval, since the factors $x^{\kappa'}$ and $x^{\kappa'-1}$ may become infinite at $x = 0$. We can however infer at once their uniform convergence throughout the interval $\epsilon \leqq x \leqq c$ where ϵ is any positive constant less than c.

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* WEIERSTRASS, Werke, vol. II, p. 202.

Since the second of the two series just written is uniformly convergent throughout the interval $\epsilon \leqq x \leqq c$, and since its terms are the derivatives of the terms of the first, it follows that it represents the derivative of the function represented by the first series at every point of the interval $\epsilon \leqq x \leqq c$, and therefore at every point of the interval $0 < x \leqq c$.

We will write:

$$y_1 = \sum_{m=0}^{m=\infty} Y_m, \qquad y_1' = \sum_{m=0}^{m=\infty} Y_m'.$$

We wish to prove that y_1 satisfies (1) at every point of the interval $0 < x \leqq c$. Consider any point x of this interval and take a positive constant ϵ so small that $\epsilon < x \leqq c$. Throughout this last interval each one of the following four series is obviously uniformly convergent:

$$\begin{aligned}
\frac{\nu}{x^2} y_1 &= \frac{\nu}{x^2} Y_0 + \frac{\nu}{x^2} Y_1 + \frac{\nu}{x^2} Y_2 + \cdots, \\
q_1 y_1 &= \qquad\quad q_1 Y_0 + q_1 Y_1 + \cdots, \\
\frac{\mu}{x} y_1' &= \frac{\mu}{x} Y_0' + \frac{\mu}{x} Y_1' + \frac{\mu}{x} Y_2' + \cdots, \\
p_1 y_1' &= \qquad\quad p_1 Y_0' + p_1 Y_1' + \cdots.
\end{aligned}$$

Adding these four equations we get by (3):

$$\left(\frac{\mu}{x} + p_1\right) y_1' + \left(\frac{\nu}{x^2} + q_1\right) y_1 = -Y_0'' - Y_1'' - Y_2'' - Y_3'' \cdots.$$

The series on the right must be uniformly convergent, since it is the sum of four uniformly convergent series. It therefore represents the function $-y_1''$, and we thus see that y_1 satisfies (1) at the point x which is any point of the interval $0 < x \leqq c$.

The results we have so far obtained may be summed up (in a slightly different form from that in which we have yet stated them) as follows:

A solution y_1 of the differential equation (1) *and its derivative y_1' may be developed about the point $x = 0$ as follows:*

$$\begin{aligned}
y_1 &= x^{\kappa'}[1 + \varphi_1(x) + \varphi_2(x) + \cdots], \\
y_1' &= x^{\kappa'-1}[\kappa' + \bar{\varphi}_1(x) + \bar{\varphi}_2(x) + \cdots],
\end{aligned}$$

these developments being valid throughout a certain interval $0 < x \leqq c$. Here the series in parenthesis converge uniformly and absolutely throughout the interval $0 \leqq x \leqq c$, and the functions φ and $\bar{\varphi}$ are continuous throughout this *last mentioned interval, and vanish when $x = 0$. Except at the point $x = 0$ these functions are determined by the recurrent formulæ:*

$$\varphi_m(x) = \frac{1}{\kappa}\Big[\int_0^x Q_m(x)dx - x^{-\kappa}\int_0^x x^{\kappa} Q_m(x)dx\Big],$$

$$\bar{\varphi}_m(x) = \frac{1}{\kappa}\Big[\kappa'\int_0^x Q_m(x)dx - \kappa'' x^{-\kappa}\int_0^x x^{\kappa} Q_m(x)dx\Big],$$

where

$$Q_m(x) = -p_1\bar{\varphi}_{m-1}(x) - xq_1\varphi_{m-1}(x), \qquad (m = 2, 3, \cdots)$$

and

$$Q_1(x) = -p_1\kappa' - xq_1.$$

§ 3. *The Smaller Exponent.*

The solution y_1 found in the last section may be written:

$$y_1 = x^{\kappa'}E_1(x),$$

where $E_1(x)$ is continuous throughout the interval $0 \leqq x \leqq c$, and $E_1(0) = 1$. We will take c so small that $E_1(x)$ does not vanish when $0 \leqq x \leqq c$.

