CERTAIN PERIODIC ORBITS OF k FINITE BODIES REVOLVING ABOUT A RELATIVELY LARGE CENTRAL MASS*

BY

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§ 1. *The problem.*

Let a system of $k+1$ finite bodies be moving in a plane, subject to the newtonian law of attraction. Let the bodies be spheres, composed of homogeneous concentric layers; and let the mass of one of them, M, be large with respect to the masses of the others, $M_1, \cdots, M_k$. In other words, the distribution of masses is such as is presented by the sun and any number of planets, or by a planet and any number of satellites. For convenience in what follows, the expression, planet and satellites, will be used, with the understanding that it covers also sun and planets.

Among all the possible variations of the configuration of the system, are there any which are periodic with an arbitrarily pre-assigned period, T; that is, do there exist initial conditions (probably dependent upon T and the masses) which result in periodic variations with the required period? It is proposed to discuss in this paper various cases in which periodic orbits of the k finite satellites exist, to show an easy method of constructing the solutions, and finally to make an application to the case of Jupiter's satellites I, II, and III. The fundamental method of analytical continuation of known orbits, due to POINCARÉ, is employed for the demonstrations of existence and convergence; for the construction, and elsewhere, certain variables and methods are used, which approximate to those employed by Professor MOULTON in his lectures on the lunar theory and in his recent contribution to the problem of three bodies.†

Let quantities $\mu, \beta_1, \cdots, \beta_k$, be defined thus:

$$(1) \qquad M\beta_i\mu = M_i \qquad (i=1, \cdots, k),$$

where one of the β's is to be selected arbitrarily. Let a system of integers without common divisor, p_i, $(i = 1, \cdots, k-1)$, and a number q_k be selected arbitrarily, save for one restriction mentioned below, and let the following defi-

* Presented to the Society April 14, 1906. Received for publication February 26, 1907.

† *A class of periodic solutions of the problem of three bodies with an application to the lunar theory*, Transactions of the American Mathematical Society, vol. 7 (1906), pp. 537–577.

nitions be made:

(2) $$\nu = \frac{2\pi}{T} = \frac{n_k}{q_k} = \frac{n_i - n_k}{p_i} \qquad (i = 1, \cdots, k-1),$$

(3) $$n_i^2 a_i^3 = \kappa^2 M \qquad (i = 1, \cdots, k),$$

where κ^2 denotes the gravitational constant, and where, of the three values of a_i satisfying (3), that one is to be selected which is real. Also let the notation be so selected that $a_1, \cdots, a_k$ are in ascending order of magnitude, the p_i being so selected that no two of the a_i are equal.

If μ were zero—that is, if the satellites were "infinitesimal"—possible orbits would be conic sections about the planet with $a_1, \cdots, a_k$ as major semi-axes; from (3) it follows that the mean angular velocities would be $n_1, \cdots, n_k$. It is quite immaterial whether any of the n_i are negative: the results obtained hold, irrespective of retrograde motion of some of the bodies. In case all the eccentricities were zero, the configuration of the system would undergo periodic variations with period T; for it follows from (2) that

$$\frac{2\pi}{n_i - n_k} = \frac{T}{p_i},$$

or each synodic period is a sub-multiple of T. This condition being satisfied,* the motion of the system would be periodic with respect to a line through M, rotating with uniform angular velocity—that of M_k, or, indeed, that of any other body,† M_j—though whether or not the system ever returns to the same position in space depends upon whether q_k is rational or irrational.

[If, in some orbit, the eccentricity were not zero, then the necessary and sufficient conditions for periodicity are that each sidereal period be a sub-multiple of T, which requires that q_k be an integer, but permits common divisors for the p_i. This case is not treated in this paper.]

In describing the periodic orbits mentioned, the infinitesimal satellites would be subject to certain initial conditions; that is, at the instant $t = t_0$ the $2k$ coördinates and their derivatives with respect to the time would have certain

* POINCARÉ, in dealing with three satellites [*Méthodes Nouvelles de la Mécanique Céleste*, vol. 1, pp. 154–6], with the apparent implication that his remarks apply equally to a larger number of bodies, formulates the condition thus: Integers, α, β, γ, mutually prime exist, such that

$$\alpha + \beta + \gamma = 0, \qquad \alpha n_1 + \beta n_2 + \gamma n_3 = 0.$$

Evidently, in the case of three satellites, this condition is equivalent to (2), since $(n_1 - n_3)/\beta = (n_2 - n_3)/(-\alpha)$; but, for a greater number, it is not so. Thus, if $n_1 = 2\sqrt{2}$, $n_2 = 2$, $n_3 = \sqrt{2}$, and $n_4 = 1$, the integers $2, -2, -1, 1$, satisfy a condition similar to POINCARÉ's; but periodicity is impossible. A re-formulation, such as (2), is necessary.

† The commensurability of $n_i - n_k$ $(i = 1, \cdots, k-1)$, evidently involves that of $n_i - n_j$ $(i = 1, \cdots, j-1, j+1, \cdots, k)$. For from $n_i - n_k = p_i\nu$ and $n_j - n_k = p_j\nu$ it follows that $n_i - n_j = (p_i - p_j)\nu$.

values, $c_{ij}(i=1,\cdots,k;\ j=1,\cdots,4)$. If the k finite satellites be subjected to the same initial conditions, the mutual disturbances, in general, destroy periodicity. But it is possible in various cases to vary the initial conditions (that is, let the initial values be $c_{ij}+\Delta c_{ij}$, where the Δc_{ij} are to be determined) so as to preserve the periodicity.

§ 2. *The differential equations.*

Let the common plane of motion of the $k+1$ bodies be selected as the ΞH-plane, the origin being at M, and $M\Xi$ and MH being rectangular axes which rotate in the plane with a uniform angular velocity, N. Let the coördinates of M_i referred to these axes be ξ_i and η_i; then the differential equations of motion are

$$(4)\quad \begin{aligned} &(a)\ \frac{d^2\xi_i}{dt^2}-2N\frac{d\eta_i}{dt}-N^2\xi_i+\kappa^2(M+M_i)\frac{\xi_i}{r_i^3}+\sum_j{}'\kappa^2M_j\left(\frac{\xi_i-\xi_j}{\rho_{ij}^3}+\frac{\xi_j}{r_j^3}\right)=0,\\ &(b)\ \frac{d^2\eta_i}{dt^2}+2N\frac{d\xi_i}{dt}-N^2\eta_i+\kappa^2(M+M_i)\frac{\eta_i}{r_i^3}+\sum_j{}'\kappa^2M_j\left(\frac{\eta_i-\eta_j}{\rho_{ij}^3}-\frac{\eta_j}{r_j^3}\right)=0,\end{aligned}$$

where

$$r_i^2=\xi_i^2+\eta_i^2,\qquad \rho_{ij}^2=(\xi_j-\xi_i)^2+(\eta_j-\eta_i)^2,$$

and $\sum_j'$ means $\sum_{j=1}^{j=k}$, $(j\neq i)$.

Except in proving a certain symmetry theorem, these coördinates are less convenient than polar coördinates referred to rotating reference lines. Besides (1), (2) and (3), let the following definitions be made:

$$(5)\quad \begin{aligned} &\alpha_{ij}=a_i/a_j,\quad \nu q_i=n_i && (i=1,\cdots,k),\\ &a_j\sigma_{ij}=\rho_{ij},\qquad \delta_{ij}=\beta_jq_1^2\alpha_{1i}\alpha_{1j}^2 && (j\neq i),\\ &\nu t=\tau,\qquad \phi_{ji}=(p_j-p_i)\tau+(\lambda_j-\lambda_i), \end{aligned}$$

where the λ_i are arbitrary constants, later to be taken as the longitudes of the M_i at the origin of time, for $\mu=0$. Let polar coördinates be introduced by the equations

$$(6)\quad \begin{aligned} &\xi_i=r_i\cos u_i, && \eta_i=r_i\sin u_i,\\ &r_i=a_ix_i, && u_i=w_i+p_i\tau+\lambda_i, \end{aligned}$$

N being chosen equal to n_k, so that w_i is the longitude of M_i referred to a line rotating with uniform speed n_i. The differential equations become

$$(7)\ (a)\ x_iw_i''+2x_i'(w_i'+q_i)+\mu\sum_j{}'\delta_{ij}x_j\sin(\phi_{ji}+w_j-w_i)\left(\frac{1}{x_j^3}-\frac{1}{\sigma_{ij}^3}\right)=0,$$

$$(7)\ (b)\ x_i'' - x_i(w_i' + q_i)^2 + \frac{q_i^2(1+\beta_i\mu)}{x_i^2}$$
$$+ \mu \sum_j{}' \delta_{ij}\left[\frac{a_{ij}x_i}{\sigma_{ij}^3} + x_{ij}\cos(\phi_{ji} + w_j - w_i)\left(\frac{1}{x_j^3} - \frac{1}{\sigma_{ij}^3}\right)\right] = 0,$$

where

$$a_j^2\sigma_{ij}^2 = a_i^2x_i^2 + a_j^2x_j^2 - 2a_ia_jx_ix_j\cos(\phi_{ji} + w_j - w_i),$$

and where the accents on the variables indicate derivatives with respect to τ.

§ 3. *Symmetry theorem.*

Definition. A *symmetrical conjunction* occurs when all k satellites are in a straight line through the planet, and are moving at right angles to the line.

THEOREM. *If a symmetrical conjunction occurs at an instant, $t = t_0$, then the orbit of each satellite before and after the conjunction is symmetrical, both with regard to geometric equality of figures and with regard to intervals of time.* The proof will be made only for the case $t_0 = 0$, which does not limit the generality since any other case is reduced to this one by the substitution $t = t_1 + t_0$. The differential equations (4) are invariant under the substitution

$$(8)\qquad \bar{\xi}_i = \xi_i, \qquad \bar{\eta}_i = -\eta_i, \qquad \bar{t} = -t.$$

Consequently every integral of (4) is transformed by (8) into some integral of (4). Moreover, the initial conditions

$$(9)\qquad \xi_i = a_i, \qquad \eta_i = 0, \qquad \frac{d\xi_i}{dt} = 0, \qquad \text{and} \qquad \frac{d\eta_i}{dt} = b_i,$$

at $t = 0$, are transformed into

$$(9')\qquad \bar{\xi}_i = a_i, \qquad \bar{\eta}_i = 0, \qquad \frac{d\bar{\xi}_i}{d\bar{t}} = 0, \qquad \text{and} \qquad \frac{d\bar{\eta}_i}{d\bar{t}} = b_i,$$

at $\bar{t} = 0$. Therefore, that solution of (4) which satisfies the initial conditions (9) is transformed by (8) into itself. Hence, if

$$(10)\qquad \xi_i = \Phi_i(t), \qquad \eta_i = \Psi_i(t), \qquad (i = 1, \cdots, k),$$

is that solution, then

$$\Phi_i(t) = \Phi_i(\bar{t}) = \Phi_i(-t), \qquad \Psi_i(t) = -\Psi_i(\bar{t}) = -\Psi_i(-t),$$

whence also

$$\xi_i(\eta_1, \cdots, \eta_k) = \xi_i(-\eta_1, \cdots, -\eta_k).$$

It will be noted that the proof holds whatever the value of N. It is geometrically evident that the symmetry, if present at all, is independent of the rate of rotation of the reference line.

§ 4. *Conditions for periodic solutions.*

Since the differential equations (7) are unchanged if τ be replaced by $\tau + 2n\pi$, or t by $t + nT$, (n being an integer), it follows that if

(11) $$x_i = x_i(\tau), \qquad w_i = w_i(\tau) \qquad (i=1, \cdots, k),$$

is a solution, then so is

(12) $$x_i = x_i(\tau + 2n\pi), \qquad w_i = w_i(\tau + 2n\pi).$$

These two will be the same solution, if the coördinates and their derivatives have the same values at $\tau = \tau_0$; i. e., if

(13) $$\begin{aligned} x_i(\tau_0 + 2n\pi) &= x_i(\tau_0), \quad & w_i(\tau_0 + 2n\pi) &= w_i(\tau_0), \\ x_i'(\tau_0 + 2n\pi) &= x_i'(\tau_0), \quad & w_i'(\tau_0 + 2n\pi) &= w_i'(\tau_0). \end{aligned}$$

If these conditions are satisfied, then, for all values of τ,

$$x_i(\tau + 2n\pi) = x_i(\tau), \qquad w_i(\tau + 2n\pi) = w_i(\tau);$$

that is, (13) are sufficient conditions for the periodicity of the solutions. That they are also necessary is obvious.

Special case. In the case of a symmetrical conjunction at $\tau = 0$, other sufficient conditions can be formulated. For, if $x_i'(0) = w_i(0) = 0$, $(i = 1, \cdots, k)$, and if every λ_i is a multiple of π, it follows from the symmetry theorem that

(14) $$\begin{aligned} x_i(\pi) &= x_i(-\pi), \quad & w_i(\pi) &= -w_i(-\pi), \\ x_i'(\pi) &= -x_i'(-\pi), \quad & w_i'(\pi) &= w_i'(-\pi). \end{aligned}$$

But, by (13), if τ_0 be put equal to $-\pi$, the conditions for periodicity of $x_i(\tau)$ and $w_i(\tau)$ are

(15) $$\begin{aligned} &(a) \quad x_i(\pi) = x_i(-\pi), \qquad &(c) \quad w_i(\pi) = w_i(-\pi), \\ &(b) \quad x_i'(\pi) = x_i'(-\pi), \qquad &(d) \quad w_i'(\pi) = w_i'(-\pi). \end{aligned}$$

Of these conditions, (a) and (d) are satisfied by virtue of (14); while (b) and (c) in connection with (14) require $x_i'(\pi) = w_i(\pi) = 0$. It may then be stated that sufficient conditions for the periodicity of x_i and w_i (with period in t equal to T), are

(16) $$\begin{aligned} &(a) \quad x_i'(0) = 0, \qquad w_i(0) = 0, \qquad \lambda_i = 0, \pi, \\ &(b) \quad x_i'(\pi) = 0, \qquad w_i(\pi) = 0. \end{aligned}$$

For a symmetrical conjunction, when conditions (16a) are imposed, conditions (b) are necessary as well as sufficient.

§ 5. *Nature of the periodicity conditions.*

POINCARÉ has shown * that for a type of differential equations including (7) the solutions are developable as power series in μ and in the Δc_{ij} [see § 1], whose coefficients are functions of τ. And, for an arbitrarily assigned interval for τ, T_1 (herein taken greater than $2n\pi$), there exist values of μ and the Δc_{ij} sufficiently small (not zero), to make the series convergent for $0 \leqq \tau \leqq T_1$. Hence the conditions (13) and (16) are that certain power series in the $4k+1$ parameters, μ, Δc_{ij} $(i = 1, \cdots, k;\ j = 1, \cdots, 4)$ shall vanish.

Now, from the theory of implicit functions it is well known † that if (1) $f_i(y_1, \cdots, y_{4k};\ \mu)$ $(i = 1, \cdots, 4k)$ are holomorphic in the vicinity of $\mu = y_j = 0$ $(j = 1, \cdots, 4k)$, (2) $f_i(0, \cdots, 0;\ 0) = 0$ $(i = 1, \cdots, 4k)$, and if (3) the functional determinant (jacobian) of the f_i with respect to the y_j is distinct from zero for $\mu = y_j = 0$ $(j = 1, \cdots, 4k)$, then there exist unique functions $y_j = y_j(\mu)$, which (*a*) are holomorphic in the vicinity of $\mu = 0$, (*b*) $f_i(y_1, \cdots, y_{4k};\ \mu) = 0$, (*c*) vanish with μ. Even when (3) is not satisfied, there sometimes exist solutions having the properties (*a*), (*b*) and (*c*).

