THE PROPERTIES OF CURVES IN SPACE WHICH MINIMIZE A DEFINITE INTEGRAL*

BY

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While the theory of the calculus of variations from the stand-point of WEIERSTRASS has been treated extensively for problems in the plane by KNESER, BOLZA and many other writers, comparatively little seems to have been published concerning the extension of the Weierstrassian theory to problems in space. It is the purpose of the present paper to discuss from as geometrical a stand-point as possible the arcs

$$(1) \qquad x = x(t), \qquad y = y(t), \qquad z = z(t) \qquad (t_0 \leqq t \leqq t_1),$$

which minimize or maximize an integral of the form

$$J = \int_{t_0}^{t_1} F(x, y, z, x', y', z')\,dt$$

involving three dependent variables, where F satisfies the homogeneity condition

$$(2) \qquad F(x, y, z, \kappa x', \kappa y', \kappa z') = \kappa F(x, y, z, x', y', z') \qquad (\kappa > 0).$$

The end-points of the minimizing curve may be fixed, or one of them may vary on a given curve or surface.

The papers hitherto published with regard to problems of this type have for the most part dealt with the derivation of necessary conditions by means of the first and second variations.† In the following discussion the necessary conditions which an arc C must satisfy in order to minimize or maximize the integral are derived without the use of the second variation. Sufficient conditions which insure a minimum or a maximum are derived with the help of the Hilbert invariant integral. No discussion of the problem which might be called

* Presented to the Society October 27, 1906. Received for publication May 12, 1908.

† In the dissertation of GERNET, the integral is taken in the form

$$\int f\left(x, y, z, \frac{dy}{dx}, \frac{dz}{dx}\right) dx,$$

and necessary conditions are derived without the use of the second variation. See N. GERNET, *Untersuchungen zur Variationsrechnung*, Dissertation, Göttingen, 1902.

complete, so far as is known to the writers, has hitherto been published, and but one writer * has discussed the problem in parametric representation. The study of the sufficient conditions for fixed and for variable end-points seems to be entirely new.

The first six sections contain the theory of the problem when both end-points are fixed, or when one is allowed to vary on a surface. The case when one end-point is allowed to vary on a curve is treated in the last two sections.

§ 1. *The Euler equations.*

In order to carry through the methods applied in this paper it will be assumed that the function F is of class C''' for all points (x, y, z) in a region R of space, and for all sets of values of (x', y', z') different from $(0, 0, 0)$.† The functions of the form (1) defining arcs along which the value of the integral is to be taken will be supposed to be of class C' and such that for every value of t in the interval $t_0 \leqq t \leqq t_1$ the values x, y, z, x', y', z' are of the kind just described. When an end-point is allowed to vary on a curve or on a surface, the functions defining the curve or surface are supposed to be of class C'.

A number of identities can be derived from (2) which will be useful in this and succeeding sections. By differentiating (2) with respect to x', it is seen that

$$F_{x'}(x, y, z, \kappa x', \kappa y', \kappa z') = F_{x'}(x, y, z, x', y', z'), \tag{3}$$

and in general it follows that a derivative of F with respect to x', y', z' is homogeneous of order $1 - n$, where n is the order of the derivative. By differentiating (2) with respect to κ and then letting $\kappa = 1$, we obtain the equation

$$x'F_{x'} + y'F_{y'} + z'F_{z'} = F. \tag{4}$$

From this equation follow immediately by differentiation

$$\begin{aligned} x'F_{x'x'} + y'F_{x'y'} + z'F_{x'z'} &= 0, \\ x'F_{y'x'} + y'F_{y'y'} + z'F_{y'z'} &= 0, \\ x'F_{z'x'} + y'F_{z'y'} + z'F_{z'z'} &= 0. \end{aligned} \tag{5}$$

From the latter set it is seen that the co-factors A_{ik} of the elements of the determinant of the first members satisfy the equations

$$x' : y' : z' = A_{11} : A_{12} : A_{13} = A_{21} : A_{22} : A_{23} = A_{31} : A_{32} : A_{33}.$$

By multiplying these three proportions by x', y', z', respectively, it follows that there exists a function F_1 such that

* VON ESCHERICH, *Die zweite Variation der einfachen Integrale*, Sitzungsberichte der Akademie der Wissenschaften in Wien, vol. CX (1901), p. 1355.

† A function is said to be of class $C^{(k)}$ if it is continuous and has continuous derivatives up to and including those of order k.

$$(6)\qquad \begin{aligned} x'^2F_1 &= A_{11} = F_{y'y'}F_{z'z'} - F^2_{y'z'},\\ y'^2F_1 &= A_{22} = F_{z'z'}F_{x'x'} - F^2_{z'x'},\\ z'^2F_1 &= A_{33} = F_{x'x'}F_{y'y'} - F^2_{x'y'},\\ y'z'F_1 &= A_{23} = F_{x'y'}F_{x'z'} - F_{x'x'}F_{y'z'}.\\ z'x'F_1 &= A_{31} = F_{y'z'}F_{y'x'} - F_{y'y'}F_{zx'}:\\ x'y'F_1 &= A_{12} = F_{z'x'}F_{z'y'} - F_{z'z'}F_{x'y'}. \end{aligned}$$

It follows also that

$$(7)\qquad \begin{vmatrix} F_{x'x'} & F_{x'y'} & F_{x'z'} & x' \\ F_{y'x'} & F_{y'y'} & F_{y'z'} & y' \\ F_{z'x'} & F_{z'y'} & F_{z'z'} & z' \\ x' & y' & z' & 0 \end{vmatrix} = -F_1(x'^2 + y'^2 + z'^2)^2,$$

and the minors of this determinant of the type

$$(8)\qquad \begin{vmatrix} F_{x'x'} & F_{x'y'} & F_{x'z'} \\ F_{y'x'} & F_{y'y'} & F_{y'z'} \\ x' & y' & z' \end{vmatrix} = F_1 z'(x'^2 + y'^2 + z'^2)$$

are similarly expressible in terms of F_1 and cannot all vanish simultaneously when F_1 is different from zero, since one at least of x', y', z' is always different from zero.

