GEOMETRY IN WHICH THE SUM OF THE ANGLES OF EVERY TRIANGLE IS TWO RIGHT ANGLES*

BY

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In his article *Die Legendre'schen Sätze über die Winkelsumme im Dreieck*,† DEHN has shown what may be stated precisely by the aid of the following abbreviations:

H_{-a} denotes all those theorems of geometry which are logical consequences of HILBERT's axiom-groups I, II, III, IV, viz., his axioms of association, of order, of parallels, and of congruence respectively.

H_{-a-p} denotes all those theorems of geometry which are logical consequences of HILBERT's axiom groups, I, II, IV.

S denotes the assumption that the sum of the angles of every triangle is two right angles.‡

What DEHN has shown is that H_{-a} does not follow from H_{-a-p} and S. I wish to show that, nevertheless, if H_{-a-p} and S are true of a space, then either this space is a H_{-a} space or it is possible so to introduce ideal points that the space thus extended will be a H_{-a} space, these ideal points, moreover, being such that no one of them is between two points of the original space.

One might state this a little more suggestively, if less accurately, as follows: "While the parallel postulate, III, and thus all of that part of Hilbertian Geometry which follows without use of his 'Axiom of ARCHIMEDES' and 'Vollständigkeit Axiom,' can not indeed be proved from his other postulates I, II, IV with III replaced by S, nevertheless it can be shown that a space concerning which these postulates (I, II, IV, S) are valid must be, if not the whole, then at least a part, of a space in which III also is true."§

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† Mathematische Annalen, vol. 53 (1900), pp. 404–439.

‡ This may be stated precisely as follows: If ABC is a triangle and C is between B and E then, in the angle ACE, there is a point D such that $\sphericalangle ACD \equiv \sphericalangle BAC$ and $\sphericalangle DCE \equiv \sphericalangle ABC$.

§ This result has an interesting connection with our spatial experience. Statements have been made to the effect that, since no human instruments, however delicate, can measure exactly enough to decide in every conceivable case whether the sum of the angles of a triangle is equal to two right angles (unless the difference between this sum and two right angles should exceed

Let us first establish a number of preliminary theorems. The theorems of H_{-a-p} are assumed and I shall take for granted, sometimes without giving exact reference, certain of those definitions and theorems of HILBERT's *Festschrift* which are founded on his axiom groups I, II, IV. Reference will also be made to HALSTED's *Rational Geometry* (R. G.) in case of theorems for which demonstrations without use of HILBERT's III are therein indicated.

For simplicity let us confine ourselves to one plane.

THEOREM I. *If two straight lines, a and b, are perpendicular to each other at a point O, then every straight line, c, in the plane ab, has a point in common with either a or b.*

*Dem.** This conclusion of course holds if c coincides with a or b. If it does not coincide with a or b, then it has (cf. HILBERT's Theorem 5 and definition following IV, 3 and Theorem 9) a point C within one of the four right angles into which a and b divide the plane ab. There exists a point A on that ray of this angle which belongs to a and a point B on that ray thereof which belongs to b. There is (cf. R. G., art. 47, theorem) a point D on c such that OD is perpendicular to c. On c there are two points, E and E', such that $DE' \equiv DE \equiv OD$ (cf. HILBERT's IV, 1 and IV, 2). Then (as may be seen with help of S), the angle DOE is one half a right angle and the angle $E'OD$ is one half a right angle. Accordingly the angle $E'OE$ is a right angle. Hence it is easily seen (cf. R. G., arts. 51 and 143) that E and E' can not both be within the right angle AOB. Therefore c, which contains both E and E', has a point within, and a point without, the angle AOB and thus must cut either OA or OB.

THEOREM 2. *If O lies between A and C and also between B and D, and if $\measuredangle AOB$ is a right angle and $\measuredangle ABD \equiv \measuredangle ACD$, then there is a circle passing through A, B, C, and D.*†

Dem. There exist (cf. R. G., 47, 51 and HILBERT's IV, 4) perpendiculars

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a certain minimum amount), it is therefore impossible to settle the question whether our space is Bolyai-Lobachewskian or Euclidean even though it be granted that it is one or the other.