Let this theorem be applied to the conditions for periodicity, whose left-hand members, being convergent power series in μ and the Δc_{ij} and vanishing for $\mu = \Delta c_{ij} = 0$, evidently satisfy (1) and (2) above. The functional determinant, evaluated for $\mu = \Delta c_{ij} = 0$, is simply the determinant of the coefficients of the linear terms. In certain cases it vanishes and in others it does not. Under the latter circumstances, at least, periodic orbits exist, since the equations of condition define the Δc_{ij} as holomorphic functions of μ. For every value of μ sufficiently small there exist initial conditions [dependent upon T, μ, q_k, the β_i, and the p_i $(i = 1, \cdots, k)$] such that the orbits described are periodic with the required period. It is evident that for smaller and smaller values of μ smaller and smaller deviations from the initial conditions of undisturbed motion are necessary in order to get periodic orbits. The periodic orbits for $\mu \neq 0$ may then be said to "grow out of" the undisturbed periodic orbits as μ grows from zero.

In this paper are discussed only those orbits which grow out of circles; and it is convenient to subdivide these into various types.

Type I. The system has a symmetrical conjunction.

Type II. The infinitesimal system has a symmetrical conjunction, but the finite system has not.

Type III. Neither system has a conjunction.

§ 6. *Integration of the differential equations as power series in parameters.*

It will be necessary to obtain the first few terms of the developments mentioned in the preceding section. Instead of increments Δc_{ij} to the initial

* Loc. cit., pp. 56–62.

† JORDAN, *Cours d'Analyse*, vol. 1, p. 82.

undisturbed values of the coördinates it will be more convenient, in finding the properties of the solutions, to employ parameters Δn_i, e_i, ω_i, τ_i, defined as follows. At $\tau = 0$ let

$$\begin{aligned} x_i &= (1 + \Delta c_{i1}) &&= (1 + \Delta n_i)^{-\frac{2}{3}}(1 - e_i \cos \theta_i), \\ \nu x_i' &= \Delta c_{i2} &&= n_i(1 + \Delta n_i)^{\frac{1}{3}} \frac{e_i \sin \theta_i}{1 - e_i \cos \theta_i}, \\ v_i - \lambda_i &= w_i = \Delta c_{i3} &&= \omega_i + \arccos \frac{\cos \theta_i - e_i}{1 - e_i \cos \theta_i} \qquad (i = 1, \cdots, k), \\ \nu w_i' &= n_i \Delta c_{i4} &&= \frac{n_i(1 + \Delta n_i)\sqrt{1 - e_i^2}}{(1 - e_i \cos \theta_i)^2} - n_i, \\ &- q_i(1 + \Delta n_i)\tau_i = \theta_i - e_i \sin \theta_i, \end{aligned} \tag{17}$$

(the v_i, being the true longitudes from a fixed reference line, equal $u_i + q_k \tau$).

It is evident that the Δc_{ij} are holomorphic functions of the Δn_i, e_i, ω_i and τ_i for sufficiently small values of the latter quantities. Consequently, solutions of (7) exist also as power series in the new parameters. Further, since the real positive values of the radicals and the smallest values of the inverse cosines are to be taken in (17), the Δc_{ij} are given uniquely in terms of the Δn_i, e_i, ω_i and τ_i. From these two facts it follows that, if the latter quantities can be determined as unique power series in μ, satisfying the conditions for periodicity, then also there exist for the Δc_{ij} unique power series in μ, satisfying the conditions. Conversely, while the jacobian of the Δc_{ij} with respect to the new parameters is zero for $\Delta n_i = e_i = \omega_i = \tau_i = 0$, yet, in the only case where discussion will be necessary (viz. for $\Delta c_{i2} = \Delta c_{i3} = 0$, whence $\omega_i = \tau_i = 0$), the solution for the Δn_i and e_i in terms of the Δc_{i1} and Δc_{i4} is unique. For the jacobian of the Δc_{i1} and Δc_{i4} with respect to the Δn_i and e_i is distinct from zero for $\Delta n_i = e_i = 0$. Hence, in this case, if the Δc_{i1} and Δc_{i4} exist as unique series in μ, satisfying the periodicity conditions, so also must the Δn_i and e_i exist as such series.

In the developments of the coördinates as power series in μ and the new parameters all those terms independent of μ may be obtained, together with a knowledge of their properties, in the following simple manner. The terms in question are those remaining when μ is put equal to zero; and are, therefore, the solution of the problem of k infinitesimal satellites when the initial conditions are (17); in other words, the solutions of k two-body problems.

The dynamical meaning of the new parameters is then evident. The orbits of the infinitesimal system, subjected to initial conditions (17), are ellipses, in which the mean angular motions, major semi-axes, eccentricities, longitudes of pericenter, and times of pericenter passage are respectively

$$n_i(1 + \Delta n_i), \qquad a_i(1 + \Delta n_i)^{-\frac{2}{3}}, \qquad e_i, \qquad \lambda_i + \omega_i, \qquad \text{and} \qquad \frac{\tau_i}{\nu}.$$

If in the development of x_i the coefficient of $\Delta n_i^f e_i^g \omega_i^h \tau_i^k$ be denoted by $x_{i,fghk}$, then, by applying TAYLOR's theorem to the well known developments of the coördinates in elliptic motion as power series in the eccentricity,* it is found that the following coefficients of first and second degree terms do not vanish:

$$
(18)\quad
\begin{array}{lll}
x_{i,0000} = 1, & x_{i,1000} = -\frac{2}{3}, & x_{i,0100} = -\cos q_i\tau, \\
x_{i,2000} = \frac{5}{9}, & x_{i,0200} = \sin^2 q_i\tau, & x_{i,0101} = -q_i \sin q_i\tau, \\
x_{i,1100} = \frac{2}{3}\cos q_i\tau + q_i\tau \sin q_i\tau, & & w_{i,1000} = q_i\tau \\
w_{i,0100} = 2\sin q_i\tau, & w_{i,0010} = 1, & w_{i,0001} = q_i, \\
w_{i,0200} = \frac{5}{4}\sin 2q_i\tau, & w_{i,0101} = -2q_i\cos q_i\tau, & w_{i,1100} = 2q_i\tau\cos q_i\tau.
\end{array}
$$

From simple dynamical considerations the following important properties may be established. Let $x_{i,fghk}$ be written $x_{i,fghk}^{(q_i\tau)}$ to indicate its dependence upon τ. Then

$$
(19)\qquad x_{i,0g00}'^{(m\pi)} = w_{i,0g00}^{(m\pi)} = 0 \qquad (i = 1, \cdots, k),
$$

where m is an integer. For, the coefficients $x'_{i,0g00}$, etc., are those of the terms which do not involve Δn_i, ω_i and τ_i, these terms being obtained by putting $\Delta n_i = \omega_i = \tau_i = 0$, in the developments. But, for these parameters equal to zero, the initial positions are apses, and the periods (in t) are $2\pi/n_i$. Hence, at $\tau = m\pi/q_i$, M_i is at an apse; and $x'_i = w_i = 0$, whatever the value of e_i. Since this is true for a range of values of e_i, it follows that the coefficient of each power of e_i in x'_i and in w_i is zero.

It is evident that in the terms independent of μ appear only those parameters whose subscript is the same as that of the coördinate developed; the terms involving μ introduce, however, the other $4(k-1)$ parameters.

Terms involving μ.

The only terms involving μ whose coefficients are needed in the sequel are μ and $\mu e_j (j = 1, \cdots, k)$. Let the coefficient of μ in the development of x_i be $x_i(0;\ \tau)$, and that of μe_j be $x_i(j;\ \tau)$, let the coefficients of the same quantities in w_i be respectively $w_i(0;\ \tau)$ and $w_i(j;\ \tau)$, $(i = 1, \cdots, k;\ j = 1, \cdots, k)$. The process of finding these depends as follows upon two properties of the solutions.

(a) Since they must satisfy the differential equations identically in the parameters, the equating of coefficients of corresponding powers on both sides furnishes sets of differential equations for the successive coefficients in the solutions.

(b) The arbitrary constants which the successive coefficients carry are determined by the condition that the solutions shall reduce identically to (17) at $\tau = 0$.

* MOULTON, *Introduction to Celestial Mechanics*, pp. 154–5.

For each pair of coefficients, $x_i(f;\ \tau)$ and $w_i(f;\ \tau)$, $(f=0, \cdots, k)$, equations (7) give two simultaneous differential equations of the second order. The one from (7a) can be integrated once immediately, and its integral combined with the equation from (7b) renders the latter a well known type,

$$(20) \qquad x_i''(f;\ \tau) + q_i^2 x_i(f;\ \tau) + \sum_{m=0}^{m=\infty} (\gamma_m \cos m\tau + \delta_m \sin m\tau) + \alpha\tau = 0 .$$

Its solution, $x_i(f;\ \tau)$, when substituted into the first integral, permits the final integration for $w_i(f;\ \tau)$. The initial conditions are

$$(21) \qquad x_i(f;\ 0) = x_i'(f;\ 0) = w_i(f;\ 0) = w_i'(f;\ 0) = 0 \qquad (i=1, \cdots, k;\ f=0, \cdots, k),$$

for the conditions (17) do not involve μ at all.

Now the form of the solution varies greatly according as a term $\cos q_i\tau$ or $\sin q_i\tau$ is or is not present in (20). In the former case the solutions contain a so-called Poisson term, $\tau \cos q_i\tau$, or $\tau \sin q_i\tau$, and in the latter case they do not. In all the $x_i(f; \tau)$ and $w_i(f; \tau)$, $(f = 1, \cdots, k)$, a Poisson term is present; they are present in the $x_i(0\ ; \tau)$, if, and only if, for some pair of the n_i, say n_f and n_g, there exists an integer, J, such that

$$(22) \qquad J \cdot (n_f - n_g) = n_g .$$

The meaning and consequences of such a relation will be discussed in § 11.

In performing the integrations it is necessary to expand

$$(1 - 2\epsilon_{ij} \cos \phi_{ji} + \epsilon_{ij}^2)^{-s/2} \qquad (s=3, 5),$$

as a cosine series; where, for the sake of a uniform notation, the following definitions are made:

$$(23) \qquad \begin{aligned} \epsilon_{ij} &= \alpha_{ji}, \quad \text{and} \quad \eta_{ij} = \alpha_{ji}, \quad \text{if } j < i, \\ \epsilon_{ij} &= \alpha_{ij}, \quad \text{and} \quad \eta_{ij} = 1, \quad \text{if } j > i. \end{aligned}$$

Also, where it is not essential to know the value of a Fourier sine or cosine series, it will be denoted simply by $F^s(\theta)$ or $F^c(\theta)$ respectively, where θ is the argument according to whose multiples the series proceeds. Now

$$(24) \qquad \begin{aligned} (1 - 2\epsilon_{ij} \cos \phi_{ji} + \epsilon_{ij}^2)^{-\frac{3}{2}} &= \sum_{m=0}^{m=\infty} F_m(\epsilon_{ij}) \cdot \cos m\phi_{ji}, \\ (1 - 2\epsilon_{ij} \cos \phi_{ji} + \epsilon_{ij}^2)^{-\frac{5}{2}} &= \sum_{m=0}^{m=\infty} G_m(\epsilon_{ij}) \cdot \cos m\phi_{ji}, \end{aligned}$$

where the F_m and G_m are well known power series in ϵ_{ij}, beginning with ϵ_{ij}^m.

Finally, the desired coefficients are, for $i = 1, \cdots, k$,

$$
(25)\quad
\begin{aligned}
x_i(0;\ \tau) &= A_{i0} + B_{i0}\cos q_i\tau + C_{i0}\sin q_i\tau + H_{i0}\tau\sin q_i\tau + \sum_j{}' F^c(\phi_{ji}),\\
w_i(0;\ \tau) &= D_{i0} - 2B_{i0}\sin q_i\tau + 2C_{i0}\cos q_i\tau\\
&\quad + 2H_{i0}\tau\cos q_i\tau + \sum_j{}' F^s(\phi_{ji}) - \frac{2H_{i0}}{q_i}\sin q_i\tau + E_{i0}\tau,
\end{aligned}
$$

and for $f \neq i$,

$$
(26)(A)\quad
\begin{aligned}
x_i(f;\ \tau) &= A_{if} + B_{if}\cos q_i\tau + C_{if}\sin q_i\tau\\
&\quad + \delta_{if}L_{if}\tau\sin(q_i\tau + \lambda_i - \lambda_f) + \delta_{if}\{F^c(\phi_{fi}) + F^s(\phi_{fi})\},\\
w_i(f;\ \tau) &= D_{if} + E_{if}\tau - 2B_{if}\cdot\sin q_i\tau + 2C_{if}\cdot\cos q_i\tau\\
&\quad - \frac{2\delta_{if}L_{if}}{q_i}\sin(q_i\tau + \lambda_i - \lambda_f) + 2\delta_{if}L_{if}\tau\cos(q_i\tau + \lambda_i - \lambda_f)\\
&\quad + \delta_{if}\{F^s(\phi_{fi}) + F^c(\phi_{fi})\},
\end{aligned}
$$

while, for $f = i$.

$$
(26)(B)\quad
\begin{aligned}
x_i(i;\ \tau) &= A_{ii} + B_{ii}\cos q_i\tau + C_{ii}\sin q_i\tau + \sum_j{}'\delta_{ij}\{F^c(\phi_{ji}) + F^s(\phi_{ji})\}\\
&\quad + \tau\sin q_i\tau\cdot[H_{ii} + \sum_j{}'\delta_{ij}M_{ij}],\\
w_i(i;\ \tau) &= D_{ii} + E_{ii}\tau - 2B_{ii}\sin q_i\tau + 2C_{ii}\cos q_i\tau\\
&\quad + \sum_j{}'\delta_{ij}\{F^s(\phi_{ji}) + F^c(\phi_{ji})\} - \frac{2}{q_i}\sin q_i\tau\sum_j{}'\delta_{ij}M_{ij}\\
&\quad + 2\tau\cos q_i\tau[H_{ii} + \sum_j{}'\delta_{ij}M_{ij}].
\end{aligned}
$$

Of the coefficients and constants of integration, only the following are of importance:

$$
\begin{aligned}
L_{if} &= -\frac{\eta^3_{if}}{2q_i}[\tfrac{9}{2}F_0 + \tfrac{1}{4}F_2 - \tfrac{9}{2}\eta^2_{if}(1+\alpha^2_{if})G_0 + \tfrac{11}{8}\alpha_{if}\eta_{if}G_1\\
&\qquad + \tfrac{3}{4}\eta^2_{if}(1+\alpha^2_{if})G_2 - \tfrac{3}{8}\alpha_{if}\eta_{if}G_3],\\
M_{ij} &= \frac{\eta^3_{ij}}{2q_i}[\alpha_{ij}F_0 + 2F_1 - \alpha_{ij}\eta^2_{ij}(\tfrac{15}{2} + 3\alpha^2_{ij})G_0 + 3\alpha^2_{ij}\eta^2_{ij}G_1 + \tfrac{9}{4}\alpha_{ij}\eta^2_{ij}G_2],
\end{aligned}
$$

(27) $H_{i0} \neq 0$, if, and only if, (22) holds,

$$
C_{i0} = \sum_j{}'\delta_{ij}\sum_{m=1}^{\infty}E_m(\epsilon_{ij})\sin m(\lambda_j - \lambda_i),
$$

$$
H_{ii} = E_{i0} + \frac{1}{q_i}\sum_j{}'\delta_{ij}\eta^3_{ij}[\alpha_{ij}F_0(\epsilon_{ij}) - \tfrac{1}{2}F_1(\epsilon_{ij})],
$$

the $E_m(\epsilon_{ij})$ being certain linear functions of the $F_m(\epsilon_{ij})$. Thus,

$$(27') \quad \begin{aligned} q_i[m^2(q_j-q_i)^2-q_i^2]E_m(\epsilon_{ij}) &= m(q_j-q_i)\eta_{ij}^3(\alpha_{ij}F_m-\tfrac{1}{2}[F_{m+1}+F_{m-1}]) \\ &\quad - q_i\eta_{ij}^3(F_{m+1}-F_{m-1}), \quad \text{if } m \neq 1, \\ q_i[(q_j-q_i)^2-q_i^2]E_1(\epsilon_{ij}) &= (q_j-3q_i)+(q_j-q_i)\eta_{ij}^3(\alpha_{ij}F_1-\tfrac{1}{2}[F_2+2F_0]) \\ &\quad - q_i\eta_{ij}^3(F_2-2F_0). \end{aligned}$$

§ 7. *Existence of periodic orbits. Type I.*

For all values of μ and the Δn_i, e_i, ω_i and τ_i, sufficiently small, the solutions of differential equations (7), with initial conditions (17), are given by series in those parameters, the first few coefficients of which have been tabulated in (18), (25) and (26).