Suppose now that an arc C is at hand which minimizes the integral J. Then by a method quite similar to that used by WHITTEMORE,* it can be shown that along the arc C the three equations

$$(9)\quad F_{x'} = \int_{t_0}^{t} F_x\,dt + a, \qquad F_{y'} = \int_{t_0}^{t} F_y\,dt + b, \qquad F_{z'} = \int_{t_0}^{t} F_z\,dt + c,$$

must be satisfied, where a, b, c are constants. It is evident that a, b, c must have the same values all along C since x, y, z, x', y', z' are continuous functions of t. Let us now consider the second members of equations (9) as functions of t, and the first members as functions of x', y', z', and t, the latter occurring in the functions $x(t)$, $y(t)$, $z(t)$. The parameter t may be chosen so as to represent the length of arc along C, and then if we assume $F_1 \neq 0$ in the expression (8), it follows from the theory of implicit functions that two of the equations (9) can always be solved with

$$x'^2 + y'^2 + z'^2 = 1,$$

* *Lagrange's Equation in the Calculus of Variations and the Extension of a Theorem of Erdmann*, Annals of Mathematics, 2d series, vol. 2 (1900–1901), p. 130.

for x', y', z' in terms of t in the neighborhood of any set of values t, x', y', z' on the arc C. The resulting functions $x'(t)$, $y'(t)$, $z'(t)$, have continuous derivatives with respect to t, and on account of the uniqueness of such solutions must coincide with the values of x', y', z' along C. By differentiating equations (9) we have the following theorem:

If an arc C is of class C' and minimizes the integral J, the parameter can always be so chosen that the representation of C is of class C'' and satisfies the three Euler differential equations

$$(10) \qquad \frac{dF_{x'}}{dt} - F_x = 0, \qquad \frac{dF_{y'}}{dt} - F_y = 0, \qquad \frac{dF_{z'}}{dt} - F_z = 0.$$

These three equations are not independent, for by multiplying their first members by x', y', z'; respectively, and adding, one finds the expression

$$\frac{d}{dt}(x'F_{x'} + y'F_{y'} + z'F_{z'}) - (x''F_{x'} + y''F_{y'} + z''F_{z'} + x'F_x + y'F_y + z'F_z),$$

which with the aid of equation (4) reduces to zero. It is evident then that any one of the equations (10) is a result of the other two.

The nature of the solutions of Euler's equations may be shown by reducing the equations to a system of differential equations in canonical form. It may be assumed without loss of generality that the length of arc is the parameter, so that

$$(11) \qquad x'x'' + y'y'' + z'z'' = 0.$$

The three equations (10), with (11), define x'', y'', z'' as functions of x, y, z, x', y', z'. This may be shown most symmetrically, perhaps, by making use of an auxiliary variable u. The equations

$$(12) \quad \frac{dF_{x'}}{dt} - F_x + x'u = 0, \qquad \frac{dF_{y'}}{dt} - F_y + y'u = 0, \qquad \frac{dF_{z'}}{dt} - F_z + z'u = 0,$$

and (11), are linear in x'', y'', z'', u, with the determinant (7) as their functional determinant. The expressions

$$(13) \quad x'' = A(x, y, z, x', y', z'), \quad y'' = B(x, y, z, x', y', z'), \quad z'' = C(x, y, z, x', y', z'),$$

found by solving them are solutions also of equations (10), for by multiplying equations (12) by x', y', z', respectively, it is seen that the corresponding value of u must vanish identically. We assume now $F_1 \neq 0$ for all values of x, y, z, x', y', z' for which it is defined. Then the determinant (7) is different from zero and the second members of equations (13) are of class C' on account of the original assumptions on the function F.

The existence theorems for differential equations in the form (13) justify the following statement, the curves defined by equations (10) being called as usual extremals:

If the problem under consideration is a regular one, that is, if the function F_1 is different from zero for all values of its arguments, then through any point (x_0, y_0, z_0) of the region R there passes one and only one extremal in a given direction (x_0', y_0', z_0'). These extremals extend from boundary to boundary of R and are defined analytically by equations of the form

$$(14)\qquad \begin{aligned} x &= \phi(s, x_0, y_0, z_0, x_0', y_0', z_0'), \\ y &= \psi(s, x_0, y_0, z_0, x_0', y_0', z_0'), \\ z &= \chi(s, x_0, y_0, z_0, x_0', y_0', z_0'), \end{aligned}$$

where

$$(15)\qquad x_0'^2 + y_0'^2 + z_0'^2 = 1,$$

and for $s = 0$ the following initial conditions are satisfied,

$$(16)\qquad \begin{aligned} &\phi(0, x_0, y_0, z_0, x_0', y_0', z_0') = x_0, \qquad &&\phi_s(0, x_0, y_0, z_0, x_0', y_0', z_0') = x_0', \\ &\psi(0, x_0, y_0, z_0, x_0', y_0', z_0') = y_0, &&\psi_s(0, x_0, y_0, z_0, x_0', y_0', z_0') = y_0', \\ &\chi(0, x_0, y_0, z_0, x_0', y_0', z_0') = z_0, &&\chi_s(0, x_0, y_0, z_0, x_0', y_0', z_0') = z_0'. \end{aligned}$$

The functions ϕ, ϕ_s are of class C' for all values of their arguments defining (x, y, z)-points in the region R. and similar statements hold true for ψ and χ.

§ 2. *Conditions of transversality.*

If it is desired to minimize the integral J with respect to curves having one end-point $1(x_1, y_1, z_1)$ fixed, while the other $0(x_0, y_0, z_0)$ may vary on a fixed curve or surface, a second necessary condition upon the direction of the minimizing curve C at the point 0, corresponding to the well-known transversality condition in the plane, must be satisfied. The case where the end-point is variable on a fixed curve L will first be considered.