A second solution is given by the well known formula:

$$y_2 = y_1\int \frac{e^{-\int p dx}}{y_1^2}dx.$$

The constants of integration here involved may be explicitly written as follows:

$$y_2 = ky_1\int_c^x \frac{x^{-\mu}e^{-\int_0^x p_1 dx}}{y_1^2}dx + Cy_1$$

$$= kx^{\kappa'}E_1(x)\int_c^x x^{-\kappa-1}\frac{e^{-\int_0^x p_1 dx}}{\{E_1(x)\}^2}dx + Cy_1.$$

Let us write:

$$\frac{e^{-\int_0^x p_1 dx}}{\{E_1(x)\}^2} = 1 + F(x).$$

It will be noticed that $F(x)$ is continuous throughout the interval $0 \leqq x \leqq c$, and that $F(0) = 0$. We may now write:

$$y_2 = -\frac{k}{\kappa}x^{\kappa''}E_1(x) + \frac{k}{\kappa}c^{-\kappa}x^{\kappa'}E_1(x) + kx^{\kappa'}E_1(x)\int_c^x x^{-\kappa-1}F(x)dx + Cy_1.$$

In order now to get as simple a function y_2 as possible, we will choose the as yet undetermined constants k and C as follows:

$$k = -\kappa, \quad C = c^{-\kappa}.$$

Then:

$$y_2 = x^{\kappa''}E_1(x)\left\{1 - \kappa x^{\kappa}\int_c^x x^{-\kappa-1}F(x)dx\right\}.$$

The second term which stands here in brackets is evidently continuous when $0 < x \leqq c$, and approaches 0 with x.* We may therefore write:

$$y_2 = x^{\kappa''}E_2(x)$$

where $E_2(x)$ is continuous throughout the interval $0 \leqq x \leqq c$, and $E_2(0) = 1$.

In the last section we obtained for the derivative of y_1 a value which may be written:

$$y_1' = x^{\kappa'-1}H_1(x)$$

where $H_1(x)$ is continuous throughout the interval $0 \leqq x \leqq c$ and $H_1(0) = \kappa'$. We can get a similar expression for y_2'; for we may write:

$$y_2 = x^{-\kappa}y_1 - \kappa x^{\kappa'}\int_c^x x^{-\kappa-1}F(x)dx.$$

Differentiating this and replacing y_1 and y_1' by the values $x^{\kappa'}E_1(x)$ and $x^{\kappa'-1}H_1(x)$ respectively we get:

$$y_2' = -\kappa x^{\kappa''-1}E_1(x) + x^{\kappa''-1}H_1(x) - \kappa x^{\kappa''-1}F(x) - x^{\kappa''-1}\left(\kappa\kappa' x^{\kappa}\int_c^x x^{-\kappa-1}F(x)dx\right).$$

And this finally may be written:

$$y_2' = x^{\kappa''-1}H_2(x),$$

where $H_2(x)$ is continuous throughout the interval $0 \leqq x \leqq c$ and $H_2(0) = \kappa''$.

Summing up the principal results of this section and the last we have the theorem:

If the exponents κ' and κ'' are unequal, the equation (1) always has two solutions y_1 and y_2 which together with their derivatives may be written:

$$y_1 = x^{\kappa'}E_1(x), \quad y_2 = x^{\kappa''}E_2(x), \quad y_1' = x^{\kappa'-1}H_1(x), \quad y_2' = x^{\kappa''-1}H_2(x).$$

where $E_1(x)$, $E_2(x)$, $H_1(x)$, $H_2(x)$ are continuous throughout the interval $0 \leqq x \leqq c$, and $E_1(0) = E_2(0) = 1$, $H_1(0) = \kappa'$, $H_2(0) = \kappa''$.

The case in which the coefficients p and q of (1) are real deserves special mention. Here μ and ν are real, and therefore κ' and κ'' are either real or conjugate imaginary. In the first case it is clear that all four functions E_1, E_2, H_1, H_2 are real.

* A proof of this will be found in the Bulletin for October, 1898, p. 25.