For $\mu = 0$, a solution, with period 2π (or T in t), is

$$(28) \qquad x_i = 1, \qquad w_i = 0;$$

whence

$$r_i = a_i, \qquad u_i = \lambda_i + p_i\tau,$$

where $\Delta n_i = e_i = \omega_i = \tau_i = 0$ $(i = 1, \cdots, k)$. For $\mu \neq 0$ these $4k$ parameters can be so determined that the solutions are still periodic with the required period.

Let $\omega_i = \tau_i = 0$, and $\lambda_i = 0, \pi$ $(i = 1, \cdots, k)$. The satellites have, then, at $\tau = 0$, a symmetrical conjunction; therefore, by (16), the necessary and sufficient conditions for periodicity are

$$(29) \quad \begin{aligned} &(a)\ 0 = \Delta n_i(q_i\pi) + e_i(2\sin q_i\pi) + \mu w_i(0;\pi) + \cdots, \\ &(b)\ 0 = \Delta n_i(0) + e_i(q_i\sin q_i\pi)\mu x_i'(0;\pi) + \Delta n_i e_i x'_{i,1100}(\pi) \\ &\qquad + \mu\sum_{f=1}^{f=k} e_f x_i'(f;\pi) + \cdots \quad (i=1,\cdots,k). \end{aligned}$$

Evidently these functions $w_i(\pi)$ and $x_i'(\pi)$ satisfy conditions (1) and (2) of § 5; it remains simply to examine condition (3). The jacobian of the $w_i(\pi)$ and $x_i'(\pi)$ with respect to the Δn_i and e_i, taken for $\Delta n_i = e_i = \mu = 0$, will be denoted by Δ_1, and has the value

$$(30) \qquad \Delta_1 \equiv \pi^k \cdot \prod_{i=1}^{i=k} q_i^2 \sin(q_i\pi),$$

so that its vanishing depends upon the q_i. Since, from (2) and (5), $q_i = q_k + p_i$ $(i = 1, \cdots, k-1)$, it follows that Δ_1 vanishes if, and only if, the arbitrary q_k be selected as an integer.

Case I: *q_k is not an integer.* Since $\Delta_1 \neq 0$, condition (3) of § 5 is satisfied; hence, periodic orbits exist, the Δn_i and e_i (and, therefore, the Δc_{i1} and Δc_{i4})

existing as unique power series in μ. The presence of q_k and of the arbitrary integers p_i shows that for every value of μ (sufficiently small) and for every given set of mass-ratios, β_i/β_k there exist a k-fold infinitude of periodic orbits of this type.

Case II: q_k *is an integer.* Here $\Delta_1 = 0$. Nevertheless the jacobian of the $w_i(\pi)$ with respect to the Δn_i, taken at $\Delta n_i = e_i = \mu = 0$ is

$$(31) \qquad \Delta_2 = \pi^k \prod_{i=1}^{i=k} q_i \neq 0.$$

[The only possibility for $q_i = 0$ is $p_i = -q_k$ which requires $n_i = 0$. The infinitesimal system would have one body, the major axis of whose orbit is infinite, or else the body moves in a straight line through the planet. Let such a selection for p_i be excluded.] Hence (29a) can be solved for the Δn_i as power series in the e_i and μ, converging for sufficiently small values of the latter quantities. Now, by (19), every term in (29a), (b) has either μ or some Δn_i as a factor; hence the solutions have the form

$$(32) \qquad \Delta n_i = \mu \cdot P_i(e_j, \mu) \qquad (i=1, \cdots, k; j=1, \cdots, k).$$

If the power series (32) be substituted into (29b), the resulting series converge for sufficiently small values of μ and e_j, and contain μ as a factor. (This merely means that $\Delta n_j = \mu = 0$ satisfy the periodicity conditions whatever be the e_j, and is nothing new.) If μ is divided out,* relations are obtained among the e_j and μ, of the form

$$(29b') \quad 0 = x_i'(0;\pi) - e_i \frac{1}{q_i \pi} x'_{i,1100}(\pi) \cdot w_i(0;\pi) + \sum_{j=1}^{j=k} e_j x_i'(j;\pi) + \mu\{\ \}_i + \cdots.$$

At this point two questions of importance arise; viz. as to the vanishing of the $x_i'(0;\pi)$, and as to the vanishing of the determinant of the coefficients of the linear terms, e_j. By (25) and (27) every $x_i'(0;\pi)$ is zero unless the relation (22) holds for some pair of the n_i. When such a relation does hold, the equations (29b') are not satisfied by $e_j = \mu = 0$, so that solutions for the e_j in μ, vanishing with μ, do not exist. Hence, periodic orbits of Type I, "growing out of circular orbits" do not exist, if, for any n_f and n_g, $J(n_f - n_g) = n_g$, where J is an integer.

When no such relation exists,† the equations (29b') are satisfied by $e_j = \mu = 0$; it remains to examine the determinant, Δ_3, of the first degree terms in e_j. Δ_3 involves in each of its elements power series in the ϵ_{ij}, or a_i/a_j, a_j/a_i; and it is not known that there are no sets of values of the ϵ_{ij} for which $\Delta_3 = 0$. It can,

* It is precisely this step which makes the selection of the parameters Δn_i, e_i, especially advantageous.

† That q_k an integer does not involve the existence of such a relation is shown in § 11.

however, be shown that there is an infinite number (a range) of values for which the determinant is distinct from zero.

Let P_{if} be any element of Δ_3, the first subscript indicating the row and the second the column; then

$$(33)\qquad \begin{aligned} P_{if} &= x_i'(f;\pi) = q_i(-1)^{q_i}\pi\delta_{if}L_{if}, \qquad f \neq i, \\ P_{ii} &= x_i'(i;\pi) - q_i(-1)^{q_i}w_i(0;\pi) \\ &= q_i(-1)^{q_i}\pi\sum_j{}'\delta_{ij}[q_iM_{ij} + \eta_{ij}^3\{\alpha_{ij}F_0(\epsilon_{ij}) - \tfrac{1}{2}F_1(\epsilon_{ij})\}], \end{aligned}$$

the simplification in P_{ii} resulting from the last equation of (27). The ϵ_{ij} depend upon ν, q_k, the β_i and the p_i, which were arbitrarily chosen. It will now be shown that for a fixed selection of the β_i, ν and q_k there is an infinite number of selections of the p_i (viz. all for which the ϵ_{ij} are "sufficiently small") for which $\Delta_3 \neq 0$. For convenience, let all the ϵ_{ij} be expressed in terms of a single parameter; thus,

$$(34)\qquad a_i = b_i^2\alpha^{2(k-i)}a_k \qquad (i = 1, \cdots, k-1),$$

where the b_i, ν and q_k (hence also a_k and n_k), are constants independent of α. Every element of Δ_3 is, then, a power series in α; for

$$\epsilon_{ij} = \frac{b_j^2}{b_i^2}\alpha^{2(i-j)}, \qquad \text{and} \qquad \eta_{ij} = \frac{b_j^2}{b_i^2}\alpha^{2(i-j)}, \qquad \text{if } j < i,$$

$$\epsilon_{ij} = \frac{b_i^2}{b_j^2}\alpha^{2(j-i)}, \qquad \text{and} \qquad \eta_{ij} = 1, \qquad \text{if } j > i,$$

$$(35)\qquad \begin{aligned} q_i &= \frac{q_k}{b_i^3}\alpha^{3(i-k)}, \qquad P_{ii} = \alpha^{(6i+4-6k)}Q_{ii}(\alpha), \\ P_{if} &= \alpha^{(10i-4f-6k)}\cdot Q_{if}(\alpha), \qquad \text{if } f < i, \\ P_{if} &= \alpha^{(8f-2i)}\cdot Q_{if}(\alpha), \qquad \text{if } f > i, \end{aligned}$$

where the $Q_{if}(\alpha)$ $(f = 1, \cdots, k)$ are power series in α, *beginning with a constant term.* [In P_{ii} under the sign $\sum_j'$, the lowest power of α for $j < i$ is $(10i - 4j - 6k)$; for $j > i$, the lowest power is $6j - 6k$. For all values of j taken in the summation, the lowest power is evidently $6i + 4 - 6k$]. The lowest power of α in any of the P_{if} is then

$$(36)\qquad \begin{aligned} &\text{for } f < i, \quad 6i + 4 - 6k, \quad \text{viz. for } f = i - 1, \\ &\text{for } f = i, \quad 6i + 4 - 6k, \\ &\text{for } f > i, \quad 6i + 8 - 6k, \quad \text{viz. for } f = i + 1. \end{aligned}$$

Hence, if the factor $\alpha^{6i+4-6k}$ be removed from the ith row $(i = 1, \cdots, k)$, a determinant Δ_4 is obtained such that $\Delta_4 \cdot \alpha^{-k(3k-7)} = \Delta_3$, and each of its elements is a power series in α. In the ith row of Δ_4 two elements, those in columns $i-1$ and i, have present a constant term in their series. The series of every other element begins with a power of α not lower than the fourth. Each term in the development of Δ_4 will be a power series in α, and the only term beginning with a constant is the main diagonal product. For every term must contain one, and only one, element from each row and column. To get a term containing a constant, one must select in the first row the element P_{11}; the only possible selection from the second row is then P_{22}, since P_{21} would be a second element from the first column. Continuing thus, the only selection possible in the ith row is P_{ii}. When expanded, Δ_4 can be re-arranged as a single power series in α, whose constant term (being merely the product of those in the main diagonal) is distinct from zero. Hence, for $\alpha = 0$, $\Delta_4 \neq 0$, and from this it follows that for all values of α, sufficiently small, $\Delta_4 \neq 0$. Also, Δ_4 being a power series, vanishes, if at all, at a finite number of points within the circle of convergence.

But Δ_3 can vanish only with Δ_4. Hence in general, even in the case where q_k is an integer, it is possible to solve ($29b'$) for the e_i in terms of μ (vanishing with μ). The substitution of these solutions into (32) gives the Δn_i, as well as the e_i, as holomorphic functions of μ, for μ sufficiently small. Periodic orbits with the required period, therefore, exist.

In drawing this last conclusion, however, a point of some delicacy arises. The p_i are functions of α in the foregoing argument, and obviously the p_i are not integers (as the formulation of the problem requires that they be), for all values of α on any interval. The question arises as to whether there are, indeed, any values of α, "sufficiently small," for which the p_i are integers. From (35) and $p_i = q_i - q_k$, it follows that

$$(37) \qquad p_i = q_k\left(\frac{1}{b_i^3\alpha^{3(k-i)}} - 1\right) \qquad (i=1,\cdots,k-1).$$

The present discussion will be confined to exhibiting a selection of the b_i, such that there are an infinite number of values of α less than any assigned quantity, for each of which the p_i are integers without common divisor. Let the assigned value be α_0, and let the b_i be defined by

$$(38) \qquad b_i = \frac{1}{\alpha_0^{k-i}}\sqrt[3]{\frac{q_k}{q_k+k-i}} \qquad (i=1,\cdots,k-1);$$

and consider (37) for $\alpha = \alpha_0\sqrt[3]{1/n}$, where n is an integer. Evidently, since q_k is an integer, and since

$$(39) \qquad p_i = (q_k + k - i)n^{k-i} - q_k \qquad (i=1,\cdots,k-1),$$

the p_i are integers. Consider the possibility of a common factor. If p_{k-1} and

p_{k-2} have a common factor, their difference has the same factor. Thus, if there is a factor common to $(q_k+1)n-q_k$ and $(q_k+2)n^2-q_k$, it is also a factor of $n^2+n(n-1)(q_k+1)$. Similarly, if p_{k-2} and p_{k-3} have a factor in common, it must be a factor of $n^3+n^2(n-1)(q_k+2)$. Hence, if p_{k-1}, p_{k-2}, and p_{k-3} have a common factor, it must divide $n^2[n+(n-1)(q_k+2)]$ and $n[n+(n-1)(q_k+1)]$; and if it be prime to n, it must divide $[n+(n-1)(q_k+2)]$ and $[n+(n-1)(q_k+1)]$. But if these two integers have a common factor, it must divide their difference, $n-1$. Now n and $n-1$ are mutually prime; consequently there is no factor of $(n-1)$ which divides $[n+(n-1)(q_k+1)]$, and hence no factor common to $p_{k-1}, p_{k-2}, p_{k-3}$, unless it divides n. Let n be chosen a prime number. There are an infinite number of primes; hence the p_i have the stated property.

The periodic solutions exist then, and might be obtained as series in μ alone (convergent for sufficiently small values), by substituting into the original series in μ and the Δn_i and e_i the values of the latter $2k$ parameters, as obtained in terms of μ from the periodicity conditions. A far more advantageous method is, however, available.

§ 8. *Method of construction. Type I.*

It has been shown that, for μ sufficiently small, there exist series

$$(40)\qquad x_i(\tau)=1+\sum_{n=1}^{n=\infty}x_{i,n}(\tau)\cdot\mu^n,\quad w_i(\tau)=\sum_{n=1}^{n=\infty}w_{i,n}(\tau)\cdot\mu^n\qquad(i=1,\cdots,k),$$

which (*a*) converge for $0\leqq\tau\leqq2\pi$, (*b*) satisfy differential equations (7), (*c*) satisfy $x_i'(0)=w_i(0)=0$ identically in μ, and (*d*) make $x_i(\tau+2\pi)-x_i(\tau)=w_i(\tau+2\pi)-w_i(\tau)=0$, identically in μ.

From (*d*) and (*a*) follows the permanent convergence of series (40). From (*c*) and (*d*) follow respectively

$$(41)\qquad x_{i,n}'(0)=w_{i,n}(0)=0,$$

$$(42)\qquad x_{i,n}(\tau+2\pi)-x_{i,n}(\tau)=w_{i,n}(\tau+2\pi)-w_{i,n}(\tau)=0.$$

These equations (41) and (42) will determine the constants of integration arising at each step.

First order terms.