Now the geometry which I wish to show to be a consequence of H_{-a-p} and S is one concerning whose resemblance to Euclidean space one may say even more. One may say (possibly a little roughly): "While, as DEHN has shown, it is indeed true that a space of which H_{-a-p} and S were true would not necessarily be strictly Hilbertian (in the sense of H_{-a}), nevertheless no human being confined therein could ever distinguish it from a Hilbertian space even though he were supplied with instruments which could decide for him whether any two segments were exactly equal."

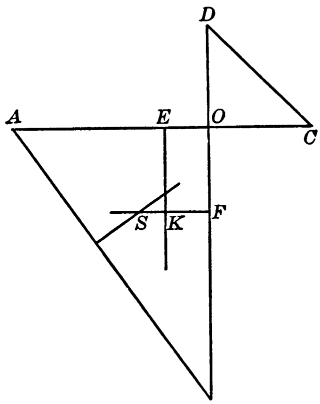
* "*Dem.*" means an *indication* of a demonstration. Likewise "by theorem" does not necessarily mean that this is the only theorem used in the demonstration. Similar statements may be made about the use, in this paper, of the words "therefore," "hence," etc.

Capital letters of the English alphabet designate points.

The sign "$=$" means "is, or are, identical with."

† This theorem gives a sufficient condition that three given points should lie on a circle, also a sufficient condition that 4 given points should lie on a circle.

LE and MF to AC and BD at their middle* points E and F respectively. As LE cannot meet BD (cf. R. G., art. 47), according to Theorem 1 it must meet FM at a point K. Now, by S and Theorem 1, the perpendicular to AB at its middle point must meet either KE or KF. Suppose it meets KF at a



point S. Then $SD \equiv SB \equiv SA$ (by definitions of a right angle and of a perpendicular, HILBERT's Theorem 10, and statement preceding Theorem 13). So there is a circle with S as center passing through A, B, and D. By R. G., art. 57, $\measuredangle SAD \equiv \measuredangle SDA$ and $\measuredangle SAB \equiv \measuredangle SBA$. It may then be seen, even without further use of S, that SAD and SAB are acute and hence SA is not perpendicular to AC and therefore, by R. G., art 138, that the circle which passes through A, B, D must have another point P in common with AC also that $\measuredangle DPA \equiv \measuredangle DBA$ (cf. S and R. G., art. 133); but this, in view of the hypothesis that $\measuredangle DBA \equiv \measuredangle DCA$, is impossible (cf. R. G., 66) unless P is C. The circle therefore through A, B, and D passes also through C. Similarly if the perpendicular to AB at its middle point met EK it would follow that there would be a circle through A, B, and C, and this circle would necessarily pass through D.

Convention. A point O, two straight lines perpendicular to each other at this point, a point I on one of these lines and points Q and S on the other, such that O is between Q and S, are selected once for all and considered as fixed throughout this discussion.

As an aid to the exhibition of an analytic geometry I wish to develop a calculus of those rays (half lines) which start from I towards the O-side of IK where IK is perpendicular to OI. These rays are called "leftward rays." If a leftward ray cuts QS it is called a "cutting ray." Small letters of the English alphabet are used to denote cutting rays.

* See R. G., articles 82, 64 and 66.

If M is a point of QS, $|IM|$ means IT where T is a point of the ray OQ such that $OM \equiv OT$.

The ray IO is designated by the symbol 0 (it is a "zero ray"). If I' is a point on the ray OQ such that $OI' \equiv OI$, then the ray II' is designated by the symbol 1 (it is a "unit ray").

Definition 1 (sum of two cutting rays). If L and M are two points (or the same point) on QS, ray IL + ray IM means ray IN where N is a point of QS such that $ON = OL + OM$ in the vector sense.