The case in which κ' and κ'' are conjugate imaginary is particularly interesting because then $R\kappa' = R\kappa''$ and we can therefore get *both* solutions y_1 and y_2 by the method of successive approximations of §2. It is readily seen that in this case the functions E_1 and E_2 are conjugate imaginary functions:

$$E_1(x) = E(x) + iF(x), \quad E_2(x) = E(x) - iF(x).$$

Here E and F are continuous and real throughout the interval $0 \leqq x \leqq c$ and $E(0) = 1$, $F(0) = 0$.

The two solutions y_1 and y_2 will now be complex. The expressions:

$$y_3 = \frac{y_1 + y_2}{2}, \quad y_4 = \frac{y_1 - y_2}{2i}$$

give us however real solutions, which, if we let:

$$\kappa' = \alpha + \beta i, \ \kappa'' = \alpha - \beta i$$

have the values:

$$y_3 = x^\alpha\{\cos(\beta \log x)E(x) - \sin(\beta \log x)F(x)\},$$
$$y_4 = x^\alpha\{\sin(\beta \log x)E(x) + \cos(\beta \log x)F(x)\}.$$

§ 4. *The Case of Equal Exponents.*

We come now to the case, which has so far been excluded, in which $\kappa' = \kappa''$. We shall find here that the assumption that $|p_1|$ and $x|q_1|$ can be integrated up to the point $x = 0$ is no longer sufficient. *We will therefore now assume that*

$$\int_b^\epsilon |p_1|(\log x)^2 dx, \qquad \int_b^\epsilon x|q_1|(\log x)^2 dx$$

approach finite limits as ϵ approaches zero.

In order to get two independent solutions of (2) we must now write:

$$\eta_1 = x^{\kappa'}, \qquad \eta_2 = x^{\kappa'} \log x.$$

Either of these two functions may be used as our first approximation. We will consider these two cases in succession:

(*a*) *We start from the function* $Y_{01} = \eta_1$, and compute the functions Y_{11}, Y_{21}, Y_{31}, $\cdots$ by means of (3). This gives as before:

$$Y_{m1} = f_{m1}^{(1)}\eta_1 + f_{m1}^{(2)}\eta_2, \qquad Y'_{m1} = f_{m1}^{(1)}\eta'_1 + f_{m1}^{(2)}\eta'_2,$$

where $f_{m1}^{(1)}$ and $f_{m1}^{(2)}$ are given by formulæ (4) which now reduce not to (4′) but to:

$$(4'') \qquad \begin{aligned} f_{m1}^{(1)} &= -\int_0^x x^{1-\kappa'} \log x \, P_m dx, \\ f_{m1}^{(2)} &= \int_0^x x^{1-\kappa'} P_m dx. \end{aligned}$$

Let us now restrict the length of the interval $0c$ by requiring that $1/c \geqq e$ (the exponential base). It follows that:

$$|\log x| \geqq 1 \qquad (0 < x \leqq c).$$

We further introduce the two quantities C and M:

$$C > |\kappa'|, \quad C > 1, \quad M(x) = \int_0^x \{ C\,|p_1| + x\,|q_1| \}\,|\log x|\, dx.$$

We now establish the convergence of the integrals (4″) and at the same time prove that Y_{m1} satisfies the following inequalities:

$$(5') \qquad \begin{aligned} |Y_{m1}| &\leqq \{3M(x)\}^m C^m x^{R\kappa'}, \\ |Y'_{m1}| &\leqq \{3M(x)\}^m C^{m+1} x^{R\kappa'-1}. \end{aligned}$$

These formulæ evidently hold when $m = 0$. Let us assume that the integrals (4″) converge and that formulæ (5′) hold when $m \leqq m_1 - 1$. It remains merely to prove that the same is true when $m = m_1$. We have:

$$|P_{m_1}| \leqq |p_1|\,|Y'_{m_1-1\,1}| + |q_1|\,|Y_{m_1-1\,1}| \leqq \{3M(x)\}^{m_1-1} C^{m_1-1} x^{R\kappa'-1} \{ C\,|p_1| + x\,|q_1| \}.$$

