

ISOTHERMAL SYSTEMS OF GEODESICS*

BY

EDWARD KASNER

The object of this note is to determine the surfaces upon which there exist one or more isothermal systems composed of ∞^1 geodesic lines. Such a system of geodesics together with the orthogonal trajectories † may be regarded as dividing the surface into infinitesimal squares. The solution of the question leads in particular to an interesting characteristic property of the surfaces of constant curvature.

Upon any surface of revolution there is certainly one system of the kind described, namely, the meridians. The same is true for the surfaces which are applicable on a surface of revolution, i. e., for the class of surfaces whose linear element may be expressed in the form

$$(1) \quad ds^2 = \lambda(x)(dx^2 + dy^2).$$

We now prove the converse result :

If a surface possesses an isothermal system of geodesics, then it is applicable on a surface of revolution, the geodesics corresponding to the meridians.

For assume the system of geodesics to be the parameter lines $y = \text{const.}$, and the orthogonal trajectories to be the parameter lines $x = \text{const.}$ ‡ Then the linear element may be written

$$ds^2 = E(x, y)(dx^2 + dy^2).$$

Since the differential equation of the geodesics,

$$y'' = \frac{\partial E}{\partial x}(y' - y'^3) + \frac{\partial E}{\partial y}(y'^2 - 1),$$

is to be satisfied by $y = \text{const.}$, i. e., by $y' = 0$, it follows that $\partial E/\partial y = 0$, and hence that ds^2 is of the form (1) which is characteristic of the class of surfaces described in the theorem.

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† These form a system of (geodesically) parallel curves. The only surfaces which can be divided into infinitesimal squares by two systems of geodesics are the developable surfaces.

‡ This choice of parameter lines is legitimate, unless the system of geodesics in question is one of the systems of minimal lines. This case however is of no interest and is excluded from the following discussion.

We proceed now to examine whether more than one system of isothermal geodesics can exist upon a surface. It will be convenient to transform the element (1) by introducing the parameters u, v of the minimal lines. The new form may be written

$$(1') \quad ds^2 = 2e^{f(u+v)} du dv.$$

The equation of the geodesics is now

$$(2) \quad v'' = g(u+v)(v'^2 - v'),$$

where the function g is the negative derivative of the function f ,

$$(2') \quad g(t) = -f'(t).$$

One solution of (2) is evidently $v' = 1$ or

$$(3) \quad u - v = \text{const.},$$

which corresponds to the meridians in the case of a surface of revolution.

Every isothermal system may be assumed in the form

$$(4) \quad U_1 - V_1 = \text{const.},$$

where U_1 is a function of u alone, and V_1 is a function of v alone. If this is to represent geodesics, then

$$V_1'^2 U_1'' - U_1'^2 V_1'' = g U_1' V_1' (U_1' - V_1').$$

Dividing* both sides by $U_1'^2 V_1'^2$, this may be written

$$(5) \quad \frac{U' - V'}{U - V} = g(u + v),$$

where U and V are defined

$$(6) \quad U = \frac{1}{U_1'}, \quad V = \frac{1}{V_1'}.$$

Since the partial derivatives g_u and g_v are equal, we find from (5)

$$(7) \quad UV'' - U''V = U'^2 - UU'' - V'^2 + VV''.$$

Finally, differentiating both sides with respect to u and v ,

$$(8) \quad U'V''' - V'U''' = 0.$$

It will be convenient to consider first the case where either U' or V' vanishes. If for example $V' = 0$, then V is a constant. Hence from (5), U is also a constant. If U and V are equal to the same constant, system (4) is identical with system (3), so that no second system of the required kind is obtained. If U and V are different constants, then $g = 0$; this will be considered later.

*This is allowable, since neither U_1' nor V_1' can vanish unless (4) represents merely one of the minimal systems.

We may assume now that neither U' nor V' vanishes, and rewrite (8) in the form

$$(9) \quad \frac{U'''}{U'} = \frac{V'''}{V'} = c^2.$$

Discussion of (9) for $c = 0$.

Here we have $U''' = V''' = 0$, and hence

$$(10) \quad U = au^2 + a_1u + a_2, \quad V = bv^2 + b_1v + b_2.$$

Substituting these functions in (7), we find that the coefficients must fulfill the conditions

$$(11) \quad a = b, \quad 4a(a_2 - b_2) = a_1^2 - b_1^2.$$

Suppose first that $a \neq 0$. Then

$$a_2 - b_2 = \frac{a_1^2 - b_1^2}{4a}.$$

Hence from (5), after some calculation,

$$\frac{1}{g} = \frac{u+v}{2} + \frac{a_1 + b_1}{4a}.$$

Suppose next that $a = 0$. Then from (11) either $a_1 - b_1 = 0$ or else $a_1 + b_1 = 0$. The first relation gives

$$U = a_1u + a_2, \quad V = a_1v + b_2; \quad g = 0.$$

The second relation, on the other hand, gives

$$U = a_1u + a_2, \quad V = -a_1v + b_2; \quad \frac{1}{g} = \frac{u+v}{2} + \frac{a_2 - b_2}{2a_1}.$$

Finally if $a_1 = 0$, U and V reduce to constants, a case already considered.

Discussion of (9) for $c \neq 0$.