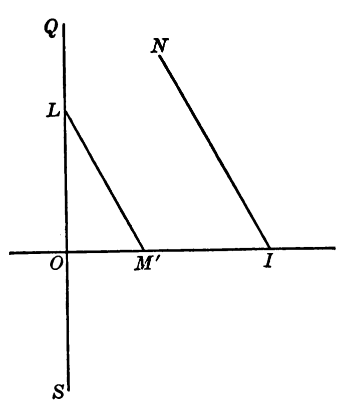
Definition 2. If IM and IN are two leftward rays, then "$IM > IN$" means that IN precedes IM in the sense $IS - IO - IQ$; and "$IM < IN$" means $IN > IM$.

THEOREM 3. *If α and β are two leftward rays, then of the three statements $\alpha > \beta$, $\alpha = \beta$, or $\alpha < \beta$ one and only one is true. If $\alpha < \beta$ and $\beta < \gamma$, then $\alpha < \gamma$.*

THEOREM 4. *The set of all cutting rays is a commutative group with respect to the operation of addition* $(+)$.

Dem. See Definition 1.

Definition 3 (quotient of two cutting rays). If L and M are points of the straight line QS and M is a point of ray OI such that $OM \equiv OM'$ and if IN is a ray such that $\measuredangle OIN \equiv \measuredangle OM'L$, if further IN lies on the Q- or the S-side of OI according as M and L are on the same side, or opposite sides, of O, then $IL/IM = IN$.



Definition 4 (product of two cutting rays), $ab = c$ if and only if $c/a = b$.

THEOREM 5. *If $a/b = c/b$, then $a = c$.*

THEOREM 6. *$b/1 = b$ and $b \times 0 = 0$.*

THEOREM 7. *If $a/b = c/d$, then $b/a = d/c$.*

THEOREM 8. *If A, B, C, D are points on QS and A and B are on the same side or opposite sides of O according as C and D are on the same side* *or on opposite sides of* O, *then the existence of four points* A', B', C', D' *on a circle and another point* O' *such that* O' *is between* B' *and* C', *and between* A' *and* D', *angle* $A'O'B'$ *is a right angle,* $OA \equiv O'A'$, $OB \equiv O'B'$, $OC \equiv O'C'$, *and* $OD \equiv O'D'$, *is a necessary and sufficient condition that* $IA/IB = IC/ID$.

Dem. Use Definition 3 and Theorem 2.

THEOREM 9. *If* $a/b = c/d$, *then* $a/c = b/d$.

Dem. Use Theorem 8.

THEOREM 10. *If* $a/b = c/d$ *and* $a/e = f/d$, *then* $b/e = f/c$.

*Dem.** Consider three cases —

I. Suppose $|b| < |e|$. Evidently there exist an n such that $b/n = e/d$, and an x such that $b/x = e/c$ whence also $b/e = x/c$ (see Theorem 9). Now $a/f = e/d$ (see hypothesis and Theorem 9), so that $a/f = b/n$. Also $b/e = n/d$ (see Theorem 9), therefore $n/d = x/c$ and thus, by Theorem 9, $x/n = c/d$. But since, by hypothesis, $a/b = c/d$, we have $a/b = x/n$. But from $a/f = b/n$, by Theorem 9, $a/b = f/n$ and thus $x/n = f/n$; therefore, by Theorem 5, $x = f$. But as $b/e = x/c$, it follows finally that $b/e = f/c$.

II. Suppose $e < b$. Use similar reasoning to show that $e/b = c/f$ and thence conclude (by use of Theorem 7) that $b/e = f/c$.

III. Suppose $e = b$. In this case, by Theorem 5, $f = c$ and by Theorem 9 the theorem is true.

THEOREM 11. *If* $ab = x$† *and* $a'b' = ab$, *then* $a/a' = b'/b$; *and conversely, if* $ab = x$ *and* $a/a' = b'/b$, *then* $a'b' = ab$.