Let the equations of the curve L be

$$(L)\qquad x = x(u), \qquad y = y(u), \qquad z = z(u),$$

the functions being of class C' as specified in § 1. The minimizing extremal C can be imbedded in a one-parameter family of curves

$$(V)\qquad x = f(t, u), \qquad y = g(t, u), \qquad z = h(t, u),$$

which intersect the curve L for $t = t_0$, pass through the point 1 for $t = t_1$, and contain the extremal C for the particular parameter value u_0. Analytically these properties are expressed by the following identities in u:

$$\begin{aligned} x(u) &= f(t_0, u), \qquad & y(u) &= g(t_0, u), \qquad & z(u) &= h(t_0, u), \\ x_1 &= f(t_1, u), & y_1 &= g(t_1, u), & z_1 &= h(t_1, u). \end{aligned}$$

The value of the integral J taken along a member V of this family of curves from the point of intersection 0 of V with L to the fixed point 1, is a function of the parameter u,

$$J(u) = \int^{t_1} F(f, g, h, f_t, g_t, h_t)\, dt.$$

If C minimizes the integral, the function $J(u)$ has a minimum for $u = u_0$. Its derivative is

$$\frac{dJ}{du} = F_{x'}f_u + F_{y'}g_u + F_{z'}h_u\Big]_{t_0}^{t_1} - \int_{t_0}^{t_1} (Pf_u + Qg_u + Rh_u)\, dt,$$

where P, Q, R stand for the first members of the Euler equations (10). Since C is an extremal and all of the curves V pass through the point 1 for $t = t_1$, it follows that

$$\left[\frac{dJ}{du}\right]_{u=u_0} = -\,[F_{x'}f_u + F_{y'}g_u + F_{z'}h_u]^{t=t_0},$$

where the arguments of the derivatives of F are the values of x, y, z, x', y', z' on the extremal C at the point 0. A second necessary condition for a minimum is therefore the following:

The direction of the extremal C at its point of intersection 0 with the fixed curve L must satisfy the equation

$$(17) \qquad F_{x'}x_u + F_{y'}y_u + F_{z'}z_u = 0,$$

where the arguments of the derivatives of F are the values of x, y, z, x', y', z' on the curve C, and x_u, y_u, z_u define the direction of the curve L.

From this condition one may easily derive the equations which must hold when the end-point is allowed to vary on a surface S,

$$(S) \qquad x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v).$$

For if C minimizes J with respect to all curves joining the point 1 to the surface S, it must certainly minimize J with respect to all curves which join 1 with either of the two parameter lines of the surface through the point 0. The transversality condition for the end-point varying on a surface is therefore the following:

The direction of the extremal C at its point of intersection 0 with the surface S must satisfy the equations

$$(18) \qquad F_{x'}x_u + F_{y'}y_u + F_{z'}z_u = 0, \qquad F_{x'}x_v + F_{y'}y_v + F_{z'}z_v = 0,$$

where the arguments of the derivatives of F are the values x, y, z, x', y', z' on the curve C, and x_u, y_u, z_u; x_v, y_v, z_v define the directions of the parameter lines of the surface S.

It is evident that if these equations are satisfied, any curve of the surface which passes through the point 0 will also be transversal to C at 0, since the direction cosines of any curve on the surface are proportional to expressions of the form

$$x_u u' + x_v v',\ y_u u' + y_v v',\ z_u u' + z_v v'.$$

Furthermore the derivatives $F_{x'}$, $F_{y'}$, $F_{z'}$ are proportional to the direction cosines of the normal to the surface, as is easily seen from equations (18).

§ 3. *The family of extremals through a given point, or to which a given surface is transversal.*

Before proceeding to derive the equations of a family of extremals through a given point, let us consider for a moment the quotient U/s^2, where

$$U = \begin{vmatrix} \phi_s & \phi_{x_0'} & \phi_{y_0'} & \phi_{z_0'} \\ \psi_s & \psi_{x_0'} & \psi_{y_0'} & \psi_{z_0'} \\ \chi_s & \chi_{x_0'} & \chi_{y_0'} & \chi_{z_0'} \\ 0 & x_0' & y_0' & z_0' \end{vmatrix}.$$

This quotient is a finite and continuous function of its arguments even when $s=0$. For by applying the mean value theorem for the variable s to the elements of the minor of the element 0 in the determinant, and by using the relations (16), it is seen that U/s^2 approaches 1 as s approaches zero. If U/s^2 remains different from zero along an extremal arc C_{01} with the initial element $(x_0, y_0, z_0, x_0', y_0', z_0')$, it will also remain different from zero on C_{01} when $(x_0, y_0, z_0, x_0', y_0', z_0')$ are replaced by the values $(x_2, y_2, z_2, x_2', y_2', z_2')$ corresponding to a point 2 chosen on C_{01} sufficiently near to 0, and in the order 201 with the points 0 and 1.

The family of extremals through the initial point 0 of an extremal arc C_{01} can be found by making the constants x_0, y_0, z_0 in equations (14) equal to the coördinates of the point 0. If x_0', y_0', z_0' are replaced by functions $x'(u, v)$, $y'(u, v)$, $z'(u, v)$ which reduce to the coördinates of the initial direction of C_0 for $u = u_0$, $v = v_0$, and which satisfy equation (15) identically, the equations (14) take the form

$$(19) \qquad x = \phi(s, u, v), \qquad y = \psi(s, u, v), \qquad z = \chi(s, u, v).$$

These equations represent the extremal C_{01} for $u = u_0$, $v = v_0$, and satisfy also the identities

$$x_0 = \phi(0, u, v), \qquad y_0 = \psi(0, u, v), \qquad z_0 = \chi(0, u, v),$$

which express the fact that all of the extremals pass through the point (x_0, y_0, z_0) for $s = 0$. It is evident that the derivatives ϕ_u, ψ_u, χ_u and the similar ones for v all vanish identically for $s = 0$.

The functional determinant

$$\Delta(s, u, v) = \begin{vmatrix} \phi_s & \phi_u & \phi_v \\ \psi_s & \psi_u & \psi_v \\ \chi_s & \chi_u & \chi_v \end{vmatrix}$$

is readily seen to be equal to the product

$$U \begin{vmatrix} x' & x'_u & x'_v \\ y' & y'_u & y'_v \\ z' & z'_u & z'_v \end{vmatrix}.$$

The functions $x'(u, v)$, $y'(u, v)$, $z'(u, v)$ can always be chosen so that the second factor is different from zero for $u = u_0$, $v = v_0$, and under those circumstances $\Delta(s, u, v)$ vanishes for $s = 0$ on C_{01}, but is different from zero in the neighborhood of $s = 0$ on account of the properties of U derived above.

If the family of extremals (19) *through the point* 0 *includes the extremal* C_{01} *and has a determinant* Δ *which is different from zero everywhere on the arc* C_{10} *except at the point* 0, *then the quotient* U/s^2 *does not vanish on* C_{01}, *and a point* 2 *can be selected in the manner described in the first paragraph of this section, such that the family of extremals through* 2 *will have a determinant* Δ *which does not vanish anywhere on the arc* C_{01}.