Since (40) must satisfy (7) identically in μ, $x_{i,1}(\tau)$ and $w_{i,1}(\tau)$ must satisfy

$$(43)\quad\begin{aligned}&(a)\ w_{i,1}''+2q_ix_{i,1}'+\sum_j{}'\delta_{ij}\sin\phi_{ji}\left(1-\eta_{ij}^3\sum_{m=0}^{\infty}F_m(\epsilon_{ij})\cos m\phi_{ji}\right)=0,\\&(b)\ x_{i,1}''-2q_iw_{i,1}'-3q_i^2x_{i,1}+q_i^2\beta_i\\&\qquad+\sum_j{}'\delta_{ij}\left[\cos\phi_{ji}+\eta_{ij}^3(\alpha_{ij}-\cos\phi_{ji})\sum_{m=0}^{\infty}F_m(\epsilon_{ij})\cos m\phi_{ji}\right]=0.\end{aligned}$$

Since every ϕ_{ji} is a multiple of τ, plus a multiple of π, equations (43) are of the type

$$
(44)\quad
\begin{aligned}
&(a) \qquad w''_{i,1} + 2q_i x'_{i,1} + \sum_{m=1}^{m=\infty} D^{(m)}_{i,1} \sin m\tau = 0, \\
&(b) \quad x''_{i,1} - 2q_i w'_{i,1} - 3q_i^2 x_{i,1} + E^{(0)}_{i,1} + \sum_{m=1}^{m=\infty} E^{(m)}_{i,1} \cos m\tau = 0,
\end{aligned}
$$

where the $D^{(m)}_{i,1}$ and $E^{(m)}_{i,1}$ are linearly related to the F_m, and can be expressed in terms of the latter as soon as the p_i are chosen. The solutions are

$$
(45)\quad
\begin{aligned}
x_{i,1}(\tau) &= -\frac{1}{q_i^2}(2q_i c^{(1)}_{i,1} + E^{(0)}_{i,1}) + c^{(2)}_{i,1} \cos q_i\tau \\
&\qquad + c^{(3)}_{i,1} \sin q_i\tau + \sum_{m=1}^{\infty} A^{(m)}_{i,1} \cos m\tau, \\
w_{i,1}(\tau) &= \frac{1}{q_i}(3q_i c^{(1)}_{i,1} + 2E^{(0)}_{i,1})\tau + c^{(4)}_{i,1} - 2c^{(2)}_{i,1} \sin q_i\tau \\
&\qquad + 2c^{(3)}_{i,1} \cos q_i\tau + \sum_{m=1}^{\infty} B^{(m)}_{i,1} \sin m\tau,
\end{aligned}
$$

where the $c^{(j)}_{i,1}$ $(j = 1, \cdots, 4;\ i = 1, \cdots, k)$ are the constants of integration and

$$
(46)\quad (m^2 - q_i^2) A^{(m)}_{i,1} = E^{(m)}_{i,1} - \frac{2q_i}{m} D^{(m)}_{i,1}, \qquad m^2 B^{(m)}_{i,1} = D^{(m)}_{i,1} - 2mq_i A^{(m)}_{i,1}.
$$

Poisson terms do not appear in (45); for, since no relation (22) holds, no term in $\cos q_i\tau$ or $\sin q_i\tau$ is present in (44). Now by (41) and (42)

$$
c^{(3)}_{i,1} = c^{(4)}_{i,1} = 0, \qquad c^{(1)}_{i,1} = -\frac{2}{3q_i} E^{(0)}_{i,1},
$$

so that the k constants, $c^{(2)}_{i,1}$, alone remain to be determined. And here arise two cases, just as in the existence proof.

Case I. q_k *is not an integer.* Here, q_i not being an integer, $\cos q_i\tau$ does not have the period 2π; consequently by (42), $c^{(2)}_{i,1} = 0$ $(i = 1, \cdots, k)$.

Case II. q_k *is an integer.* Here $\cos q_i\tau$ has the period 2π, and (42) is satisfied for the arbitrary $c^{(2)}_{i,1} \neq 0$. These k constants remain undetermined until the second order terms are found, when the $c^{(2)}_{i,1}$ are uniquely determined in avoiding Poisson terms.

Terms of any order. *Case* I.

Assume that for $n = 1, \cdots, (h-1)$, $x_{i,n}(\tau)$ and $w_{i,n}(\tau)$ have been found, the constants being determined, and have the form

$$
(47)\quad x_{i,n}(\tau) = \sum_{m=0}^{m=\infty} A^{(m)}_{i,n} \cos m\tau, \qquad w_{i,n}(\tau) = \sum_{m=0}^{m=\infty} B^{(m)}_{i,n} \sin m\tau \quad (i = 1, \cdots, k).
$$

An induction will show that $x_{i,h}(\tau)$ and $w_{i,h}(\tau)$ have the form (47), that, moreover, the differential equations for $x_{i,h}$ and $w_{i,h}$ are of the type (44), and that the constants of integration are determined just as in the case preceding, $n = 1$. The differential equations are [see (7)]

$$(a)\ \sum_{j+k=h} x_{i,j} w''_{i,k} + 2\sum_{j+k=h} x'_{i,j} w'_{i,k} + 2q_i x'_{i,h}$$

$$(48)\qquad + \sum_j{}' \delta_{ij} \sum_{k+l+m=h-1} x_{j,k}(\sin \overline{\phi_{ji} + w_j - w_i})_l [x_j^{-3} - \sigma_{ij}^{-3}]_m = 0,$$

$$(b)\ x''_{i,h} - \sum_{j+k+l=h} x_{i,j}(w'_{i,k} w'_{i,l} + 2q_i w'_{i,k}) - q_i^2 x_{i,h} + q_i^2 (x_i^{-2})_h$$

$$+ q_i^2 \beta_i (x_i^{-2})_{h-1} + \sum_j{}' \delta_{ij} \{\alpha_{ij} \sum_{k+l=h-1} x_{i,k}(\sigma_{ij}^{-3})_l$$

$$+ \sum_{k+l+m=h-1} x_{j,k}(\cos \overline{\phi_{ji} + w_j - w_i})_l [x_j^{-3} - \sigma_{ij}^{-3}]_m \} = 0,$$

where an expression in a parenthesis, having a subscript, l, outside, denotes the sum of all those terms in the expression which involve μ^l. It is to be shown (α) that the variables in (48) whose second subscript is h enter in the same form as the $x_{i,1}$ and $w_{i,1}$ enter (44), and (β) that the remaining terms of (48) (a) and (b) reduce respectively to a sine series and a cosine series in multiples of τ.

Evidently in $(48a)$ the only terms involving the $x_{i,h}$ and $w_{i,h}$ $(i = 1, \cdots, k)$ are $w''_{i,h} + 2q_i x'_{i,h}$; and in $(48b)$, aside from $(x_i^{-2})_h$, they are $x''_{i,h} - 2q_i w'_{i,h} - q_i^2 x_{i,h}$. Now it can be easily shown by induction that

$$\left.\frac{d^h x_i^{-2}}{d\mu^h}\right]_{\mu=0} = \sum_\nu N_\nu (-1)^{\nu_0} \prod_{f=0}^{f=h} x_i^{-(2+\nu_0)} \left(\frac{d^f x_i}{d\mu^f}\right)^{\nu_f}\Bigg]_{\mu=0},$$

where the N_ν are positive numbers and the ν_f are positive integers (or zero) satisfying the conditions

$$(49)\qquad \sum_{f=1}^{h} \nu_f = \nu_0, \qquad \sum_{f=1}^{h} f\nu_f = h.$$

Now $x_{i,h}$ enters only through $(d^h x_i/d\mu^h)^{\nu_h}$, and for this term $N_\nu = 2$, $\nu_0 = 1$, $\nu_h = 1$, and $\nu_f = 0$, $(f = 1, \cdots, h-1)$; hence, in $(x_i^{-2})_h$, $x_{i,h}$ appears with the coefficient -2. Therefore, in $(48b)$, the terms involving $x_{i,h}$ and $w_{i,h}$ are $x''_{i,h} - 2q_i w'_{i,h} - 3q_i^2 x_{i,h}$, and this establishes statement (α) above.

Using again the notation $F^c(\tau)$ and $F^s(\tau)$ to designate respectively a cosine series and a sine series in multiples of τ, it is evident that

$$F_1^s \cdot F_2^s = F^c, \qquad F_1^c \cdot F_2^c = F^c, \qquad F^c \cdot F^s = F^s,$$

$$(F^c)^n = F^c, \qquad (F^s)^{2n} = F^c, \qquad (F^s)^{2n+1} = F^s.$$

Hence, as every $x_{i,n}$ and $w'_{i,n}$ $(n = 1, \cdots, h-1)$ is a $F^c(\tau)$, so also are all sums of products of these quantities, and also all polynomials in the $x_{i,n}$, such as $(x_i^{-\lambda})_l$ and $(\sigma_{ij}^{-3})_l$. Similarly the sums of all products $\overline{x_{i,j} w''_{i,k}}$ and $\overline{x'_{i,j} w'_{i,j}}$ are $F^s(\tau)$. There remain in (48) only the terms $(\sin \overline{m_{ij} + w_j - w_i})_l$ where $m_{ij} = \phi_{ji}$ in (48a), and $m_{ij} = \phi_{ji} + \pi/2$ in (48b). Let, for the moment, $z_{ij} = m_{ij} + w_j - w_i$. Then, since $[d^f z_{ij}/d\mu^f]_{\mu=0} = w_{j,f} - w_{i,f}$, it can be shown by a simple induction that

$$\left[\frac{d^l \sin z_{ij}}{d\mu^l}\right]_{\mu=0} = \sum_\nu N_\nu \prod_{f=0}^{l} \sin\left(m_{ij} + \nu_0 \frac{\pi}{2}\right)(w_{j,f} - w_{i,f})^{\nu_f},$$

where the N_ν are numbers and the ν_f satisfy (49) after h is replaced by l. Now since every $(w_{j,f} - w_{i,f})$, $(f = 1, \cdots, h-1)$ is a $F^s(\tau)$, the product $\Pi_{f=1}^{f=l}(w_{j,f} - w_{i,f})^{\nu_f}$ is a $F^s(\tau)$ or $F^c(\tau)$ according as $\sum_{f=1}^{f=l} \nu_f$ is odd or even. But when this sum is odd, ν_0 is odd; and when even, ν_0 is even. The entire product $\Pi_{f=0}^{f=l}$ is then always a $F^s(\tau)$ if $m_{ij} = \phi_{ji}$, or a $F^c(\tau)$ if $m_{ij} = \phi_{ji} + \pi/2$.

The differential equations (48) are, therefore, of the form

$$(50) \quad \begin{aligned} &(a) \quad w''_{i,h} + 2q_i x'_{i,h} + \sum_{m=1}^{m=\infty} D_{i,h}^{(m)} \sin m\tau = 0, \\ &(b) \quad x''_{i,h} - 2q_i w'_{i,h} - 3q_i^2 x_{i,h} + E_{i,h}^{(0)} + \sum_{m=1}^{m=\infty} E_{i,h}^{(m)} \cos m\tau = 0, \end{aligned}$$

where no term in $\cos q_i\tau$ or $\sin q_i\tau$ occurs under the summation sign, since q_i is not an integer. Obviously the integration of (50) is the same problem as that of (44), so that the solutions are

$$(51) \quad \begin{aligned} x_{i,h}(\tau) &= -\frac{1}{q_i^2}(2q_i c_{i,h}^{(1)} + E_{i,h}^{(0)}) + c_{i,h}^{(2)} \cos q_i\tau + c_{i,h}^{(3)} \sin q_i\tau + \sum_{m=1}^{m=\infty} A_{i,h}^{(m)} \cos m\tau, \\ w_{i,h}(\tau) &= \frac{1}{q_i}(3q_i c_{i,h}^{(1)} + 2E_{i,h}^{(0)})\tau + c_{i,h}^{(4)} - 2c_{i,h}^{(2)} \sin q_i\tau + 2c_{i,h}^{(3)} \cos q_i\tau \\ &\qquad + \sum_{m=1}^{m=\infty} B_{i,h}^{(m)} \sin m\tau \quad (i = 1, \cdots, k), \end{aligned}$$

where

$$(52) \quad (m^2 - q_i^2) A_{i,h}^{(m)} = E_{i,h}^{(m)} - \frac{2q_i}{m} D_{i,h}^{(m)}, \qquad m^2 B_{i,h}^{(m)} = D_{i,h}^{(m)} - 2mq_i A_{i,h}^{(m)}.$$

Also, by (41) and (42),

$$(53) \quad c_{i,h}^{(1)} = -\frac{2}{3q_i} E_{i,h}^{(0)}, \qquad c_{i,h}^{(2)} = c_{i,h}^{(3)} = c_{i,h}^{(4)} = 0 \qquad (i = 1, \cdots, k),$$

so that the induction is completely established. The coefficients of the successive $x_{i,n}$ and $w_{i,n}$ are obtained *without integration*, by simply applying (52) and the first equations of (53). It is merely necessary to compute at each step the $D_{i,h}^{(m)}$ and the $E_{i,h}^{(m)}$.

Terms of any order. Case II.

Assume, as in case I, that for $n = 1, \cdots, (h-1)$ the $x_{i,n}(\tau)$ and $w_{i,n}(\tau)$ have the form (47), all the constants of integration having been determined with the exception of the $c_{i,h-1}^{(2)}$. The differential equations for the $x_{i,h}$ and $w_{i,h}$ are, then, (50a) and (b); where, however, since q_i is an integer, $\cos q_i\tau$ and $\sin q_i\tau$ may occur under the summation sign. These terms arise from two sources: from the terms $c_{i,h-1}^{(2)}\cos q_i\tau$ and $c_{i,h-1}^{(2)}\sin q_i\tau$ in the $x_{i,h-1}$ and $w_{i,h-1}$, and from similar terms in the earlier $x_{i,n}$ and $w_{i,n}$, as well as (usually) from combinations of the ϕ_{ji} in the coefficients.

When (50a) are integrated and combined with (50b), equations for the $x_{i,h}$ are obtained; to avoid Poisson terms in the $x_{i,h}$, the coefficients of the terms in $\cos q_i\tau$ in these last equations must be made to vanish. These coefficients, $E_{i,h}^{(q_i)} - D_{i,h}^{(q_i)}$, involve the $c_{i,h-1}^{(2)}$ and various known constants; e. g., the $c_{i,n}^{(2)}$ $(n=1,\cdots,h-2)$; and the vanishing of the $E_{i,h}^{(q_i)} - D_{i,h}^{(q_i)}$ will usually determine the $c_{i,h-1}^{(2)}$.

In the first place, the only terms of (48) in which the $x_{i,h-1}$ and $w_{i,h-1}$ appear are

$$
(54)\qquad
\begin{aligned}
(a)\quad & x_{i,h-1}w''_{i,1} + x_{i,1}w''_{i,h-1} + 2x'_{i,h-1}w'_{i,1} + 2x'_{i,1}w'_{i,h-1}\\
& + \sum_j{}' \delta_{ij}\{[x_{j,h-1}\sin\phi_{ji}(1-\eta_{ij}^3\sigma_{ij,0}^{-3})]\\
& + \sin\phi_{ji}[-3x_{j,h-1} - 3\eta_{ij}^5\sigma_{ij,0}^{-5}(\alpha_{ij}^2x_{i,h-1} + x_{j,h-1}\\
& - \alpha_{ij}\overline{x_{j,h-1}+x_{i,h-1}}\cos\phi_{ji} + \alpha_{ij}\overline{w_{j,h-1}-w_{i,h-1}}\sin\phi_{ji})]\\
& + (1-\eta_{ij}^3\sigma_{ij,0}^{-3})\cos\phi_{ji}(w_{j,h-1}-w_{i,h-1})\},\\
(b)\quad & -2q_ix_{i,h-1}w'_{i,1} - 2q_ix_{i,1}w'_{i,h-1} + 6q_i^2x_{i,1}x_{i,h-1}\\
& - 2q_i^2\beta_ix_{i,1}x_{i,h-1} + \sum_j{}' \delta_{ij}\{[\alpha_{ij}x_{i,h-1}\eta_{ij}^3\sigma_{ij,0}^{-3}]\\
& - 3\eta_{ij}^5\sigma_{ij,0}^{-5}(\alpha_{ij}-\cos\phi_{ji})(\alpha_{ij}^2x_{i,h-1} + x_{j,h-1}\\
& - \alpha_{ij}\overline{x_{j,h-1}+x_{i,h-1}}\cos\phi_{ji} + \alpha_{ij}\overline{w_{j,h-1}-w_{i,h-1}}\sin\phi_{ji})\\
& - \eta_{ij}^3\sigma_{ij,0}^{-3}[x_{j,h-1}\cos\phi_{ji} - \overline{w_{j,h-1}-w_{i,h-1}}\sin\phi_{ji}]\\
& - [2x_{j,h-1}\cos\phi_{ji} + \overline{w_{j,h-1}-w_{i,h-1}}\sin\phi_{ji}]\},
\end{aligned}
$$

where $\sigma_{ij,0} = (1-2\epsilon_{ij}\cos\phi_{ji}+\epsilon_{ij}^2)^{\frac{1}{2}}$. Hence it is evident that the $c_{f,h-1}^{(2)}$ enter the $E_{i,h}^{(q_i)} - D_{i,h}^{(q_i)}$ linearly; and, denoting their coefficients by R_{if} $(i=1,\cdots,k;\ f=1,\cdots,k)$, it is found that

$$
(55)\qquad R_{if} = (-1)^{q_i}\frac{2}{\pi}P_{if} \qquad (i=1,\cdots,k;\ f=1,\cdots,k),
$$

where the P_{if} are the elements of the determinant Δ_4, discussed in the existence proof. Hence, when $\Delta_4 \neq 0$, so is the determinant of the coefficients of the $c_{f,h-1}^{(2)}$ in the equations $E_{i,h}^{(q_i)} - D_{i,h}^{(q_i)} = 0$. The constants $c_{f,h-1}^{(2)}$ $(f=1,\cdots,k)$ are then uniquely determined.