Dem. I. If $ab = x$ and $a'b' = ab$, then, according to Definition 4, $x/a = b$ and $x/a' = b'$. According to Theorems 6 and 7 $1/b' = a'/x$ and $1/b = a/x$. Hence, by Theorem 10, $b'/b = a/a'$.

II. If $ab = x$ and $a/a' = b'/b$, then, by Definition 4 and Theorem 2, $x/a = b/1$. Hence, by Theorem 7, $a/x = 1/b$ and therefore, by hypothesis and Theorems 10 and 6 and Definition 4, $x = a'b'$.

THEOREM 12. *If* $ab = x$, *then* $ba = ab$.

Dem. If $ab = x$, then, by Definition 4 and Theorem 6, $x/a = b/1$. Hence, by Theorems 9 and 6, $x/b = a$. Therefore, by Definition 4, $ba = x$. But since, by hypothesis, $ab = x$, $ba = ab$.

THEOREM 13. *If* $ab = x$, $(ab)c = y$, *and* $bc = z$, *then* $a(bc) = (ab)c$.

Dem. According to Theorem 11 this proposition will be established if it is proved that $ab/a = bc/c$. Now according to Definition 4 and Theorem 6 $ab/a = b/1$. Likewise $bc/c = b/1$. So $ab/a = bc/c$.

THEOREM 14. *If* $ab = x$, $ac = y$, *then* $a(b + c) = ab + ac$.

Dem. Use S in connection with 2nd figure on page 34 of HILBERT's *Grundlagen der Geometrie* (2nd edition).

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* The suggestion of this demonstration I am unable at present to trace to its source.

† i. e., ab is a cutting ray.

THEOREM 15. *If* $|a|<1$ *then for every* b, ab *exists as a cutting ray. For every* c *there exists* x *such that* $|cx|<1$.

If $|a|<|b|$ and bx exists as a cutting ray, then $|ax|<|bx|$. Given d, e, k, $e\neq 0$, $k\neq 0$, then there exist d', e' such that $|d'|<|k|$, $|e'|<|k|$, and $d/e=d'/e'$.

THEOREM 16. *If* $|a|$, $|b|$, $|c|$, $|d|$ *are all less than* 1, *and if* $a/b=e$ *and* $c/d=f$, *then* $a/b+c/d=(ad+bc)/bd$.

Dem. According to hypothesis and Definition 4, $a=be$, $c=df$; and by hypothesis and Theorems 15, 12, 13, and 14, $ad+bc=(be)d+b(df)=d(be)+b(df)=(db)e+(bd)f=(bd)e+(bd)f=bd(e+f)$. This gives, by the use of Definition 4, $(ad+bc)/bd=e+f=a/b+c/d$.

THEOREM 17. *If* $|a|$, $|b|$, $|c|$, $|d|$ *are all less than* 1, $a/b=e$, $c/d=f$ *and* $ef=y$, *then* $ac/bd=ef$.

Dem. By hypothesis and Definition 4, $c=df$, $a=be$ and, by hypothesis and Theorem 15, $ac=(be)(df)$. Hence, by hypothesis, Theorems 15, 12, and 13, $ac=(bd)(ef)$, and from this, by Definition 4, $ac/bd=ef$.

THEOREM 18. *If* $|a|$, $|b|$, $|c|$, $|d|$, $|a'|$, $|b'|$, $|c'|$, $|d'|$ *are all less than* 1, *and if* $a/b=a'/b'$, $c/d=c'/d'$, *then* $(ad+bc)/bd=(a'd'+b'c')/b'd'$.

Dem. By hypothesis and Theorems 15 and 11, $ab'=ba'$ and $cd'=dc'$. Hence (see Theorem 15) $(ab')(dd')=(ba')(dd')$ and $(cd')(bb')=(dc')(bb')$. Hence, by Theorems 12, 13, and 14, $(ad)(b'd')+(bc)(b'd')=(bd)(a'd')+(bd)(b'c')$ and thus, by Theorems 12 and 14, $(ad+bc)b'd'=bd(a'd'+b'c')$; therefore by Theorem 11, $(ad+bc)/bd=(a'd'+b'c')/b'd'$.