From this formula the convergence of (4″) when $m = m_1$ follows, and we get the inequalities:

$$\begin{aligned} |Y_{m_1 1}| \leqq{} & \{3M(x)\}^{m_1-1} C^{m_1-1} x^{R\kappa'} \int_0^x \{ C\,|p_1| + x\,|q_1| \}\,|\log x|\, dx \\ & + \{3M(x)\}^{m_1-1} C^{m_1-1} x^{R\kappa'}\,|\log x| \int_0^x \{ C\,|p_1| + x\,|q_1| \}\, dx \\ \leqq{} & 2M(x)\{3M(x)\}^{m_1-1} C^{m_1-1} x^{R\kappa'} \leqq \{3M(x)\}^{m_1} C^{m_1} x^{R\kappa'}, \\ |Y'_{m_1 1}| \leqq{} & \{3M(x)\}^{m_1-1} C^{m_1-1}\,|\kappa'|\, x^{R\kappa'-1} \int_0^x \{ C\,|p_1| + x\,|q_1| \}\,|\log x|\, dx \\ & + \{3M(x)\}^{m_1-1} C^{m_1-1} x^{R\kappa'-1} (|\kappa'|\,|\log x| + 1) \int_0^x \{ C\,|p_1| + x\,|q_1| \}\, dx \\ \leqq{} & 3M(x)\{3M(x)\}^{m_1-1} C^{m_1} x^{R\kappa'-1} \leqq \{3M(x)\}^{m_1} C^{m_1+1} x^{R\kappa'-1}, \end{aligned}$$

and these are the inequalities (5′) when $m = m_1$.

Let us now define the functions φ_{m1} and $\bar{\varphi}_{m1}$ as follows:

$$Y_{m1} = x^{\kappa'} \varphi_{m1}(x), \qquad Y'_{m1} = x^{\kappa'-1} \bar{\varphi}_{m1}(x).$$

These functions satisfy the inequalities:

$$|\varphi_{m1}| \leqq \{3M(x)\}^m C^m, \qquad |\bar{\varphi}_{m1}| \leqq \{3M(x)\}^m C^{m+1},$$

and therefore approach zero with x when $m > 0$. We will define them as having in this case the value zero, when $x = 0$.

We will now take c so small that $3M(c)C < 1$. Then the inequalities last written show that the series:

$$\sum_{m=0}^{m=\infty} \varphi_{m\,1}(x), \qquad \sum_{m=0}^{m=\infty} \bar{\varphi}_{m\,1}(x)$$

are absolutely and uniformly convergent throughout the interval $0 \leqq x \leqq c$. We then prove, precisely as on pp. 44, 45, that if

$$y_1 = \sum_{m=0}^{m=\infty} Y_{m\,1}, \quad y_1' = \sum_{m=0}^{m=\infty} Y'_{m\,1},$$

then y_1 is a solution of (1) and y_1' is its derivative.

(*b*) *We start from the function* $Y_{0\,2} = \eta_2$ and compute $Y_{1\,2}$, $Y_{2\,2}$, $\cdots$ by means of (3). This gives:

$$Y_{m\,2} = f^{(1)}_{m\,2}\eta_1 + f^{(2)}_{m\,2}\eta_2, \qquad Y'_{m\,2} = f^{(1)}_{m\,2}\eta_1' + f^{(2)}_{m\,2}\eta_2',$$

where $f^{(1)}_{m\,2}$ and $f^{(2)}_{m\,2}$ are given by the two integrals in $(4'')$.

As in the first part of this section we assume that $1/c \geqq e$. Besides the wo quantities C and M there introduced we need here the function:

$$N(x) = \int_0^x \{ C\,|p_1| + x\,|q_1| \} (\log x)^2 dx.$$

Here again we use the method of mathematical induction to prove that the integrals $(4'')$ converge, and also to establish the inequalities:

$$(5'') \qquad \begin{aligned} |Y_{m\,2}| &\leqq 6N(x)\{3M(x)\}^{m-1} C^{m-1} x^{R\kappa'}, \\ |Y'_{m\,2}| &\leqq 6N(x)\{3M(x)\}^{m-1} C^{m} x^{R\kappa'-1}. \end{aligned} \qquad (m = 1,\ 2,\ \ldots)$$

We first show directly that when $m = 1$ the integrals $(4'')$ converge and the formulæ $(5'')$ hold. This little piece of computation we will omit for the sake of brevity. Then we show that if when $m \leqq m_1 - 1$ the integrals $(4'')$ converge and the formulæ $(5'')$ hold, the same will be true when $m = m_1$. The details of this work we also omit as they are almost identical with the work at the corresponding point in the treatment of case (*a*).