An extremal to which a surface S is transversal can be imbedded in a two-parameter family of extremals each of which is cut transversally by the surface S. To show this it is only necessary to substitute for x_0, y_0, z_0 in the equations (14) of the extremals the functions $x(u, v)$, $y(u, v)$, $z(u, v)$ which define the surface, and for x'_0, y'_0, z'_0 the direction cosines of the directions to which the surface is transversal determined by equations (18). That such a direction is uniquely determined at each point of the surface can be seen as follows. The equations

$$(20)\quad x'^2+y'^2+z'^2=1, \qquad F_{x'}x_u+F_{y'}y_u+F_{z'}z_u=0, \qquad F_{x'}x_v+F_{y'}y_v+F_{z'}z_v=0,$$

are satisfied by one set of values corresponding to the intersection 0 of the extremal C_{01} with the surface S, and at the point 0 the functional determinant of the first members with respect to x', y', z' is different from zero. In fact this determinant is readily seen to be equal to the product of the two determinants

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & x' & y' & z' \\ 0 & x_u & y_u & z_u \\ 0 & x_v & y_v & z_v \end{vmatrix} \quad \begin{vmatrix} 0 & x' & y' & z' \\ x' & F_{x'x'} & F_{x'y'} & F_{x'z'} \\ y' & F_{y'x'} & F_{y'y'} & F_{y'z'} \\ z' & F_{z'x'} & F_{z'y'} & F_{z'z'} \end{vmatrix}.$$

From equations (18)

$$(21)\quad y_u z_v - y_v z_u = kF_{x'}, \qquad z_u x_v - z_v x_u = kF_{y'}, \qquad x_u y_v - x_v y_u = kF_{z'},$$

so that on account of equation (4) the first determinant is equal to kF. The factor k is not zero if the surface S has not a singular point at 0, *and we add here the assumption $F \neq 0$ at the point* 0. The second determinant is different from zero on account of equation (7). From the well-known theorems on implicit functions it follows that x', y', z', are determined as functions of u and v, of class C' in the neighborhood of the point 0.

After substituting the values $x(u, v)$, $y(u, v)$, $z(u, v)$, $x'(u, v)$, $y'(u, v)$, $z'(u, v)$ as just determined for x_0, y_0, z_0, x'_0, y'_0, z'_0, the equations (14) take the form

$$(22)\qquad x = \phi(s, u, v), \qquad y = \psi(s, u, v), \qquad z = \chi(s, u, v).$$

On account of the properties of the functions (14) and of the solutions of equations (20) there will be a region

$$0 \leqq s \leqq s_1, \qquad |u - u_0| \leqq \kappa, \qquad |v - v_0| \leqq \kappa,$$

in which ϕ, ϕ_s, ψ, ψ_s, χ, χ_s are of class C', where the quantities u_0, v_0 are the particular parameter values defining the extremal C_{01}. Since for $s = 0$ the point (x, y, z) for all values of u and v lies on the surface S, the equations of S in the neighborhood of the point 0 are

$$x = \phi(0, u, v), \qquad y = \psi(0, u, v), \qquad z = \chi(0, u, v),$$

and on account of the transversality of S to the extremals of the family, we have

$$(23)\qquad \begin{aligned} F_{x'}\phi_u + F_{y'}\psi_u + F_{z'}\chi_u &= 0, \\ F_{x'}\phi_v + F_{y'}\psi_v + F_{z'}\chi_v &= 0, \end{aligned}$$

for $s = 0$, where the arguments of the derivatives of F are ϕ, ψ, χ, ϕ_s, ψ_s, χ_s.

The value of the determinant

$$\Delta(s, u, v) = \begin{vmatrix} \phi_s & \phi_u & \phi_v \\ \psi_s & \psi_u & \psi_v \\ \chi_s & \chi_u & \chi_v \end{vmatrix}$$

when $s = 0$ becomes simply kF, with the help of the equations (4) and (23), and the factor k cannot vanish since it is understood that the surface S does not have a singular point at 0. And F has been assumed to be different from zero at the point 0 at least.

It has been shown therefore that *if a surface S is transversal to an extremal C_{01} at the point* 0, *then C_{01} can be imbedded in a family of extremals* (22) *to* *each of which S is transversal. The determinant $\Delta(s, u, v)$ of the family does not vanish in the neighborhood of the point* 0.

Suppose that the determinant Δ vanishes for some point 2 different from O on the extremal C_{01}, corresponding to the values s_2, u_0, v_0. If it is assumed that at least one of the three-rowed determinants of the matrix

$$(24) \qquad \left\|\begin{matrix} \Delta_s & \Delta_u & \Delta_v \\ \phi_s & \phi_u & \phi_v \\ \psi_s & \psi_u & \psi_v \\ \chi_s & \chi_u & \chi_v \end{matrix}\right\|$$

does not vanish with Δ, for convenience the determinant

$$\left|\begin{matrix} \Delta_s & \Delta_u & \Delta_v \\ \phi_s & \phi_u & \phi_v \\ \psi_s & \psi_u & \psi_v \end{matrix}\right|,$$

then the equations

$$(25) \qquad \Delta(s, u, v) = 0, \qquad \phi(s, u, v) = x, \qquad \psi(s, u, v)$$

determine s, u, v as functions

$$(26) \qquad s = s(x, y), \qquad u = u(x, y), \qquad v = v(x, y)$$

in the neighborhood of the values x_2, y_2. By substituting these results in the third of the equations (22), a surface

$$(D) \qquad z = z(x, y)$$

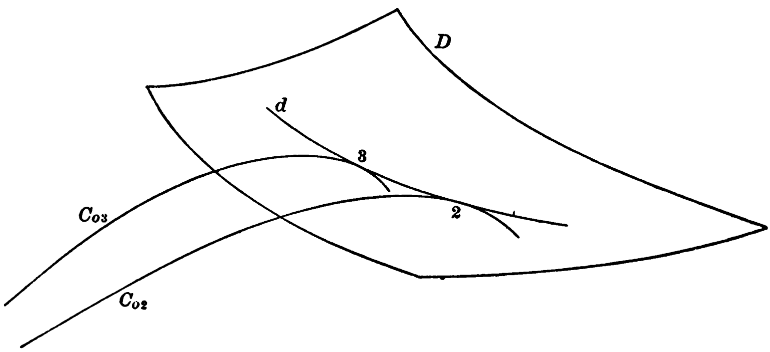

FIG. 1.