THEOREM 19. *If* $|a|$, $|b|$, $|c|$, $|d|$, $|a'|$, $|b'|$, $|c'|$, $|d'|$ *are all less than* 1 *and* $a/b=a'/b'$ *and* $c/d=c'/d'$, *then* $ac/bd=a'c'/b'd'$.

Dem. According to Theorems 15 and 11, $ab'=ba'$, $cd'=dc'$, whence follows (see Theorem 15), $(ab')(cd')=(ba')(dc')$. Hence, by Theorems 15, 12, and 13, $(ac)(b'd')=(bd)(a'c')$ and thus, by Theorem 11, $ac/bd=a'c'/b'd'$.

Definition 5. If either there is no e such that $a/b=e$ or there is no f such that $c/d=f$, then $a/b\times c/d$ means $a'c'/b'd'$ and $a/b+c/d$ means $a'd'+b'c'/b'd'$ where a', b', c', d' are in absolute value less than 1, and $a/b=a'/b'$ and $c/d=c'/d'$.

THEOREM 20. $a/b\times c/d=a'c'/b'd'$ *and* $a/b+c/d=a'd'+b'c'/b'd'$ *where* a', b', c', d' *are all less than* 1 *in absolute value and* $a'/b'=a/b$ *and* $c'/d'=c/d$.

Dem. See Theorems 16 and 17 and Definition 5.

Lemma. If $a'b$, $a'c$ and $(a'b)c$ are cutting rays, then $a'b/a'c=b/c$.

Dem.

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\begin{aligned}
(a'b)c &= (ba')c && \text{by hypothesis and Theorem 12,}\\
&= b(a'c) && \text{by Theorem 13,}\\
&= (a'c)b && \text{by Theorem 12:}
\end{aligned}
$$

Hence, by Theorem 11, $a'b/a'c=b/c$.

THEOREM 21. *If b and d are different from 0 and a and c are any cutting rays, then there exist x and y $(y \neq 0)$ such that $a/b + x/y = c/d$. If $a/b + x/y = c/d$ and $a/b + w/z = c/d$, then $x/y = w/z$.*

Dem. Use Lemma and Theorems 15, 4, 20, 11.

THEOREM 22. *If a, b, and d are different from 0 and c is any cutting ray, then there exist x and y such that $a/b \times x/y = c/d$. If $a/b \times x/y = c/d$ and $a/b \times w/z = c/d$, then $x/y = w/z$.*

Dem. I. By Theorem 15 there exist a', b', c', d' all less than 1 in absolute value, such that $a/b = a'/b'$, $c/d = c'/d'$. By Theorems 15, 20, 13, 11, and Lemma, $a/b \times b'c'/a'd' = a(b'c')/b(a'd') = (ab')c'/(ba')d' = c'/d'$.

II. Suppose $a/b \times x/y = c/d$ and $a/b \times w/z = c/d$. By Theorem 15 there exist a', b', c', d', x', y', w', z', all less than 1 in absolute value, such that $a/b = a'/b'$, $c/d = c'/d'$, $x/y = x'/y'$, $w/z = w'/z'$. By hypothesis and Theorem 20, $a'x'/b'y' = c'/d'$ and $a'w'/b'z' = c'/d'$. Hence, by Theorems 15, 11, 12, 13, $(a'd')x' = (b'c')y'$ and $(a'd')w' = (b'c')z'$. Hence, by Theorems 12 and 11, $x'/y' = b'c'/a'd' = w'/z'$. Hence $x/y = w/z$.

Convention. Any Greek letter denotes a/b where a and b are cutting rays.

THEOREM 23. *The set of all α's is a number system for which Hilbert's Theorems* 1–16 *of* § 13 *of his Grundlagen der Geometrie hold true with respect to* $+$, $\times$, *and* $>$ *as defined in this paper.**

Dem. HILBERT's Theorems 1–12 may be proved by use of my Theorems 20, 6, 4, 21, 22, 12, 15, 14, 13, and Lemma to Theorem 22. His Theorems 13–16 may be proved with the use of my Theorems 1–12 in connection with Definitions 1–5.