We next let:

$$Y_{m\,2} = x^{\kappa'} \log x\, \varphi_{m\,2}(x), \qquad Y'_{m\,2} = x^{\kappa'-1} \log x\, \bar{\varphi}_{m\,2}(x),$$

and obtain the inequalities:

$$\begin{aligned} |\log x|\,|\varphi_{m\,2}(x)| &\leqq 6N(x)\,\{3M(x)\}^{m-1} C^{m-1}, \\ |\log x|\,|\bar{\varphi}_{m\,2}(x)| &\leqq 6N(x)\,\{3M(x)\}^{m-1} C^{m}. \end{aligned} \qquad (m = 1, 2, \cdots)$$

Since these inequalities will be merely reinforced by omitting the logarithms, we infer from them at once that when $m > 0$:

$$\lim_{x=0} \varphi_{m\,2}(x) = \lim_{x=0} \bar{\varphi}_{m\,2}(x) = 0\,.$$

We will therefore define $\varphi_{m\,2}(0)$ and $\bar{\varphi}_{m\,2}(0)$ as having the value zero when $m > 0$. We see also that the series

$$\sum_{m=0}^{m=\infty} \varphi_{m\,2}(x), \qquad \sum_{m=0}^{m=\infty} \bar{\varphi}_{m\,2}(x)$$

are absolutely and uniformly convergent throughout the interval $0 \leqq x \leqq c$ provided that

$$3M(c)\,C < 1.$$

From this point on we can reason as we have done before. Without spending more time on details, which offer nothing new, we will state at once the results to which this section leads:

If $\kappa' = \kappa''$, equation (1) *has two solutions y_1 and y_2 which together with their derivatives may be written:*

$$\begin{aligned} y_1 &= x^{\kappa'} E_1(x)\,, & y_2 &= x^{\kappa'} \log x\, E_2(x)\,, \\ y_1' &= x^{\kappa'-1} H_1(x)\,, & y_2' &= x^{\kappa'\;1} \log x\, H_2(x)\,. \end{aligned}$$

Here $E_1(x)$, $E_2(x)$, $H_1(x)$, $H_2(x)$ are continuous throughout the interval $0 \leqq x \leqq c$, and $E_1(0) = E_2(0) = 1$, $H_1(0) = H_2(0) = \kappa'$.

The solution y_2 can be thrown into a different form, which not only exhibits the nature of the function in more detail, but is also closely analogous to the form in which this solution is ordinarily written when the coefficients of (1) are analytic functions. In order to obtain this form we write:

$$y_2 - x^{\kappa'} \log x = x^{\kappa'} \sum_{m=1}^{m=\infty} \log x \cdot \varphi_{m\,2}(x)\,,$$

$$y_1 \log x - x^{\kappa'} \log x = x^{\kappa'} \sum_{m=1}^{m=\infty} \log x \cdot \varphi_{m\,1}(x)\,.$$

The terms of the series on the second sides of these equations are strictly speaking not defined when $x = 0$. We will however regard them as having the value 0 at this point. It is then easily shown that each term is continuous throughout the interval $0 \leqq x \leqq c$ and that both series are uniformly convergent throughout this interval. This follows immediately from the inequalities we have already obtained in the case of the first series, while for the second series we need merely to note that:

$$\log x \cdot M(x) \leqq N(x).$$

Each of these two series, therefore, represents a function which is continuous when $0 \leqq x \leqq c$ and which vanishes when $x = 0$. The same is true of the difference of these two functions which we will denote by $F(x)$. If then we subtract the two equations last written, we get:

$$y_2 = y_1 \log x + x^{\kappa'} F(x)$$

where $F(x)$ *is continuous throughout the interval* $0 \leqq x \leqq c$ *and* $F(0) = 0$.

HARVARD UNIVERSITY,
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