is found (See Figure 1). This surface is the envelope of the extremals (22). For from equations (25) and (D) the expression

$$\chi_s - z_x \phi_s - z_y \psi_s$$

has the factor

$$\begin{vmatrix} 0 & \Delta_s & \Delta_u & \Delta_v \\ \phi_s & \phi_s & \phi_u & \phi_v \\ \psi_s & \psi_s & \psi_u & \psi_v \\ \chi_s & \chi_s & \chi_u & \chi_v \end{vmatrix} = -\Delta\Delta_s,$$

which vanishes on D on account of the first of equations (25).

If a one-parameter family be chosen arbitrarily from the set (22), this family will not in general have an enveloping curve. The condition that a curve

$$(d) \qquad x = x(\alpha), \qquad y = y(\alpha), \qquad z = z[x(\alpha), y(\alpha)] = z(\alpha)$$

on the surface D shall be an envelope of extremals, may be derived as follows. If we substitute $x(\alpha)$, $y(\alpha)$ in the functions $u(x, y)$, $v(x, y)$ defined by equations (26), two functions $u(\alpha)$, $v(\alpha)$ are determined, and a one-parameter family of extremals is defined when $u(\alpha)$ and $v(\alpha)$ are substituted in (22). These extremals are tangent to the curve d if x and y are determined as functions of α so that

$$(27) \qquad x_\alpha = m\phi_s, \qquad y_\alpha = m\psi_s, \qquad z_\alpha = z_x x_\alpha + z_y y_\alpha = m\chi_s,$$

s, u, v being thought of as functions of x and y. The three determinants of the matrix

$$\left\| \begin{matrix} x_\alpha & y_\alpha & z_x x_\alpha + z_y y_\alpha \\ \phi_s & \psi_s & \chi_s \end{matrix} \right\|$$

must therefore be zero, i. e., the three equations

$$\begin{aligned} -\psi_s z_x x_\alpha + (\chi_s - \psi_s z_y) y_\alpha &= 0, \\ (\phi_s z_x - \chi_s) x_\alpha + \phi_s z_y y_\alpha &= 0, \\ \psi_s x_\alpha - \phi_s y_\alpha &= 0 \end{aligned}$$

must be satisfied. The coefficients of x_α and y_α in these equations can not all vanish since at least one of the derivatives ϕ_s, ψ_s, χ_s is different from zero at the point 2. That any two of the equations are a consequence of the third may be shown by expanding the determinant of any pair of the equations, and using the relation

$$\chi_s - z_x \phi_s - z_y \psi_s = 0.$$

The determination of a one parameter family of extremals having an enveloping curve d is therefore to be effected by solving one of the above equations. It has the form

$$A(x, y) x_\alpha + B(x, y) y_\alpha = 0$$

when s, u, v are replaced by their values in terms of x and y from equations (26). Since this differential equation is of the first order there exists one and only one integral curve

$$(28) \qquad x = x(\alpha), \qquad y = y(\alpha),$$

in the xy-plane, passing through the point (x_2, y_2) for $\alpha = 0$. The equations of the family of extremals tangent to d are found by substituting $x(\alpha)$, $y(\alpha)$ in the expressions for u and v in terms of x and y from equations (26), and then putting the resulting functions $u(\alpha)$, $v(\alpha)$ in equations (22). A family of extremals

$$(29) \qquad x = \phi(s, \alpha), \qquad y = \psi(s, \alpha), \qquad z = \chi(s, \alpha)$$

is thus found, containing C_{01} for $\alpha = 0$, and tangent to d when $s = s(\alpha)$ from equations (28) and (26). The equations of the envelope d will then be

$$(30) \qquad x = \phi[s(\alpha), \alpha], \qquad y = \psi[s(\alpha), \alpha], \qquad z = \chi[s(\alpha), \alpha].$$

If necessary the sign of the parameter α may be changed so that the factor m in equations (27) is positive.

Consider then a family of extremals

$$x = \phi(s, u, v), \qquad y = \psi(s, u, v), \qquad z = \chi(s, u, v),$$

all of which pass through a fixed point, or which are all cut transversally by a given surface S, for $s = 0$. Suppose that on a particular extremal C_{02} the determinant Δ vanishes at the point 2 different from zero, but that one at least of the determinants of the matrix (24) is different from zero at 2. Then the family of extremals has an enveloping surface D (see Fig. 1), which touches C_{02} at 2 and for which 2 is not a singular point. On the surface D there exists a unique curve d without singular point, which passes through the point 2 and envelops a one-parameter family of extremals containing the extremal C_{02}.

§ 4. *Jacobi's necessary condition.*

By means of the one-parameter family of extremals and the enveloping curve d, found in the last section, it can be shown that an arc C_{01} which joins a fixed point 0 to another fixed point 1, or which joins 1 to a fixed surface S, and which furthermore minimizes the integral J, can not have upon it a contact point 2 with the enveloping surface D. It will be seen that the value of the integral J taken along an extremal C_{03} of the set (19) or (22) from the value $s = 0$ to the value $s(\alpha)$ at the contact point 3 of C_{03} with d, and then along d_{32} from the point 3 to the point 2, is independent of the particular extremal chosen, i. e., it is independent of the value of the parameter α defining the extremal. In fact, the derivative of $J(C_{03})$ is

$$\frac{dJ(C_{03})}{d\alpha} = \frac{d}{d\alpha}\int_{s_0}^{s_3} F(\phi, \psi, \chi, \phi_s, \psi_s, \chi_s)\,ds$$

$$= F\frac{ds}{d\alpha}\Big|^3 + \int_{s_0}^{s_3} [F_x\phi_\alpha + F_y\psi_\alpha + F_z\chi_\alpha + F_{x'}\phi_{\alpha s} + F_{y'}\psi_{\alpha s} + F_{z'}\chi_{\alpha s}]\,ds.$$