The values of the $x_{i,h}(\tau)$ and $w_{i,h}(\tau)$ are given by (51) and (52); (53) are, however, replaced by

$$(56) \qquad c_{i,h}^{(1)} = -\frac{2}{3q_i} E_{i,h}^{(0)}, \qquad c_{i,h}^{(3)} = c_{i,h}^{(4)} = 0 \qquad (i=1, \cdots, k),$$

the $c_{i,h}^{(2)}$ remaining undetermined at this step. The induction is thus established. In this case, too, the successive $x_{i,n}$ and $w_{i,n}$ are obtained *without integration* by merely computing the $D_{i,h}^{(m)}$ and $E_{i,h}^{(m)}$, applying (52) and the first equations of (56), and solving $E_{i,h+1}^{(q_i)} - D_{i,h+1}^{(q_i)} = 0$ for the $c_{i,h}^{(2)}$.

Remarks: 1. In constructing the solutions it has been tacitly assumed that the Fourier series representation of the $x_{i,n}$ and $w_{i,n}$, and the attendant manipulations of those series, are valid. This is justified by the consideration that the well-known functions encountered in the first step are representable, together with their derivatives, by uniformly convergent Fourier series. But the product of two such series is another of the same type, and also the integral of a uniformly convergent Fourier series is uniformly convergent. At every step the convergence remains uniform.

2. The question naturally arises as to whether, in any orbits of type I, the smallest period is a multiple of the period of the infinitesimal system, viz., mT. An examination of the method of constructing the solutions furnishes the answer. If mq_k is not an integer, the constants are determined precisely as in case I; if mq_k is an integer, precisely as in case II. In any event, for a given value of μ and a given set of p_i and q_k there is a *unique* solution satisfying (41) and (42) (with $2m\pi$ replacing 2π). The solutions found above, however, satisfy these conditions; hence there are no orbits whose smallest period is a multiple of T.

§ 9. *Orbits of type* II.

For type II, as for type I, all the λ_i are multiples of π; but it is proposed to ascertain whether the ω_i and τ_i exist as functions of μ (not identically zero, but vanishing with μ) so as to satisfy periodicity conditions (13). The question can be studied most easily by examining the method of construction.

Let the origin of time be selected as the instant when satellite k has an apsidal passage, and the origin of longitude at that satellite's apsidal position, so that $\omega_k = \tau_k = 0$; and let it be assumed for the moment that the ω_i and τ_i $(i = 1, \cdots, k-1)$, exist as functions of μ satisfying (13). Then there exist solutions of the type (40) satisfying (42), but not satisfying (41) for all values of n, except for $i = k$. Because of the absence of conditions (41) some of the constants in each step are left undetermined until terms of the next order are found.

For the first order terms the differential equations are again (44), and the solutions are (45), the coefficients being determined by (46). Evidently the $c_{i,1}^{(1)}$ are determined as in type I, but for the other $c_{i,1}^{(j)}$ two cases arise.

Case I. q_k *is not an integer.* Here, by (42), $c_{i,1}^{(2)} = c_{i,1}^{(3)} = 0$ $(i = 1, \cdots, k)$, but the $c_{i,1}^{(4)}$ are at present undetermined, except $c_{k,1}^{(4)} = 0$.

Case II. q_k *is an integer.* Here the terms $\cos q_i \tau$ and $\sin q_i \tau$ have the required period, 2π; and hence the $c_{i,1}^{(2)}$ and $c_{i,1}^{(3)}$, together with the $c_{i,1}^{(4)}$, remain undetermined, except $c_{k,1}^{(4)} = c_{k,1}^{(3)} = 0$.

Terms of any order. *Case* I.

Assume that the $x_{i,n}(\tau)$ and $w_{i,n}(\tau)$ $(n = 1, \cdots, h-2)$ have the form (47), and that the $x_{i,h-1}(\tau)$ and $w_{i,h-1}(\tau) - c_{i,h-1}^{(4)}$ also have that form. Further, suppose all the constants, except the $c_{i,h-1}^{(4)}(i = 1, \cdots, k-1)$, to have been determined, the $c_{i,n}^{(4)}$ $(n = 1, \cdots, h-2)$ being zero.

The differential equations for the $x_{i,h}(\tau)$ and $w_{i,h}(\tau)$ are still (48); but, because of the $c_{i,h-1}^{(4)}$, these equations now reduce, not to (50), but to

$$(57)\quad \begin{aligned} &(a) \qquad w''_{i,h} + 2q_i x'_{i,h} + \sum_{m=0}^{m=\infty} (D_{i,h}^{(m)} \sin m\tau + H_{i,h}^{(m)} \cos m\tau) = 0, \\ &(b) \quad x''_{i,h} - 2q_i w'_{i,h} - 3q_i^2 x_{i,h} + \sum_{m=0}^{m=\infty} (E_{i,h}^{(m)} \cos m\tau + J_{i,h}^{(m)} \sin m\tau) = 0, \end{aligned}$$

where the $H_{i,h}^{(m)}$ and $J_{i,h}^{(m)}$ vanish with the $c_{f,h-1}^{(4)}$. The first integration introduces non-periodic terms, $H_{i,h}^{(0)}\tau$. In order that the $x_{i,h}$ and $w_{i,h}$ may satisfy (42), these terms must disappear; their vanishing will determine the $c_{i,h-1}^{(4)}$.

In the ith equation of (57 a) let the coefficient of $c_{f,h-1}^{(4)}$ be K_{if}; then from (54 a) it is seen that the $c_{f,h-1}^{(4)}$ enter the $H_{i,h}^{(0)}$ linearly, and that

$$(58)\quad \begin{aligned} K_{if} &= -\frac{\delta_{if}\eta_{if}^3}{2}\{F_1(\epsilon_{if}) + 3\alpha_{if}\eta_{if}^2[G_0(\epsilon_{if}) - \tfrac{1}{2}G_2(\epsilon_{if})]\} && (f \neq i), \\ K_{ii} &= -\sum_j{}' K_{ij} && (i = 1, \cdots, k). \end{aligned}$$

Since the $H_{i,h}^{(0)}$ vanish with the $c_{f,h-1}^{(4)}$, it follows that, if the $x_{i,h}$ and $w_{i,h}$ are to be periodic, the following linear equations must be satisfied:

$$(59)\qquad \sum_{f=1}^{f=k-1} K_{if} c_{f,h-1}^{(4)} = 0 \qquad (i = 1, \cdots, k).$$

The necessary and sufficient condition that these k equations have a solution other than $c_{f,h-1}^{(4)} = 0$ $(f = 1, \cdots, k-1)$ is that every determinant of order $k-1$ formed from the matrix of k rows and $k-1$ columns should vanish. In this case, however, a more simple statement is possible. Since, if the summation in (59) extended from 1 to k, the determinant of all the coefficients would vanish identically (the sum of the elements of each row being zero), every solution of $k-1$ of the equations would satisfy the remaining one. Any solution of the first $k-1$ equations of (59), taken with $c_{k,h-1}^{(4)}=0$, is, however, a solution of the first $k-1$ equations in which f runs from 1 to k; and, consequently, is a solution of the last equation of (59). Hence the necessary and sufficient condition that (59) have a solution, other than $c_{f,h-1}^{(4)}=0$, is that the determinant, Δ_5, of the coefficients in the first $k-1$ equations of (59) shall vanish.

It will here be shown merely that Δ_5 cannot vanish for any values of α [defined by (34)] on a certain interval, and that Δ_5 vanishes, if at all, only for a finite number of values of α. Since, by (58) and (35),

$$(60)\qquad \begin{aligned} K_{if} &= \alpha^{(10i-4f-6k)}\cdot R_{if}(\alpha), &\quad \text{if } f<i,\\ K_{if} &= \alpha^{(6f-6k)}\cdot R_{if}(\alpha), &\quad \text{if } f>i, \end{aligned}$$

where the R_{if} are power series in α beginning with a constant term, it follows that the lowest term in α in the ith row of Δ_5 $(i=1,\cdots,k-1)$ is of degree $6i+4-6k$; and this power appears in the columns i and $i-1$. If, as in the case of Δ_3, the factor $\alpha^{(6i+4-6k)}$ be removed from the ith row $(i=1,\cdots,k-1)$, another determinant, Δ_6, of order $(k-1)$ is obtained, such that $\Delta_6\cdot\alpha^{-(k-1)(3k-4)}=\Delta_5$, and each of its elements is a power series in α. The discussion of Δ_4 applies equally to Δ_6; and therefore Δ_5 vanishes, if at all, only for a finite number of values of α.

In general, then, there is only one solution of (59), viz., $c_{f,h-1}^{(4)}=0$ $(f=1,\cdots,k-1)$; and since $c_{k,h-1}^{(4)}$ is also zero, the $x_{i,n}(\tau)$ and $w_{i,n}(\tau)$ $(n=1,\cdots,h-1)$ have the form (47). The differential equations for the $x_{i,h}(\tau)$ and $w_{i,h}(\tau)$ reduce, therefore, as in type I, to (50). The solutions are (51), where the constants are determined by (52) and (53), except the $c_{i,h}^{(4)}$ $(i=1,\cdots,k-1)$ which are left undetermined. Hence, the $x_{i,h}(\tau)$ and $w_{i,h}(\tau)-c_{i,h}^{(4)}$ have the form (47), and the induction is established.

In general, then—if indeed not always—it is impossible in case I to determine the constants of integration otherwise than as in type I.

Terms of any order. Case II.

Assume that the $x_{i,n}(\tau)$ and $w_{i,n}(\tau)$ $(n=1,\cdots,h-2)$ have the form (47), and that the $x_{i,h-1}(\tau)-c_{i,h-1}^{(3)}\sin q_i\tau$ and the $w_{i,h-1}(\tau)-c_{i,h-1}^{(4)}-2c_{i,h-1}^{(3)}\cos q_i\tau$ also have that form. Suppose, further, that all the constants of integration have been determined, except $c_{k,h-1}^{(2)}$ and $c_{i,h-1}^{(j)}$ $(j=2,3,4;\ i=1,\cdots,k-1)$, the $c_{i,n}^{(3)}$ and $c_{i,n}^{(4)}$ $(n=1,\cdots,h-2)$ being zero.

The differential equations for the $x_{i,h}(\tau)$ and $w_{i,h}(\tau)$ are again (48), but because of the $c_{i,h-1}^{(3)}$ and $c_{i,h-1}^{(4)}$ these equations reduce to (57), where now the $H_{i,h}^{(m)}$ and $J_{i,h}^{(m)}$ vanish with the $c_{f,h-1}^{(3)}$ and $c_{f,h-1}^{(4)}$. In order that the $x_{i,h}(\tau)$ and $w_{i,h}(\tau)$ be periodic, it is necessary, just as in type I, that $E_{i,h}^{(q_i)}-D_{i,h}^{(q_i)}$, the coefficient of $\cos q_i\tau$, in the ith equation of the final set for the $x_{i,n}(\tau)$, should vanish. It is equally necessary that $J_{i,h}^{(q_i)} - H_{i,h}^{(q_i)}$, the coefficients of $\sin q_i\tau$ in the same equation, should vanish. Finally, to avoid the "secular terms," $H_{i,h}^{(0)}\tau$, these coefficients must vanish. The vanishing of these three sets of coefficients will determine $c_{k,h-1}^{(2)}$ and the $c_{i,h-1}^{(j)}$ $(j = 2, 3, 4;\ i = 1, \cdots, k-1)$.

Now, from (54) and the fact that no relation (22) holds, it is evident that the $E_{i,h}^{(q_i)} - D_{i,h}^{(q_i)}$ do not involve the $c_{f,h-1}^{(3)}$ nor the $c_{f,h-1}^{(4)}$; also the $J_{i,h}^{(q_i)} - H_{i,h}^{(q_i)}$ involve neither the $c_{f,h-1}^{(2)}$ nor the $c_{f,h-1}^{(4)}$, and vanish with the $c_{f,h-1}^{(3)}$. The $H_{i,h}^{(0)}$ involve the $c_{f,h-1}^{(2)}$, the $c_{f,h-1}^{(3)}$, and the $c_{f,h-1}^{(4)}$. Further, since the $x_{i,h-1}(\tau)$ and $w_{i,h-1}(\tau)$ may be written

$$(61)\qquad \begin{aligned} x_{i,h-1}(\tau) &= c_{i,h-1}^{(2)} \sin\left(q_i\tau + \frac{\pi}{2}\right) + c_{i,h-1}^{(3)} \sin q_i\tau + \sum_{m=0}^{m=\infty} A_{i,h-1}^{(m)} \cos m\tau, \\ w_{i,h-1}(\tau) &= 2c_{i,h-1}^{(2)} \cos\left(q_i\tau + \frac{\pi}{2}\right) + 2c_{i,h-1}^{(3)} \cos q_i\tau + \sum_{m=1}^{m=\infty} B_{i,h-1}^{(m)} \sin m\tau, \end{aligned}$$

it is evident that the $c_{f,h-1}^{(3)}$ enter the $J_{i,h}^{(q_i)} - H_{i,h}^{(q_i)}$ (which are the coefficients of $\sin q_i\tau$), in precisely the same way as the $c_{f,h-1}^{(2)}$ enter the $E_{i,h}^{(q_i)} - D_{i,h}^{(q_i)}$, [which are the coefficients of $\sin(q_i\tau + \pi/2)$].

In the treatment of case II for type I it was shown that, when $\Delta_4 \neq 0$, the equations $E_{i,h}^{(q_i)} - D_{i,h}^{(q_i)} = 0$ admit a unique solution for the $c_{f,h-1}^{(2)}$ $(f = 1, \cdots, k)$. The same conclusion is evidently valid here, and it follows at once that the equations

$$(62)\qquad J_{i,h}^{(q_i)} - H_{i,h}^{(q_i)} = 0 \qquad (i = 1, \cdots, k),$$

admit a unique solution for the $c_{f,h-1}^{(3)}$ $(f = 1, \cdots, k)$ viz. $c_{f,h-1}^{(3)} = 0$. [In (62), $c_{k,h-1}^{(3)}$ does not appear, being zero; but this does not affect the conclusion, since there are obviously no more solutions for $k-1$ of the unknowns when the last is determined than there are for all k.]