The integration of the system (9) gives now

$$(12) \quad U = Ae^{cu} + A_1e^{-cu} + A_2, \quad V = Be^{cv} + B_1e^{-cv} + B_2.$$

From (7) we obtain the following conditions on the constants.

$$(13) \quad A_2 = B_2, \quad AA_1 - BB_1 = 0.$$

Suppose first that neither A nor B vanishes. Then introducing a constant $m = A_1/B = B_1/A$, we find

$$(14) \quad U = Ae^{cu} + mBe^{-cu} + A_2, \quad V = Be^{cv} + mAe^{-cv} + A_2;$$

$$g = c \frac{e^{c(u+v)} + m}{e^{c(u+v)} - m}.$$

In the second place suppose that one of the constants A, B vanishes while the other does not. Assuming for example $A = 0, B \neq 0$, it follows from (13) that $B_1 = 0$. Hence

$$U = A_1 e^{-cu} + A_2, \quad V = B e^{cv} + A_2; \quad g = c \frac{B e^{c(u+v)} + A_1}{B e^{c(u+v)} - A_1}.$$

Finally the supposition that both A and B vanish gives

$$U = A_1 e^{-cu} + A_2, \quad V = B_1 e^{-cv} + A_2; \quad g = -c.$$

Examining all the values of the function g which present themselves, we find that they are included in the three types

$$(15) \quad \alpha, \quad \frac{2}{t + \beta}, \quad c \frac{e^{ct} + \gamma}{e^{ct} - \gamma},$$

where t represents the argument $u + v$, and α, β, γ and c are constants. From (2') the corresponding types of the function f are

$$(16) \quad -at + \alpha_1, \quad \log \frac{\beta_1}{(t + \beta)^2}, \quad \log \frac{\gamma_1 e^{ct}}{(e^{ct} - \gamma)^2}.$$

Calculating now the Gaussian curvature of the element (1') by means of the formula * $K = -e^{-f} f''$, we find, corresponding to the forms (16), that K reduces to 0, $2/\beta_1, 2c^2\gamma/\gamma_1$, respectively. Thus in every case K is constant, and we reach the following conclusion:

The only surfaces upon which more than one isothermal system of geodesics can exist are the surfaces of constant (Gaussian) curvature. †

Conversely, upon any surface of constant curvature there actually exist more than one, indeed a double infinity of systems of the kind described. This is easily shown directly from the above formulas. It is sufficient to introduce the values of g given in (15) into the relation (5), thus determining U, V and hence the required systems (4).

* This may be derived from the general formula for K given on page 68 of BIANCHI, *Differentialgeometrie*, German translation of LUKAT.

† [Added December 5]. So far as this theorem is concerned, the analysis on the last two pages is obviated by the following simple proof kindly suggested by Dr. L. P. EISENHART. It has been seen that when a surface possesses an isothermal system of geodesics its element may be reduced to the form (1). For this K is a function of x alone, so that it is constant along the orthogonal trajectories of the geodesics. Now if a second system of the required kind exists, K would also be constant along its orthogonal trajectories. It follows that unless K is constant over the entire surface, the two systems must coincide.

The above analysis yields the explicit equations of the ∞^2 systems of geodesics, and proves at the same time that the element of a surface of constant curvature referred to any isothermal system of geodesics and the orthogonal trajectories takes the form

$$ds^2 = 2e^{f(2x)}(dx^2 + dy^2),$$

where f is one of the types (16).

The result however may be obtained more simply by making use of the known theory of surfaces of constant curvature.* Such a surface can be built conformally on the plane in such a manner that its geodesics are pictured by a linear two-parameter system of circles, namely, by all the circles orthogonal to a base circle of real, imaginary or zero radius according as K is positive, negative or zero.† The problem of finding isothermal systems of geodesics on the surface is thus reduced to the problem of finding isothermal systems of circles contained in the two-parameter system. It is known however that the only isothermal systems of circles are the pencils, i. e., the circles passing through a pair of real or imaginary points. Such a pencil is contained in the two parameter systems only if the two points are inverse with respect to the base circle. The number of pencils is therefore ∞^2 , so that

Upon any surface of constant curvature there exist ∞^2 isothermal systems of geodesics.

For the plane these are of course the pencils of straight lines, including the systems of parallel lines.‡ For the sphere the required systems are the great circles through a pair of antipodal points. Finally for the pseudosphere the required systems consist of the geodesics passing through any point, including as a limiting case the system of meridians, where the point may be conceived of as at infinity.

The investigation leads then to the following *classification of surfaces according to the number of isothermal systems of geodesics existing upon them*:

1°. *Surfaces applicable on no surface of revolution; no system.* 2°. *Surfaces applicable on a surface of revolution, but of variable curvature; one system.* 3°. *Surfaces of constant curvature; ∞^2 systems.*

The problem which has been solved is equivalent to the determination of isothermal systems consisting of (geodesically) parallel curves. For the orthogonal trajectories of such a system form an isothermal system of geodesics, and vice versa. On the general surface of revolution the only isothermal system of parallels is composed of the circles of latitude. On a surface of constant curvature there are ∞^2 systems, namely, the geodesic circles described about any point of the surface as center, including the limiting cases where the center may be regarded as at infinity.

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* The forms derived by DARBOUX (*Leçons*, vol. 3, p. 219) show that the geodesics through any point constitute an isothermal system; but the converse result does not appear to have been noticed.

† For references see *Transactions*, vol. 4 (1903), p. 150, note.

‡ Cf. B. O. PEIRCE, *On families of curves which are the lines of certain plane vectors either solenoidal or lamellar*, *Proceedings of the American Academy of Arts and Sciences*, vol. 38 (1903), p. 673.