After integration by parts this takes the form

$$\frac{dJ(C_{03})}{d\alpha} = F\frac{ds}{d\alpha}\Big|^3 + [F_{x'}\phi_\alpha + F_{y'}\psi_\alpha + F_{z'}\chi_\alpha]_0^3,$$

the integral part vanishing on account of the fact that the curve C_{03} is an extremal. The terms in the bracket vanish for $s = 0$ when all the extremals (19) pass through a fixed point 0, for then

$$x_0 = \phi(0, \alpha), \qquad y_0 = \psi(0, \alpha), \qquad z_0 = \chi(0, \alpha),$$

and consequently $\phi_\alpha, \psi_\alpha, \chi_\alpha$ vanish for $s = 0$. They also cancel when the extremals are cut transversally by a surface S on account of equation (17), since in that case the equations

$$x = \phi(0, \alpha), \qquad y = \psi(0, \alpha), \qquad z = \chi(0, \alpha),$$

represent a curve on the surface S which is transversal to the extremals (29). On account of (4) the resulting equation may be written in the form

$$\frac{dJ(C_{03})}{d\alpha} = \left[\left(\phi_s\frac{ds}{d\alpha} + \phi_\alpha\right)F_{x'} + \left(\psi_s\frac{ds}{d\alpha} + \psi_\alpha\right)F_{y'} + \left(\chi_s\frac{ds}{d\alpha} + \chi_\alpha\right)F_{z'}\right]^3,$$

the arguments of the derivatives of F being $\phi, \psi, \chi, \phi_s, \psi_s, \chi_s$. From equations (30), therefore,

(31) $$\frac{dJ(C_{03})}{d\alpha} = F_{x'}x_\alpha + F_{y'}y_\alpha + F_{z'}z_\alpha\Big|^3,$$

where $x_\alpha, y_\alpha, z_\alpha$ determine the direction of the curve d at the point 3. Since the positive tangent to d coincides with the positive tangent to C_{03}, we have

(32) $$\frac{dJ(C_{03})}{d\alpha} = F(x, y, z, x_\alpha, y_\alpha, z_\alpha)\Big|^3.$$

This follows because from equations (3) and (27) the coefficient $F_{x'}$ in equation (31) becomes

$$F_{x'}(\phi, \psi, \chi, \phi_s, \psi_s, \chi_s)\Big|^3 = F_{x'}(x, y, z, x_\alpha, y_\alpha, z_\alpha)\Big|^3.$$

The derivative of the integral $J(d_{32})$ is

(33) $$\frac{dJ(d_{32})}{d\alpha} = \frac{d}{d\alpha}\int_{\alpha_3}^{\alpha_2} F(x, y, z, x_\alpha, y_\alpha, z_\alpha)\,d\alpha = -F(x, y, z, x_\alpha, y_\alpha, z_\alpha)\Big|^3$$

From equations (32) and (33) therefore,

$$\frac{d}{d\alpha}\left[J(C_{03}) + J(d_{32})\right] = 0,$$

from which the following generalization of KNESER's theorem in the plane is obtained:

Suppose that an extremal C_{01} *is contained in a one-parameter family of extremals, each of which passes through a fixed point* 0 *and which have an enveloping curve touching* C_{01} *at a point* 2 *(see Figure* 1*). Then if* 3 *is a neighboring point to* 2 *on* d, *the value of* J *taken along* C_{03} *plus the value of* J *taken along* d_{32} *is always equal to the value of* J *taken along* C_{02}. *In other words*

$$J(C_{03}) + J(d_{32}) = J(C_{02}).$$

When the extremals are all cut transversally by a surface S, *the same equation is true if* 0 *is understood to denote the variable intersection of* C_{03} *with* S.

The envelope d_{32} can not satisfy Euler's equations, for it was shown in § 1 that but one solution of these equations passes through a given point in a given direction, and d_{32} would have to coincide, therefore, with the extremal C_{02} in the neighborhood of the point 2. This contradicts the hypothesis that the point 2 is the nearest point to 0 on the arc C_{01}, for which Δ vanishes. It follows that the points 3 and 2 can always be joined by a curve d'_{32} such that

$$J(C_{03}) + J(d'_{32}) < J(C_{02}).$$

If therefore an extremal C_{01} *minimizes or maximizes the integral* J, *it must not have upon it a point* 2 *which is conjugate to the initial point* 0.*

§ 5. *The necessary conditions of Weierstrass and Legendre.*

It has been shown in the preceding sections that if a curve C_{01} joins two fixed points, or a fixed surface with a fixed point, and minimizes the integral J, it must be a solution of Euler's equations and must satisfy Jacobi's necessary condition. A further condition similar to the one discovered by LEGENDRE for a weak minimum, and extended by WEIERSTRASS for the case of a strong minimum, must also be satisfied.

Let us consider an extremal arc C_{01},

$$x = x(t), \qquad y = y(t), \qquad z = z(t),$$

and a curve V

$$x = x(u), \qquad y = y(u), \qquad z = z(u),$$

* The theorem has been proved for conjugate points 2 at which not all of the determinants of the matrix (24) vanish with Δ. By using the so-called second variation it can be shown that even without this restriction the point 2 must not lie between 0 and 1 on C_{01}. The case where the point 2 coincides with 1 and all determinants of the matrix (24) vanish is an exceptional one, which has been discussed for problems in the plane but not for problems in space.

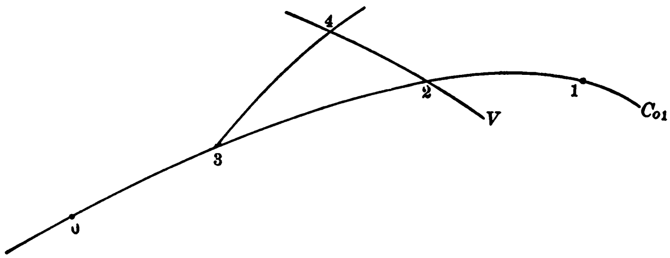

FIG. 2.