Definition 6. The length of a segment AB means the ray IT where T is a point on the ray OQ such that $AB \equiv OT$.

The length of AB is denoted by "$e(AB)$."

Lemma I. If in the triangles ABC, $A'B'C'$, $\measuredangle A \equiv \measuredangle A'$, $\measuredangle B \equiv \measuredangle B'$, $\measuredangle C \equiv \measuredangle C'$, then

$$\frac{e(AB)}{e(A'B')} = \frac{e(BC)}{e(B'C')} = \frac{e(CA)}{e(C'A')}.$$

Dem. Use Definitions 6 and 3, Theorem 23, etc., in connection with Theorem 22 of HILBERT's *Grundlagen der Geometrie*.

Lemma II. If BAC is a right angle, then $e^2(AB) + e^2(AC) = e^2(BC)$.

Dem. There is (cf. R. G., Art. 47) a point D on BC such that AD is perpendicular to BC. D is between B and C. Otherwise (as may easily be seen, cf. R. G., Art. 66, etc.), either AD and AC or AD and AB could not meet on the A side of BC. According to S, $\measuredangle ACB \equiv \measuredangle BAD$ and

* See Definitions 1–5.

$\measuredangle CAD \equiv \measuredangle ABC$. It follows by Lemma I, Theorem 23 and Definitions 6 and 1, that

$$e^2(AB) + e^2(AC) = e(BD)e(BC) + e(CD)e(BC)$$

$$= \{e(BD) + e(CD)\}e(BC) = e^2(BC).$$

THEOREM 24. *Given α, β there exists γ such that $\gamma^2 = \alpha^2 + \beta^2$.*

Dem. This theorem is evidently true for the case in which α or $\beta = 0$. If α and β are different from 0, then by Theorems 6 and 15 there exist a, b, c, d, all less than 1 in absolute value, such that $\alpha = a/b$, $\beta = c/d$. By Theorem 23 $\alpha^2 + \beta^2 = [(ad)^2 + (bc)^2]/(bd)^2$. But, by Theorem 15 and Definition 6, ad and bc are lengths of segments. Hence there exists a right-angled triangle BAC such that the length of AB is ad and the length of AC is bc. According to Lemma 2, $e^2(BC) = (ad)^2 + (bc)^2$. Therefore, by Theorem 23, $\alpha^2 + \beta^2 = e^2(BC)/(bd)^2 = [e(BC)/bd]^2$.

Convention. Two straight lines OX and OY perpendicular to each other are selected as axes of coördinates. If P is any point and D and E are the feet of the perpendiculars from P to OX and OY respectively, then x_P means 0, $e(PE)$, or $-e(PE)$ according as P is on OY, on the X-side of OY, or on the non-X-side of OY, and y_P means 0, $e(PD)$, or $-e(PD)$, according as P is on OX, on the Y-side of OX, or on the non-Y-side of OX.

As a result of this convention we have the theorem:

THEOREM 25. *Every point P is represented by one and only one sensed pair of cöordinates (x_P, y_P) and every sensed pair of cutting rays represents one and only one point.*

THEOREM 26. *Given a straight line AB, there exist α, β, γ (α, β not both 0) such that the x and y of every point on AB satisfy the equation $\alpha x + \beta y + \gamma = 0$.*

Dem. Use S, Theorems 1 and 23, lemma I, Definitions 6 and 1, also R. G., articles 47 and 49, and HILBERT's axiom IV, 4, Axiom IV, 1, Theorem 11, etc.

THEOREM 27. *If B is between A and C and $x_A \neq x_B$, then $x_A < x_B < x_C$ or $x_A > x_B > x_C$. If B is between A and C and $x_A = x_B$, then $y_A > y_B > y_C$ or $y_A < y_B < y_C$.*

Dem. Use Theorems 1 and 23, R. G., 47, and HILBERT's Theorem 5 et seq.