From the manner in which the $c_{f,h-1}^{(2)}$ enter the $H_{i,h}^{(0)}$, it follows that, when the $c_{f,h-1}^{(3)}$ vanish, the $H_{i,h}^{(0)}$ reduce to the values they have in case I above. But, since equations (59) have in general no other solution $c_{f,h-1}^{(4)} = 0$ $(f = 1, \cdots, k-1)$, there is in general only one way to determine the $c_{f,h-1}^{(j)}$ $(j = 2, 3, 4;\ f = 1, \cdots, k)$, viz. as in type I. The differential equations for the $x_{i,h}$ and $w_{i,h}$ reduce, therefore, to (50).

The solutions are given by (51) and (52), together with

$$c_{k,h}^{(3)} = c_{k,h}^{(4)} = 0, \qquad c_{i,h}^{(1)} = -\frac{2}{3q_i} E_{i,h}^{(0)} \qquad (i = 1, \cdots, k),$$

the $c_{i,h}^{(j)}$ $(j = 2, 3, 4;\ i = 1, \cdots, k-1)$ and $c_{k,h}^{(2)}$ remaining undetermined. The $x_{i,h}(\tau) - c_{i,h}^{(3)} \sin q_i\tau$ and $w_{i,h}(\tau) - c_{i,h}^{(4)} - 2c_{i,h}^{(3)} \cos q_i\tau$ have the form (47); the induction is, then, established. Since at every step, as in case I, the constants must be determined precisely as in type I, *there are in general no periodic solutions of type* II.

Remark. If either of the determinants Δ_3 and Δ_5 vanishes for some value of α, there is at least a single infinitude of solutions for the constants of integration, which would render the solutions periodic in form. Whether the series converge for this value of α is, however, unknown; so that the mere vanishing of Δ_3 or Δ_5 would not warrant the conclusion that orbits of type II exist.

§ 10. *Orbits of type* III.

It will next be inquired whether periodic orbits exist growing out of circular orbits of an infinitesimal system which has no "grand conjunction"; that is, whether there are periodic solutions of (7) when not all the λ_i are multiples of π. Let the initial conditions be (17); and let the origin of time be selected at an instant when M_k is at any apse, and the origin of longitude at the apsidal position of M_k, both for $\mu = 0$ and for $\mu \neq 0$. Then $\lambda_k = \omega_k = \tau_k = 0$.

Now the conditions for periodicity are (13). But equations (7) admit two integrals, an examination of which shows that two equations of (13), viz. $x_k(2\pi) = x_k(0)$, and $x'_k(2\pi) = x'_k(0)$, are a consequence of the other $4k - 2$ equations. Hence equations (13) become

$$(63)\quad \begin{aligned}
&(a)\ \ 0 = e_i(1 - \cos 2q_i\pi) + \mu \cdot X_{i,0} + e_i\tau_i(-\sin 2q_i\pi) \\
&\qquad + \mu \sum_{f=1}^{f=k} \{ \Delta n_f X^{(1)}_{i,f} + e_f X^{(2)}_{i,f} + \omega_f X^{(3)}_{i,f} + \tau_f X^{(4)}_{i,f} \} + \cdots, \\
&(b)\ \ 0 = e_i(q_i \sin 2q_i\pi) + \mu X'_{i,0} + e_i\tau_i(q_i^2 \overline{1 - \cos 2q_i\pi}) \\
&\qquad + \mu \sum_{f=1}^{f=k} \{ \Delta n_f X'^{(1)}_{i,f} + e_f X'^{(2)}_{i,f} + \omega_f X'^{(3)}_{i,f} + \tau_f X'^{(4)}_{i,f} \} + \cdots \qquad (i = 1, \cdots, k-1), \\
&(c)\ \ 0 = \Delta n_i(2q_i\pi) + e_i(2 \sin 2q_i\pi) + \mu W_{i,0} + e_i\tau_i(2q_i \overline{1 - \cos 2q_i\pi}) \\
&\qquad + \mu \sum_{f=1}^{f=k} \{ \Delta n_f W^{(1)}_{i,f} + e_f W^{(2)}_{i,f} + \omega_f W^{(3)}_{i,f} + \tau_f W^{(4)}_{i,f} \} + \cdots, \\
&(d)\ \ 0 = e_i(-2q_i \overline{1 - \cos 2q_i\pi}) + \mu W'_{i,0} + e_i\tau_i(q_i^2 \sin 2q_i\pi) \\
&\qquad + \mu \sum_{f=1}^{f=k} \{ \Delta n_f W'^{(1)}_{i,f} + e_f W'^{(2)}_{i,f} + \omega_f W'^{(3)}_{i,f} + \tau_f W'^{(4)}_{i,f} \} + \cdots,
\end{aligned}$$

where

$$X_{i,0} = x_i(0;\, 2\pi) - x_i(0;\, 0), \cdots, \qquad W'_{i,0} = w'_i(0;\, 2\pi) - w'_i(0;\, 0),$$

$$X^{(2)}_{i,f} = x_i(f;\, 2\pi) - x_i(f;\, 0), \cdots, \qquad W'^{(2)}_{i,f} = w'_i(f;\, 2\pi) - w'_i(f;\, 0),$$

and the $X^{(j)}_{i,f}$, $(j = 1, 3, 4)$, etc., are constants whose values will not be needed here.

While the determinant of the linear terms of the Δn_i, e_i, ω_i, and τ_i in (63) is zero; yet, when q_k is not an integer, solutions of (63) (c) and (d) exist for the Δn_i and e_i as power series in μ and the ω_i and τ_i. Properties of the series (63), similar to (19), are easily established, which show that the solutions for the Δn_i and e_i, and also the series obtained by substitution of these solutions into (63) (a) and (b), carry μ as a factor. After this substitution and a division by μ, (63) (a) and (b) become

$$(64)\quad \begin{aligned} &(a)\ 0 = X_{i,0} + \frac{1}{2q_i} W'_{i,0} + \sum_{f=1}^{f=k} \left\{ \omega_f \left[X^{(3)}_{i,f} + \frac{1}{2q_i} W'^{(3)}_{i,f} \right] \right. \\ &\qquad\qquad \left. + \tau_f \left[X^{(4)}_{i,f} + \frac{1}{2q_i} W'^{(4)}_{i,f} \right] \right\} + \cdots, \\ &(b)\ 0 = X'_{i,0} + \frac{\sin 2q_i\pi}{2(1-\cos 2q_i\pi)} W'_{i,0} + \sum_{f=1}^{f=k} \left\{ \omega_f \left[X'^{(3)}_{i,f} + \frac{\sin 2q_i\pi}{2(1-\cos 2q_i\pi)} W'^{(3)}_{i,f} \right] \right. \\ &\qquad\qquad \left. + \tau_f \left[X'^{(4)}_{i,f} + \frac{\sin 2q_i\pi}{2(1-\cos 2q_i\pi)} W'^{(4)}_{i,f} \right] \right\} + \tau_i \frac{q_i W'_{i,0}}{1-\cos 2q_i\pi} + \cdots \end{aligned}$$

The constant term in (64a) vanishes, since, for q_k not an integer, no relation (22) holds; the constant term in (64b) reduces to $-2q_i C_{i,0}$. If any of the $C_{i,0}$ are distinct from zero, (64) are not satisfied by $\omega_i = \tau_i = \mu = 0$; hence ω_i and τ_i do not exist as holomorphic functions of μ vanishing with μ, or periodic orbits of the type sought do not exist. The necessary condition for periodicity, viz., that the $C_{i,0}$ vanish, is, by (27),

$$(65)\qquad \sum_j{}' \delta_{ij} \sum_{m=1}^{m=\infty} E_m(\epsilon_{ij}) \sin m(\lambda_j - \lambda_i) = 0 \qquad (i = 1, \cdots, k).$$

Of these k equations in the quantities λ_j $(j = 1, \cdots, k-1)$, one is evidently a consequence of the others; for, before λ_k is put equal to zero, the jacobian of the $C_{i,0}$ with respect to the λ_j $(i = 1, \cdots, k;\ j = 1, \cdots, k)$ is identically zero.

If particular values be assigned to all, save one, of the λ_j, the last λ can still be given an infinitude of values for which any one $C_{i,0}$ is distinct from zero; hence (65) impose very special conditions upon the λ_j. Whether there are any sets of λ's, other than multiples of π, which satisfy (65) is here left unsettled; it will, however, be shown that there are no others "in the vicinity of" multiples of π, provided α be sufficiently small.

If the $C_{i,0}$ are developed as power series in the $\lambda_f - J_f\pi$ $(J_f = 0, 1)$, the coefficients of the linear terms are simply $[\partial C_{i,0}/\partial\lambda_f]_{\lambda_i = J_i\pi}$. Evidently, from (27),

$$(66)\quad \begin{aligned} \left[\frac{\partial C_{i,0}}{\partial \lambda_f}\right]_{\lambda_i=J_i\pi} &= \delta_{if}\sum_{m=1}^{\infty} mE_m(\epsilon_{ij})\cos m(J_f-J_i)\pi, & f \neq i,\\ \left[\frac{\partial C_{i,0}}{\partial \lambda_i}\right]_{\lambda_i=J_i\pi} &= -\sum_j{}' \delta_{ij}\sum_{m=1}^{\infty} mE_m(\epsilon_{ij})\cos m(J_j-J_i)\pi, & f = i,\end{aligned}$$

Denoting by S_{if} the coefficient of λ_f in the ith equation of (65), and introducing α by (34), the S_{if} are obtained as power series in α. From (27′) it is found that the lowest exponent of α present in S_{if} is $(f-i)$ if $f<i$, and $4(f-i)$ if $f>i$. Hence, of all the S_{if} $(f<i)$, S_{i1} carries the lowest power of α, viz., the $(1-i)$th; and of all the S_{if} $(f>i)$, $S_{i,i+1}$ carries the lowest, viz., the 4th. Consequently, if $\alpha^{(1-i)}$ be removed as a factor from all the S_{if} $(f=1,\cdots,k-1)$, a determinant Δ_7 is obtained which is equal to the determinant of the coefficients S_{if} multiplied by a power of α; and, in the ith row of Δ_7, all elements save those of the first and ith columns begin with a power of α. The discussion of Δ_4 applies also to Δ_7; showing that, for α sufficiently small, $\Delta_7 \neq 0$.

Since this determinant is distinct from zero, there exist quantities Λ_f $(f=1,\cdots,k-1)$ such that equations (65) are not satisfied for any values of the λ_f, for which $|\lambda_f - J_f\pi| < \Lambda_f$.

In considering the general question of the existence of any sets of the λ_f satisfying (65), if it be assumed for the moment that orbits of type III exist, and the method of construction be examined, equations related to (65) are encountered. Thus, in finding the second order terms, the $H_{i,2}^{(0)}$ must be made to vanish, and these involve the λ_f. In order that the k equations, $H_{i,2}^{(0)}=0$, be satisfied by the $k-1$ quantities $c_{i,1}^{(4)}$ $(i=1,\cdots,k-1)$, it is necessary that k determinants vanish, each of which involves Fourier series in the $\lambda_f-\lambda_i$.

When q_k is an integer, somewhat different difficulties arise in the existence proof; the same difficulty is, however, encountered in the construction. While a general negative conclusion is not yet warranted, it is evident that if any orbits of type III exist, they must satisfy very special conditions.

In every case, however, periodic orbits of the type sought do not exist if (22) holds.

§ 11. *Concerning lacunary spaces.*

The relation (22) may be expressed in the form $J(p_f-p_g)=p_g+q_k$; hence this relation can hold only if q_k is an integer. But the converse holds true only when $k=2$. For example, in the following selection q_k is an integer but no relation (22) holds:

When $k = 2$, (22) always holds* when q_k is an integer; for, since $q_2(p_1 - 0)/p_1 = 0 + q_2$, the relation holds if q_2/p_1 is an integer. If $p_1 \neq 1$, the period of the finite system is a multiple of that of the infinitesimal system; but no such orbits exist.

Since $n_k T = 2q_k\pi$, the case q_k an integer is the one where the consecutive conjunctions of the infinitesimal system occur at the same absolute longitude. And since, by taking T_{fg} to be the synodic period of the two infinitesimal satellites M_f and M_g,

$$n_g T_{fg} = n_g \frac{2\pi}{n_f - n_g},$$

it follows that, when (22) holds, the consecutive conjunctions of the infinitesimal pair M_f and M_g occur all in the same absolute longitude.

Moreover, since all the w_i vanish at the beginning and end of each period, all the "grand conjunctions" of the finite system occur at the same longitudes as those of the infinitesimal system, and intermediate conjunctions of any finite pair occur at very nearly the same longitudes as those of the corresponding infinitesimal pair.

These facts suggest a physical reason for the non-existence of periodic orbits under certain circumstances. The greater part of the mutual disturbances of two bodies occurs while they are near conjunction; and, if the consecutive grand conjunctions occur at exactly the same longitude, the perturbations of the elements would tend to be cumulative. Nevertheless, if there be more than two bodies, the mutual disturbances may so balance each other as to yield periodic orbits, especially if the bodies be far apart (i. e., a sufficiently small), unless (22) holds. But if two bodies have conjunctions between the grand conjunctions, all occurring very near the same longitude, the other bodies can not counterbalance the large perturbations of the two.

More exactly, there exists a range of values of the masses and the e_i, including zero, for which periodic orbits are impossible; so that, unless the orbits for $\mu = 0$ are eccentric rather than circular, there are for small values of μ no periodic plane orbits of k satellites when (22) holds.

So far as this result extends, it would indicate that no asteroids would be found, whose periods compared to Jupiter's are in the ratios $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, etc. Those whose periods are nearly in any such ratio should be found to be subject to very great perturbations.

It is well known that lacunary spaces of the sort just mentioned do occur among the asteroids. That there are such spaces also when the ratio of the periods is $\frac{3}{5}$, and other such values, is not surprising, as in such a case the

* In POINCARÉ's discussion of the problem of three bodies, therefore, the case where q_k is an integer and (22) does not hold can not arise.

slightest deviation from the correct initial values destroys periodicity, there being Poisson terms in the solutions.

§ 12. *Jupiter's satellites* I, II and III.

Of Jupiter's longer known satellites, the innermost three move almost exactly in a plane, apparently in periodic orbits having symmetrical conjunctions; and their masses with respect to that of the planet are very small. Since for orbits of type I the increase of the longitude of M_i during a period is independent of μ, being equal to $n_i T$, the average angular velocity of each satellite for a period may be taken as the corresponding n.

The unit of time being the sidereal day, and the unit of mass, the mass of Jupiter, the observational data are:*

$$\lambda_1 = \pi, \qquad \lambda_2 = 0, \qquad \lambda_3 = 0,$$

$$(67) \quad M_1 = 0.000017, \qquad M_2 = 0.000023, \qquad M_3 = 0.000088,$$

$$n_1 = 203^\circ.48895528, \qquad n_2 = 101^\circ.37472396, \qquad n_3 = 50^\circ.31760833.$$

Since

$$\frac{n_1 - n_3}{3} = 51^\circ.05711565, \qquad n_2 - n_3 = 51^\circ.05711563,$$

it follows that the n_i of (67) satisfy, far beyond observational accuracy, the equations (2), where $\nu = 51^\circ.05711564$. Taking arbitrarily $\mu = .0001$, one obtains

$$\beta_1 = .17, \qquad \beta_2 = .23, \qquad \beta_3 = .88,$$

$$(68) \quad p_1 = 3, \qquad p_2 = 1, \qquad q_3 = .985516077,$$

$$\phi_{12} = \pi + 2\tau, \qquad \phi_{13} = \pi + 3\tau, \qquad \phi_{23} = \tau,$$

$$\log \alpha_{12} = \bar{1}.79825923, \qquad \log \alpha_{13} = \bar{1}.59545277, \qquad \log \alpha_{23} = \bar{1}.79719354.$$

It seems desirable, however, to retain the β's in the computations, inasmuch as a new determination of the masses may render it necessary to use other values than those given above. The δ_{ij} are then given by:

$$\log \frac{\delta_{12}}{\beta_2} = 0.7974876, \qquad \log \frac{\delta_{21}}{\beta_1} = 0.9992284, \qquad \log \frac{\delta_{31}}{\beta_1} = 0.7964219,$$

$$(69)$$

$$\log \frac{\delta_{13}}{\beta_3} = 0.3918747, \qquad \log \frac{\delta_{23}}{\beta_3} = 0.1901339, \qquad \log \frac{\delta_{32}}{\beta_2} = 0.3929403.$$

Using the values of α_{ij} given in (68) the $F_m(\epsilon_{ij})$ and $G_m(\epsilon_{ij})$ are readily obtained from the tables of coefficients given by LEVERRIER.† To obtain the $D_{i,1}^{(m)}$ and

* TISSERAND, *Traité de Mécanique Céleste*, vol. 4, p. 2.