intersecting C_{01} at a point 2. Through a point 3 on C_{01} near to 2, there can always be found a family of curves

$$(34)\qquad x = f(t, u), \qquad y = g(t, u), \qquad z = h(t, u),$$

containing C_{32} for $u = u_2$, which furthermore pass through 3 for the parameter value $t = t_3$ and intersect V for $t = t_2$. Analytically, then,

$$(35)\qquad \begin{aligned} f(t_2, u) &\equiv x(u), \qquad g(t_2, u) \equiv y(u), \qquad h(t_2, u) \equiv z(u), \\ x_3 &\equiv f(t_3, u), \qquad y_3 \equiv g(t_3, u), \qquad z_3 \equiv h(t_3, u). \end{aligned}$$

The sum $J_{34} + J_{42}$ is now a function of the parameter u, whose derivative for $u = u_2$ can be readily calculated. The derivative of J_{34} is

$$\frac{dJ_{34}}{du} = \int_{t_3}^{t_2} \{F_x f_u + F_y g_u + F_z h_u + F_{x'} f_{tu} + F_{y'} g_{tu} + F_{z'} h_{tu}\} dt,$$

where the arguments of F and its derivatives are f, g, h, f_t, g_t, h_t. For $u = u_2$, this becomes, by the usual integration by parts of the calculus of variations and with the help of the identities (35),

$$\left[\frac{dJ_{34}}{du}\right]_{u=u_2} = F_{x'} x_u + F_{y'} y_u + F_{z'} z_u \Big|^2,$$

since C_{32} is an extremal. Here the arguments x, y, z, x', y', z' of the derivatives of F refer to C_{32} since the curves (34) go over into C_{32} for $u = u_2$, and x_u, y_u, z_u define the direction of V. The derivative of J_{42} is simply

$$\left[\frac{dJ_{42}}{du}\right]_{u=u_2} = -F(x, y, z, x_u, y_u, z_u)\Big|^2.$$

The derivative of the sum $J_{34} + J_{42}$ is therefore

$$(36)\qquad \left[\frac{d}{du}(J_{34} + J_{42})\right]_{u=u_0} = -E(x, y, z, x', y', z', x_u, y_u, z_u)\Big|^2,$$

where

$$(37)\qquad \begin{aligned} E = F(x, y, z, x_u, y_u, z_u) &- x_u F_{x'}(x, y, z, x', y', z') \\ &- y_u F_{y'}(x, y, z, x', y', z') - z_u F_{z'}(x, y, z, x', y', z'). \end{aligned}$$

From equation (36) the following theorem similar to that of WEIERSTRASS for the plane case, can be derived.

If the arc C_{01} furnishes J a strong minimum, the function E must be $\geqq 0$ for any set of values x, y, z, x', y', z' on C_{01} and for all directions x_u, y_u, z_u; while if C_{01} furnishes only a weak minimum, the E-function must similarly be $\geqq 0$ at least for directions x_u, y_u, z_u in some neighborhood of the directions x', y', z' on C_{01}. (For a maximum E must be $\leqq 0$.)

The theorem is evidently true, for if the E-function were negative at a point 2 for some direction x_u, y_u, z_u, then on account of equation (36) the inequality

$$J_{34} + J_{42} < J_{32}$$

would hold for values of u sufficiently near to u_2.

On account of the homogeneity of the function F, the expression (37) may be written in the form

$$\begin{aligned} E(x, y, z, x', y', z', x_u, y_u, z_u) &= F(x, y, z, x_u, y_u, z_u) - F \\ &\quad - (x_u - x')F_{x'} - (y_u - y')F_{y'} - (z_u - z')F_{z'} \\ (38) \qquad &= \tfrac{1}{2}\{(x_u - x')^2 F_{x'x'} + (y_u - y')^2 F_{y'y'} + (z_u - z')^2 F_{z'z'} \\ &\quad + 2(y_u - y')(z_u - z')F_{y'z'} + 2(z_u - z')(x_u - x')F_{z'x'} + 2(x_u - x')(y_u - y')F_{x'y'}\}, \end{aligned}$$

the arguments of the second derivatives of F being

$$x, y, z, x' + \theta(x_u - x'), y' + \theta(y_u - y'), z' + \theta(z_u - z'), (0 < \theta < 1).$$

We can derive therefore the following Legendre condition:

If the extremal arc C_{01} furnishes either a strong or a weak minimum, the quadratic form

$$(39) \qquad Q = F_{x'x'}\xi^2 + F_{y'y'}\eta^2 + F_{z'z'}\zeta^2 + 2\eta\zeta F_{y'z'} + 2\zeta\xi F_{z'x'} + 2\xi\eta F_{x'y'},$$

must be $\geqq 0$ for all values of x, y, z, x', y', z' on the arc C_{01}, and for all values of ξ, η, ζ.

For if ξ, η, ζ were a set of values making Q negative at some point of C_{01}, the values x_u, y_u, z_u could be taken so that

$$x_u - x' = \epsilon\xi,\ y_u - y' = \epsilon\eta,\ z_u - z' = \epsilon\zeta,$$

and for a sufficiently small ϵ the expression (38) would also be negative.

These necessary conditions may also be stated in terms of the functions F_1, $F_{x'x'}, F_{y'y'}, F_{z'z'}$. From (39) and the equations (6) it follows that F_1 can not be negative. For in that case the binary form obtained from Q by putting $\xi = 0$, for example, would be an indefinite form. The derivatives $F_{x'x'}, F_{y'y'}$, $F_{z'z'}$, if different from zero, must have the same sign as that of Q, as is easily seen by putting $\eta = \zeta = 0$ in (39). Conversely if $F_1 > 0$, and $F_{x'x'}$, $F_{y'y'}$, $F_{z'z'}$ are not all zero, the identities

$$
(40) \qquad \begin{aligned}
F_{x'x'} Q &= [F_{x'x'}\xi + F_{x'y'}\eta + F_{x'z'}\zeta]^2 + (y'\zeta - z'\eta)^2 F_1, \\
F_{y'y'} Q &= [F_{y'x'}\xi + F_{y'y'}\eta + F_{y'z'}\zeta]^2 + (z'\xi - x'\zeta)^2 F_1, \\
F_{z'z'} Q &= [F_{z'x'}\xi + F_{z'y'}\eta + F_{z'z'}\zeta]^2 + (x'\eta - y'\xi)^2 F_1,
\end{aligned}
$$

which are easily verified, show that such of the quantities Q, $F_{x'x'}$, $F_{y'y'}$, $F_{z'z'}$ as are different from zero have the same sign at all points of the extremal C_{01}, whatever be the values of ξ, η, ζ.