THEOREM 28. *The length $e(AB)$ equals $\sqrt{(x_A-x_B)^2+(y_A-y_B)^2}$ and thus the relation $e(AB) = e(A'B')$ is a necessary and sufficient condition for the equality*

$$(x_A - x_B)^2 + (y_A - y_B)^2 = (x_{A'} - x_{B'})^2 + (y_{A'} - y_{B'})^2.$$

Dem. Use lemma 2 in connection with convention preceding Theorem 25.

THEOREM 29. *The existence of α, β, γ, δ satisfying the relations*

$$x_{A'} = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} x_A - \frac{\beta}{\sqrt{\alpha^2+\beta^2}} y_A + \gamma,$$

$$x_{B'} = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} x_B - \frac{\beta}{\sqrt{\alpha^2+\beta^2}} y_B + \gamma,$$

$$y_{A'} = \frac{\beta}{\sqrt{\alpha^2+\beta^2}} x_A + \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} y_A + \delta,$$

$$y_{B'} = \frac{\beta}{\sqrt{\alpha^2+\beta^2}} x_B + \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} y_B + \delta,$$

is a necessary and sufficient condition for the congruence $AB \equiv A'B'$.

In view of Theorems 23–27 and Theorem 29 it may be easily seen that one could proceed in a manner similar to that indicated by HILBERT in § 9 of his *Grundlagen der Geometrie* and show that all of his axioms hold true of our geometry, were it not for one obstacle, namely that there is not necessarily a perfect one-to-one correspondence between the set of all points and the set of all sensed pairs of α's. If α is not a cutting ray then (α, β) does not correspond to a point in manner indicated in Theorem 25.

It is natural then to fill up this gap by means of the following definitions:

Definition 7. Every sensed pair (α, β) is called an ideal point and "ideal point" means such a sensed pair; α and β are called its coördinates.

Definition 8.* An ideal straight line means the set of all existent points, real and ideal, whose coördinates satisfy an equation of the form $\alpha x + \beta y + \gamma = 0$.

Definition 9. If of the points A, B, C at least one is ideal then ABC means $x_A > x_B > x_C$ or $x_A < x_B < x_C$ unless $x_A = x_B$, in which case ABC means $y_A > y_B > y_C$ or $y_A < y_B < y_C$.

Definition 10. If one of the points A, A', B, B' is ideal then "$AB \equiv A'B'$" means there exist α, β, γ, δ satisfying the relations

$$x_{A'} = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} x_A - \frac{\beta}{\sqrt{\alpha^2+\beta^2}} y_A + \gamma,$$

$$x_{B'} = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} x_B - \frac{\beta}{\sqrt{\alpha^2+\beta^2}} y_B + \gamma,$$

$$y_{A'} = \frac{\beta}{\sqrt{\alpha^2+\beta^2}} x_A + \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} y_A + \delta,$$

$$y_{B'} = \frac{\beta}{\sqrt{\alpha^2+\beta^2}} x_B + \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} y_B + \delta.$$

* In this definition and in what follows, small letters of the English alphabet are not always used, according to earlier convention, to designate cutting rays exclusively.

In view of Theorems 25, 26, 27, 29 and the corresponding Definitions 6, 7, 8, 9 it is evident that our real and ideal points and straight lines form a system which, with respect to order, congruence and association, is related to our set of α's as HILBERT's geometry of § 9 of his *Festschrift* is related to his number system of that paragraph, and moreover in view of Theorems 23 and 24 it is evident that the set of α's satisfy with respect to the operations of addition $(+)$, multiplication $(\times)$, and the relation $>$, sufficient conditions to enable us to proceed according to the method indicated by HILBERT to prove that our geometry satisfies his axioms of groups I–IV. It remains to be shown that no ideal point is between two real points. This may be proved by use of Definitions 7 and 1 and Theorem 23.

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