† Annales de l'Observatoire de Paris. (Mémoires) V. 2. Suppl.

$E_{i,1}^{(m)}$ of (44), equations (43) may be written

$$
(70)\quad \begin{aligned} & w_{i,1}'' + 2q_i x_{i,1}' + \sum_j{}' \delta_{ij} \sum_{m=1}^{\infty} U_{i,j}^{(m)} \sin m\phi_{ji} = 0\,, \\ & x_{i,1}'' - 2q_i w_{i,1}' - 3q_i^2 x_{i,1} + q_i^2 \beta_i + \sum_j{}' \delta_{ij} \sum_{m=0}^{\infty} V_{i,j}^{(m)} \cos m\phi_{ji} = 0\,, \end{aligned} \qquad (i = 1, \cdots, k),
$$

where

$$
U_{i,j}^{(1)} = 1 + \tfrac{1}{2}\eta_{i,j}^3 (F_2 - 2F_0), \qquad U_{i,j}^{(m)} = \frac{\eta_{ij}^3}{2}(F_{m+1} - F_{m-1}) \quad (m \neq 1),
$$

$$
V_{i,j}^{(1)} = 1 + \tfrac{1}{2}\eta_{ij}^3 [2\alpha_{ij} F_1 - (F_2 + 2F_0)], \qquad V_{i,j}^{(m)} = \frac{\eta_{ij}^3}{2}[2\alpha_{ij} F_m - (F_{m+1} + F_{m-1})] \quad (m \neq 1).
$$

Then: for $m =$ a prime, $D_{3,1}^{(m)} = U_{3,2}^{(m)}$ and $E_{3,1}^{(m)} = V_{3,2}^{(m)}$; $D_{2,1}^{(m)} = -U_{2,3}^{(m)}$, $E_{2,1}^{(m)} = V_{2,2}^{(m)}$; $D_{1,1}^{(m)} = E_{1,1}^{(m)} = 0$. Similarly, for $m = 2k$, where k is prime, $D_{1,1}^{(m)}$ and $E_{1,1}^{(m)}$ involve only $U_{1,2}^{(k)}$ and $V_{1,2}^{(k)}$; $D_{2,1}^{(m)}$ and $E_{2,1}^{(m)}$ involve both $U_{2,1}^{(k)}$ and $U_{2,3}^{(m)}$, $V_{2,1}^{(k)}$ and $V_{2,3}^{(m)}$; $D_{3,1}^{(m)}$ and $E_{3,1}^{(m)}$ involve only $U_{3,2}^{(m)}$ and $V_{3,2}^{(m)}$; etc.

In obtaining the successive $A_{i,n}^{(m)}$ and $B_{i,n}^{(m)}$ from (52), the smallest divisor introduced is $1 - q_3^2$, or 0.02875806. This divisor decreases materially the effectiveness of the small value of μ; nevertheless, the terms above those of the second order seem rather insignificant, and will not be computed here.

From the x_i given below the r_i are to be obtained by multiplying by the a_i, and from the w_i the v_i are obtained by adding the $n_i t$, whose values depend upon the values of the gravitational constant and of Jupiter's mass. The following values are found for the $x_{i,1}(\tau)$ and $w_{i,1}(\tau)$:

$$
\begin{aligned}
x_{1,1}(\tau) = {} & (.33333\beta_1 - .07237\beta_2 - .01231\beta_3) - .44852\beta_2 \cos 2\tau - .08224\beta_3 \cos 3\tau \\
& - 94.43952\beta_2 \cos 4\tau + (.38582\beta_2 - .09643\beta_3) \cos 6\tau - .11181\beta_2 \cos 8\tau \\
& + .01380\beta_3 \cos 9\tau + .04361\beta_2 \cos 10\tau - (.01930\beta_2 + .00312\beta_3) \cos 12\tau \\
& + .00743\beta_2 \cos 14\tau + .00083\beta_3 \cos 15\tau - .00457\beta_2 \cos 16\tau \\
& + (.00235\beta_2 - .00024\beta_3) \cos 18\tau - .00124\beta_2 \cos 20\tau + .00007\beta_3 \cos 21\tau \\
& + .00067\beta_2 \cos 22\tau - (.00036\beta_2 + .00002\beta_3) \cos 24\tau \\
& + .00020\beta_2 \cos 26\tau + .00001\beta_3 \cos 27\tau - .00011\beta_2 \cos 28\tau \\
& + .00006\beta_2 \cos 30\tau - .00004\beta_2 \cos 32\tau + .00002\beta_2 \cos 34\tau \\
& - .00001\beta_2 \cos 36\tau + .00001\beta_2 \cos 38\tau + \cdots.
\end{aligned}
$$

$$
\begin{aligned}
w_{1,1}(\tau) = {} & - .73709\beta_2 \sin 2\tau - .10042\beta_3 \sin 3\tau - 94.32411\beta_2 \sin 4\tau \\
& + (.33625\beta_2 - .08575\beta_3) \sin 6\tau - .08890\beta_2 \sin 8\tau + .01088\beta_3 \sin 9\tau \\
& + .03248\beta_2 \sin 10\tau - (.01368\beta_2 + .00227\beta_3) \sin 12\tau + .00576\beta_2 \sin 14\tau \\
& + .00057\beta_3 \sin 15\tau - .00302\beta_2 \sin 16\tau + (.00152\beta_2 - .00016\beta_3) \sin 18\tau \\
& - .00078\beta_2 \sin 20\tau + .00005\beta_3 \sin 21\tau + .00041\beta_2 \sin 22\tau
\end{aligned}
$$

$$- (.00022\beta_2 + .00001\beta_3) \sin 24\tau + .00012\beta_2 \sin 26\tau - .00007\beta_2 \sin 28\tau$$
$$+ .00004\beta_2 \sin 30\tau - .00002\beta_2 \sin 32\tau + .00001\beta_2 \sin 34\tau$$
$$- .00001\beta_2 \sin 36\tau + \cdots.$$

$$x_{2,1}(\tau) = (.49162\beta_1 + .33333\beta_2 - .07159\beta_3) - .49616\beta_3 \cos \tau$$
$$- (65.88537\beta_1 + 54.22316\beta_3) \cos 2\tau - .42278\beta_3 \cos 3\tau$$
$$+ (.47648\beta_1 - .11992\beta_3) \cos 4\tau - .04601\beta_3 \cos 5\tau$$
$$- (.11925\beta_1 + .02011\beta_3) \cos 6\tau - .00946\beta_3 \cos 7\tau$$
$$+ (.04236\beta_1 - .00467\beta_3) \cos 8\tau - .00238\beta_3 \cos 9\tau$$
$$- (.01756\beta_1 + .00125\beta_3) \cos 10\tau - .00067\beta_3 \cos 11\tau$$
$$+ (.00831\beta_1 - .00036\beta_3) \cos 12\tau - .00020\beta_3 \cos 13\tau$$
$$- (.00382\beta_1 + .00011\beta_3) \cos 14\tau - .00006\beta_3 \cos 15\tau$$
$$+ (.00191\beta_1 - .00004\beta_3) \cos 16\tau - .00002\beta_3 \cos 17\tau$$
$$- (.00098\beta_1 + .00001\beta_3) \cos 18\tau - .00001\beta_3 \cos 19\tau$$
$$+ .00052\beta_1 \cos 20\tau - .00028\beta_1 \cos 22\tau + .00015\beta_1 \cos 24\tau$$
$$- .00008\beta_1 \cos 26\tau + .00005\beta_1 \cos 28\tau - .00003\beta_1 \cos 30\tau$$
$$+ .00001\beta_1 \cos 32\tau - .00001\beta_1 \cos 34\tau + .00001\beta_1 \cos 36\tau \cdots.$$

$$w_{2,1}(\tau) = .83119\beta_3 \sin \tau + (64.53215\beta_1 - 54.05343\beta_3) \sin 2\tau - .35832\beta_3 \sin 3\tau$$
$$- (.32599\beta_1 + .09204\beta_3) \sin 4\tau - .03302\beta_3 \sin 5\tau$$
$$+ (.07104\beta_1 - .01374\beta_3) \sin 6\tau - .00623\beta_3 \sin 7\tau$$
$$- (.02363\beta_1 + .00298\beta_3) \sin 8\tau - .00152\beta_3 \sin 9\tau$$
$$+ (.00945\beta_1 - .00076\beta_3) \sin 10\tau - .00040\beta_3 \sin 11\tau$$
$$- (.00425\beta_1 + .00021\beta_3) \sin 12\tau - .00012\beta_3 \sin 13\tau$$
$$+ (.00173\beta_1 - .00006\beta_3) \sin 14\tau - .00004\beta_3 \sin 15\tau$$
$$- (.00098\beta_1 + .00002\beta_3) \sin 16\tau - .00001\beta_3 \sin 17\tau$$
$$+ (.00050\beta_1 - .00001\beta_3) \sin 18\tau - .00026\beta_1 \sin 20\tau$$
$$+ .00014\beta_1 \sin 22\tau - .00008\beta_1 \sin 24\tau + .00004\beta_1 \sin 26\tau$$
$$- .00002\beta_1 \sin 28\tau + .00001\beta_1 \sin 30\tau - .00001\beta_1 \sin 32\tau + \cdots.$$

$$x_{3,1}(\tau) = (.37879\beta_1 + .49038\beta_2 + .33333\beta_3) + 87.67296\beta_2 \cos \tau$$
$$+ .52186\beta_2 \cos 2\tau - (.64483\beta_1 - .12770\beta_2) \cos 3\tau + .04461\beta_2 \cos 4\tau$$
$$+ .01826\beta_2 \cos 5\tau + (.01205\beta_1 + .00819\beta_2) \cos 6\tau + .00389\beta_2 \cos 7\tau$$
$$+ .00193\beta_2 \cos 8\tau - (.00224\beta_1 - .00099\beta_2) \cos 9\tau + .00052\beta_2 \cos 10\tau$$
$$+ .00028\beta_2 \cos 11\tau + (.00053\beta_1 + .00015\beta_2) \cos 12\tau$$
$$+ .00008\beta_2 \cos 13\tau + .00005\beta_2 \cos 14\tau - (.00014\beta_1 - .00003\beta_2) \cos 15\tau$$
$$+ .00001\beta_2 \cos 16\tau + .00001\beta_2 \cos 17\tau + .00004\beta_1 \cos 18\tau$$
$$- .00001\beta_1 \cos 21\tau + \cdots.$$

$$w_{3,1}(\tau) = -85.53233\beta_2 \sin\tau - .34477\beta_2 \sin 2\tau - (.11320\beta_1 + .07379\beta_2)\sin 3\tau$$
$$- .02377\beta_2 \sin 4\tau - .00939\beta_2 \sin 5\tau - (.00535\beta_1 + .00413\beta_2)\sin 6\tau$$
$$- .00195\beta_2 \sin 7\tau - .00096\beta_2 \sin 8\tau + (.00099\beta_1 - .00049\beta_2)\sin 9\tau$$
$$- .00025\beta_2 \sin 10\tau - .00014\beta_2 \sin 11\tau - (.00023\beta_1 + .00007\beta_2)\sin 12\tau$$
$$- .00004\beta_2 \sin 13\tau - .00002\beta_2 \sin 14\tau + (.00006\beta_1 - .00001\beta_2)\sin 15\tau$$
$$- .00001\beta_2 \sin 16\tau - .00002\beta_1 \sin 18\tau + .00001\beta_1 \sin 21\tau + \cdots.$$

Since the second order terms carry the factor μ^2, they need not be obtained accurate to as many decimals; but almost as many multiples of τ are necessary.

The $x_{i,2}$ and $w_{i,2}$ [which are to be multiplied by .0001, if compared with the above terms] are:

$$x_{1,2}(\tau) = (-.1\beta_1^2 + 20.9\beta_1\beta_2 - .2\beta_1\beta_3 - 15.1\beta_2^2 + 12.2\beta_2\beta_3) - 60.4\beta_2\beta_3 \cos\tau$$
$$+ (687.2\beta_1\beta_2 - 85.3\beta_2^2 + 243.7\beta_2\beta_3)\cos 2\tau + (1.3\beta_1\beta_3 - .4\beta_2\beta_3$$
$$- 1.2\beta_3^2)\cos 3\tau + (49138.2\beta_1\beta_2 - 7844.0\beta_2^2 + 20059.1\beta_2\beta_3)\cos 4\tau$$
$$+ 10.3\beta_2\beta_3 \cos 5\tau - (228.2\beta_1\beta_2 - .6\beta_1\beta_3 + 22.5\beta_2^2 + 171.3\beta_2\beta_3)\cos 6\tau$$
$$+ 9.8\beta_2\beta_3 \cos 7\tau + (67.2\beta_1\beta_2 + 4475.5\beta_2^2 + 50.2\beta_2\beta_3)\cos 8\tau$$
$$-(.6\beta_1\beta_3 - 5.9\beta_2\beta_3 + .8\beta_3^2)\cos 9\tau - (27.9\beta_1\beta_2 + 4.0\beta_2^2 + 19.3\beta_2\beta_3)\cos 10\tau$$
$$+ 2.4\beta_2\beta_3 \cos 11\tau + (12.8\beta_1\beta_2 + 25.5\beta_2^2 + 14.2\beta_2\beta_3)\cos 12\tau$$
$$- 1.9\beta_2\beta_3 \cos 13\tau - (6.4\beta_1\beta_2 + 13.2\beta_2^2 + 8.3\beta_2\beta_3)\cos 14\tau$$
$$- .1\beta_2\beta_3 \cos 15\tau + (3.7\beta_1\beta_2 + 8.0\beta_2^2 + 4.9\beta_2\beta_3)\cos 16\tau + .2\beta_2\beta_3 \cos 17\tau$$
$$- (1.9\beta_1\beta_2 + 4.5\beta_2^2 + 2.4\beta_2\beta_3)\cos 18\tau - .3\beta_2\beta_3 \cos 19\tau + (1.0\beta_1\beta_2$$
$$+ 2.7\beta_2^2 + 1.2\beta_2\beta_3)\cos 20\tau - (.6\beta_1\beta_2 + 1.6\beta_2^2 + .6\beta_2\beta_3)\cos 22\tau$$
$$+ (.3\beta_1\beta_2 + 1.0\beta_2^2 + .4\beta_2\beta_3)\cos 24\tau - (.2\beta_1\beta_2 + .6\beta_2^2 + .3\beta_2\beta_3)\cos 26\tau$$
$$+ (.1\beta_1\beta_2 + .4\beta_2^2 + .2\beta_2\beta_3)\cos 28\tau - (.1\beta_1\beta_2 + .2\beta_2^2 + .1\beta_2\beta_3)\cos 30\tau$$
$$+ .2\beta_2^2 \cos 32\tau - .1\beta_2^2 \cos 34\tau + \cdots.$$

$$w_{1,2}(\tau) = 466.4\beta_2\beta_3 \sin\tau - (2409.0\beta_1\beta_2 - 271.2\beta_2^2 + 892.8\beta_2\beta_3)\sin 2\tau$$
$$- (3.0\beta_1\beta_2 - .9\beta_2\beta_3 - 3.3\beta_3^2)\sin 3\tau - (97989.6\beta_1\beta_2 - 15618.9\beta_2^2$$
$$+ 40019.6\beta_2\beta_3)\sin 4\tau - 7.0\beta_2\beta_3 \sin 5\tau + (351.8\beta_1\beta_2 - .9\beta_1\beta_3$$
$$+ 28.0\beta_2^2 + 262.6\beta_2\beta_3)\sin 6\tau + (.2\beta_2^2 - 18.6\beta_2\beta_3)\sin 7\tau - (89.6\beta_1\beta_2$$
$$+ 7818.4\beta_2^2 + 66.8\beta_2\beta_3)\sin 8\tau + (.6\beta_1\beta_3 - 5.4\beta_2\beta_3 + .7\beta_3^2)\sin 9\tau$$
$$+ (33.6\beta_1\beta_2 + 10.4\beta_2^2 + 19.2\beta_2\beta_3)\sin 10\tau - 2.8\beta_2\beta_3 \sin 11\tau$$
$$- (14.4\beta_1\beta_2 + 34.2\beta_2^2 + 16.3\beta_2\beta_3)\sin 12\tau + 2.8\beta_2\beta_3 \sin 3\tau$$
$$+ (7.1\beta_1\beta_2 + 17.2\beta_2^2 + 9.1\beta_2\beta_3)\sin 14\tau + .1\beta_2\beta_3 \sin 15\tau - (3.9\beta_1\beta_2$$
$$+ 9.5\beta_2^2 + 5.5\beta_2\beta_3)\sin 16\tau - .3\beta_2\beta_3 \sin 17\tau + (2.0\beta_1\beta_2 + 5.1\beta_2^2$$
$$+ 2.5\beta_2\beta_3)\sin 18\tau + .2\beta_2\beta_3 \sin 19\tau - (1.1\beta_1\beta_2 + 3.1\beta_2^2$$
$$+ 1.3\beta_2\beta_3)\sin 20\tau - .1\beta_2\beta_3 \sin 21\tau + (.6\beta_1\beta_2 + 2.0\beta_2^2$$

$$+ .7\beta_2\beta_3)\sin 22\tau - (.3\beta_1\beta_2 + 1.1\beta_2^2 + .5\beta_2\beta_3)\sin 24\tau + (.2\beta_1\beta_2 + .7\beta_2^2 + .3\beta_2\beta_3)\sin 26\tau - (.1\beta_1\beta_2 + .4\beta_2^2 + .2\beta_2\beta_3)\sin 28\tau + .2\beta_2^2\sin 30\tau - .1\beta_2^2\sin 32\tau + \cdots.$$

$$\begin{aligned} x_{2,2}(\tau) = {} & (2770.2\beta_1^2 - 17.0\beta_1\beta_2 - .2\beta_2^2 + 4730.0\beta_1\beta_3 + 27.9\beta_2\beta_3 - 6.9\beta_3^2) \\ & - (289.2\beta_1\beta_3 + 757.4\beta_2\beta_3 + 74.9\beta_3^2)\cos\tau + (46102.6\beta_1^2 \\ & - 31911.6\beta_1\beta_2 + 17666.4\beta_1\beta_3 + 41787.2\beta_2\beta_3 - 16064.0\beta_3^2)\cos 2\tau \\ & + (141.9\beta_1\beta_3 - 29.9\beta_2\beta_3 + 170.2\beta_3^2)\cos 3\tau + (519.3\beta_1^2 - 51.8\beta_1\beta_2 \\ & + 4573.9\beta_1\beta_3 + 92.9\beta_2\beta_3 + 992.1\beta_3^2)\cos 4\tau + (35.5\beta_1\beta_3 + 38.6\beta_2\beta_3 \\ & + 35.1\beta_3^2)\cos 5\tau + (64.3\beta_1^2 + 36.8\beta_1\beta_2 + 41.8\beta_1\beta_3 + 17.9\beta_2\beta_3 \\ & + 10.9\beta_3^2)\cos 6\tau + (4.2\beta_1\beta_3 + 8.4\beta_2\beta_3 + 7.7\beta_3^2)\cos 7\tau - (22.8\beta_1^2 \\ & + 26.1\beta_1\beta_2 + 16.7\beta_1\beta_3 - 1.5\beta_2\beta_3 - 1.4\beta_3^2)\cos 8\tau + (.7\beta_1\beta_3 \\ & + 2.3\beta_2\beta_3 + 2.4\beta_3^2)\cos 9\tau + (9.5\beta_1^2 + 15.4\beta_1\beta_2 + 9.6\beta_1\beta_3 + 1.2\beta_2\beta_3 \\ & + 1.6\beta_3^2)\cos 10\tau + (.4\beta_1\beta_3 + .8\beta_2\beta_3 + .9\beta_3^2)\cos 11\tau - (5.0\beta_1^2 + 8.0\beta_1\beta_2 \\ & + 5.0\beta_1\beta_3 - .4\beta_2\beta_3 - .6\beta_3^2)\cos 12\tau + (.2\beta_2\beta_3 + .3\beta_3^2)\cos 13\tau \\ & + (2.8\beta_1^2 + 5.3\beta_1\beta_2 + 2.9\beta_1\beta_3 + .2\beta_2\beta_3 + .2\beta_3^2)\cos 14\tau + (.1\beta_2\beta_3 \\ & + .1\beta_3^2)\cos 15\tau - (1.4\beta_1^2 + 3.0\beta_1\beta_2 + 1.6\beta_1\beta_3 - .1\beta_3^2)\cos 16\tau \\ & + .1\beta_3^2\cos 17\tau + (.8\beta_1^2 + 1.8\beta_1\beta_2 + .9\beta_1\beta_3)\cos 18\tau - (.5\beta_1^2 + 1.0\beta_1\beta_2 \\ & + .5\beta_1\beta_3)\cos 20\tau + (.3\beta_1^2 + .8\beta_1\beta_2 + .3\beta_1\beta_3)\cos 22\tau - (.2\beta_1^2 \\ & + .6\beta_1\beta_2 + .2\beta_1\beta_3)\cos 24\tau + (.1\beta_1^2 + .4\beta_1\beta_2 + .1\beta_1\beta_3)\cos 26\tau \\ & - (.1\beta_1^2 + .2\beta_1\beta_2 + .1\beta_1\beta_3)\cos 28\tau + .1\beta_1\beta_2\cos 30\tau + \cdots. \end{aligned}$$

$$\begin{aligned} w_{2,2}(\tau) = {} & (1229.4\beta_1\beta_3 + 2487.1\beta_2\beta_3 + 327.4\beta_3^2)\sin\tau - (91766.9\beta_1^2 \\ & - 63695.3\beta_1\beta_2 + 35062.7\beta_1\beta_3 + 83097.4\beta_2\beta_3 - 21988.4\beta_3^2)\sin 2\tau \\ & - (170.3\beta_1\beta_3 - 112.6\beta_2\beta_3 + 254.0\beta_3^2)\sin 3\tau + (1161.5\beta_1^2 + 53.3\beta_1\beta_2 \\ & - 4555.3\beta_1\beta_3 - 125.8\beta_2\beta_3 - 1730.8\beta_3^2)\sin 4\tau - (35.4\beta_1\beta_3 \\ & + 49.0\beta_2\beta_3 + 52.2\beta_3^2)\sin 5\tau - (84.2\beta_1^2 + 48.0\beta_1\beta_2 + 47.5\beta_1\beta_3 \\ & + 21.3\beta_2\beta_3 + 13.5\beta_3^2)\sin 6\tau - (3.7\beta_1\beta_3 + 9.6\beta_2\beta_3 + 9.6\beta_3^2)\sin 7\tau \\ & + (26.7\beta_1^2 + 22.4\beta_1\beta_2 + 17.9\beta_1\beta_3 - 3.6\beta_2\beta_3 - 3.6\beta_3^2)\sin 8\tau - (.8\beta_1\beta_3 \\ & + 2.6\beta_2\beta_3 + 3.0\beta_3^2)\sin 9\tau - (10.6\beta_1^2 + 13.2\beta_1\beta_2 + 9.8\beta_1\beta_3 + 1.4\beta_2\beta_3 \\ & + 1.8\beta_3^2)\sin 10\tau - (.5\beta_1\beta_3 + .8\beta_2\beta_3 + 1.1\beta_3^2)\sin 11\tau + (4.9\beta_1^2 \\ & + 7.4\beta_1\beta_3 + 4.7\beta_1\beta_3 - .5\beta_2\beta_3 - .6\beta_3^2)\sin 12\tau - (.3\beta_2\beta_3 + .3\beta_3^2)\sin 13\tau \\ & - (2.6\beta_1^2 + 4.5\beta_1\beta_2 + 2.7\beta_1\beta_3 + .2\beta_2\beta_3 + .2\beta_3^2)\sin 14\tau - .1\beta_3^2\sin 15\tau \\ & + (1.4\beta_1^2 + 2.7\beta_1\beta_2 + 1.4\beta_1\beta_3 - .1\beta_3^2)\sin 16\tau - (.7\beta_1^2 + 1.6\beta_1\beta_2 \\ & + .9\beta_1\beta_3)\sin 18\tau + (.4\beta_1^2 + .9\beta_1\beta_2 + .5\beta_1\beta_3)\sin 20\tau - (.3\beta_1^2 + .6\beta_1\beta_2 \\ & + .3\beta_1\beta_3)\sin 22\tau + (.2\beta_1^2 + .4\beta_1\beta_2 + .2\beta_1\beta_3)\sin 24\tau - (.1\beta_1^2 + .3\beta_1\beta_2 \\ & + .1\beta_1\beta_3)\sin 26\tau + .1\beta_1\beta_2\sin 28\tau + \cdots. \end{aligned}$$

$$x_{3,2}(\tau)=(.3\beta_1^2-69.3\beta_1\beta_3+5126.0\beta_2^2-202.4\beta_2\beta_3-10.0\beta_3^2)-(82613.5\beta_1\beta_2$$
$$+7650.0\beta_1\beta_3+553.6\beta_2^2+36924.0\beta_2\beta_3-6499.0\beta_3^2)\cos\tau+(18.2\beta_1\beta_2$$
$$-84.5\beta_1\beta_3+1144.6\beta_2^2-182.5\beta_2\beta_3-25.7\beta_3^2)\cos 2\tau+(.4\beta_1^2$$
$$-28.4\beta_1\beta_2+7.7\beta_1\beta_3-44.0\beta_2^2-67.0\beta_2\beta_3-11.8\beta_3^2)\cos 3\tau$$
$$-(20.3\beta_1\beta_2+9.1\beta_1\beta_3+7.5\beta_2^2+23.8\beta_2\beta_3+11.6\beta_3^2)\cos 4\tau$$
$$-(5.5\beta_1\beta_2+4.2\beta_1\beta_3+2.7\beta_2^2+11.1\beta_2\beta_3+4.8\beta_3^2)\cos 5\tau-(.1\beta_1^2$$
$$+.5\beta_1\beta_2+1.9\beta_1\beta_3+1.3\beta_2^2+5.2\beta_2\beta_3+4.2\beta_3^2)\cos 6\tau-(15.9\beta_1\beta_2$$
$$+1.0\beta_1\beta_3+.6\beta_2^2+3.0\beta_2\beta_3+2.3\beta_3^2)\cos 7\tau+(1.0\beta_1\beta_2$$
$$-.6\beta_1\beta_3-.3\beta_2^2-1.6\beta_2\beta_3-1.3\beta_3^2)\cos 8\tau-(.1\beta_1\beta_2+.4\beta_1\beta_3+1\beta_2^2$$
$$+.9\beta_2\beta_3+.7\beta_3^2)\cos 9\tau-(.7\beta_1\beta_2+.1\beta_1\beta_3+.5\beta_2\beta_3+.1\beta_2^2$$
$$+.4\beta_3^2)\cos 10\tau-(.3\beta_1\beta_2+.1\beta_1\beta_3+.1\beta_2^2+.3\beta_2\beta_3+.3\beta_3^2)\cos 11\tau$$
$$-(.2\beta_2\beta_3+.2\beta_3^2)\cos 12\tau+(.2\beta_1\beta_2-.1\beta_2\beta_3-.1\beta_3^2)\cos 13\tau$$
$$+(.1\beta_1\beta_2-.1\beta_2\beta_3-.1\beta_3^2)\cos 14\tau-.1\beta_1\beta_2\cos 16\tau+\cdots.$$

$$w_{3,2}(\tau)=(163608.6\beta_1\beta_2+15078.4\beta_1\beta_3+1357.9\beta_2^2+72947.9\beta_2\beta_3$$
$$-12809.7\beta_3^2)\sin\tau+(25.6\beta_1\beta_2+83.2\beta_1\beta_3+1791.2\beta_2^2+204.1\beta_2\beta_3$$
$$+25.3\beta_3^2)\sin 2\tau+(.1\beta_1^2+57.4\beta_1\beta_2-5.1\beta_1\beta_3+82.3\beta_2^2+64.4\beta_2\beta_3$$
$$+7.8\beta_3^2)\sin 3\tau-(6.7\beta_1\beta_2-4.5\beta_1\beta_3-24.2\beta_2^2+18.0\beta_2\beta_3$$
$$+5.7\beta_3^2)\sin 4\tau+(8.2\beta_1\beta_2+1.7\beta_1\beta_3+9.1\beta_2^2+8.6\beta_2\beta_3$$
$$+1.9\beta_3^2)\sin 5\tau+(1.8\beta_1\beta_2+.7\beta_1\beta_3+4.9\beta_2^2+4.1\beta_2\beta_3$$
$$+1.4\beta_3^2)\sin 6\tau-(.8\beta_1\beta_2-.3\beta_1\beta_3-2.4\beta_2^2-2.7\beta_2\beta_3-.7\beta_3^2)\sin 7\tau$$
$$-(.5\beta_1\beta_2-.1\beta_1\beta_3-1.4\beta_2^2-1.5\beta_2\beta_3-.3\beta_3^2)\sin 8\tau+(.3\beta_1\beta_2$$
$$+.1\beta_1\beta_3+.7\beta_2^2+.9\beta_2\beta_3+.2\beta_3^2)\sin 9\tau+(.6\beta_1\beta_2+.4\beta_2^2+.6\beta_2\beta_3$$
$$+.1\beta_3^2)\sin 10\tau+(.3\beta_1\beta_2+.3\beta_2^2+.3\beta_3\beta_2+.1\beta_3^2)\sin 11\tau$$
$$+(.1\beta_1\beta_2+.2\beta_2^2+.2\beta_2\beta_3)\sin 12\tau-(.1\beta_1\beta_2-.1\beta_2^2-.1\beta_2\beta_3)\sin 13\tau$$
$$+(.1\beta_1\beta_2+.1\beta_2^2+.1\beta_2\beta_3)\sin 14\tau+.1\beta_2\beta_3\sin 15\tau+.1\beta_1\beta_2\sin 16\tau$$
$$+\cdots.$$

The above coefficients, when multiplied by the proper powers of μ, give the values of the coördinates, so far as they depend upon the terms of the first and second orders, accurate to nine decimal places. Which places would be affected by terms of higher orders cannot be definitely stated without information as to (1) whether the series (40) converge for $\mu=.0001$, and (2) how rapidly they converge. Although the question of convergence for this value has not yet been answered, it is not improbable that the radius of convergence, whose existence has been proved above, is greater than .0001.