It should be noticed that when $F_1 > 0$ the derivative $F_{x'x'}$ vanishes when and only when $y' = z' = 0$, that is, in the direction of the x-axis. That it is zero for these arguments is seen from (5) since $x' \neq 0$. But from (6) it follows that y' and z' must vanish with $F_{x'x'}$ since $F_1 > 0$. Furthermore $F_{x'x'}$ cannot assume opposite signs, since it is possible to pass continuously from one set of values (x, y, z, x', y', z') to any other without passing through values for which $y' = z' = 0$. The Legendre necessary condition may be restated, then, in the following form:

If for a regular problem ($F_1 \neq 0$) the extremal C_{01} furnishes either a maximum or a minimum for the integral J, F_1 must be positive. Furthermore the functions $F_{x'x'}$, $F_{y'y'}$, $F_{z'z'}$, which vanish only in the direction of the x, y, and z-axes, respectively, must be positive for a minimum and negative for a maximum.

§ 6. *The invariant integral. Sufficient conditions.*

In § 3 it was shown that an extremal arc which does not contain a conjugate point 2 to 0, either between 0 and 1 or at 1, can be imbedded in a two-parameter family of extremals through a point 3 near to 0, whose determinant Δ does not vanish anywhere on C_{01}. Consider either this family or the family (22) to which the surface S is transversal. Suppose that the extremal C corresponds to the values $u = u_0$, $v = v_0$, and that the determinant $\Delta(s, u_0, v_0)$ does not vanish for any point of the arc C_{01}, i. e., that the enveloping surface D does not touch the arc C_{01}. Then the set of extremals forms a field about the extremal C. A positive constant δ may be chosen so small that the extremals of the set for which $|u - u_0| \leqq \delta$, $|v - v_0| \leqq \delta$ sweep out a portion of space which contains C in its interior, and through each point of which there passes one and only one extremal of the set. The constant δ can be so restricted that in this field the determinant Δ does not vanish, and the equations

$$
(41) \qquad \phi(s, u, v) = x, \qquad \psi(s, u, v) = y, \qquad \chi(s, u, v) = z
$$

have unique solutions

$$(42)\qquad s = s(x, y, z), \qquad u = u(x, y, z), \qquad v = v(x, y, z)$$

of class C'.*

Consider a curve K

$$(43)\qquad x = \bar{x}(t), \qquad y = \bar{y}(t), \qquad z = \bar{z}(t)$$

of class D', which lies entirely in the field. By means of equations (42) and (43) u and v are determined as functions of t. If these functions be substituted in the equations (41) there results a one-parameter family of extremals,

$$(44)\qquad x = \Phi(s, t), \qquad y = \Psi(s, t), \qquad z = \mathrm{X}(s, t),$$

having as parameter of the family the parameter t of the curve K. For $s = 0$ these extremals all pass through the point 3 or else intersect the surface S, which is transversal to all the extremals of the set (41) along a curve k.

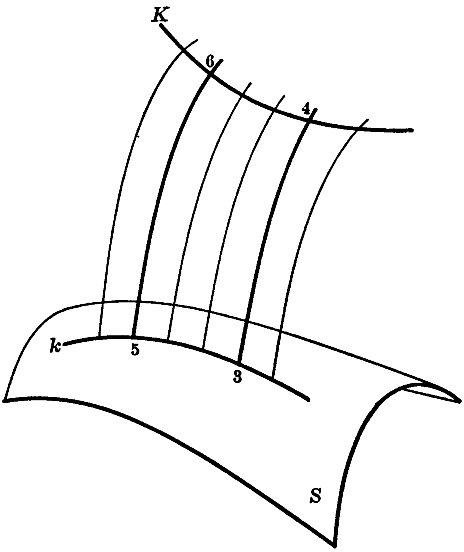

FIG. 3.

The value of the integral J, carried along an extremal of the set (44) from the point 3 where $s = 0$ to its point of intersection 4 with K, is a function of t, the parameter of the set. Using the same reasoning as that of the preceding section, the derivative of this function is found to be

$$\frac{dJ_{34}}{dt} = \frac{d}{dt}\int_0^{s_4} F(\Phi, \Psi, \mathrm{X}, \Phi', \Psi', \mathrm{X}')\,ds = \bar{x}'F_{x'} + \bar{y}'F_{y'} + \bar{z}'F_{z'}\Big|^4.$$

The value of the second member vanishes at the lower limit 3, since all the

* The proof of these statements may be made by a method analogous to that given by BOLZA for the case of the plane: *Lectures on the Calculus of Variations*, Chicago, 1904, § 34.

extremals of the set pass through a fixed point 3 or else are cut transversally by the curve k. The arguments of the derivatives of F are the functions $\Phi, \Psi, X, \Phi', \Psi', X'$ at the point 4 on K, or what is the same thing, the values of $\phi, \psi, \chi, \phi', \psi', \chi'$ with s, u, v expressed as functions of t from equations (42) and (43). Let t_4 be the value of t which gives the extremal connecting 3 with 4, and t_6 the value giving the extremal which connects a point 5 of k with the point 6 of K, and let the above equation be integrated from $t = t_4$ to $t = t_6$. Then

$$J_{56} - J_{34} = \int_{t_4}^{t_6} \{ \bar{x}' F_{x'} + \bar{y}' F_{y'} + \bar{z}' F_{z'} \} \, dt.$$

Now the first member of this equation is entirely independent of the form of the curve K. It follows that this is true of the second member also, in other words: *The value of the integral*

$$J^* = \int \{ \bar{x}' F_{x'} + \bar{y} F_{y'} + \bar{z}' F_{z'} \} \, dt,$$

taken along a curve K in the field, is independent of the path K, and depends only on the endpoints. The derivatives of F are here thought of as functions of x, y, z by means of (42). The integral has the further properties *that $J^* = 0$ if the curve K is at each point transversal to the extremal of the field, and that J^* reduces to J if K is an extremal.* These properties follow easily from equations (17) and (4).

The integral J^* is the generalization of Hilbert's invariant integral. It is useful in comparing the value $J(C_{01})$ taken along the extremal C with the value of J taken along any other curve $\bar{C}$

$$x = \bar{x}(t), \qquad y = \bar{y}(t), \qquad z = \bar{z}(t)$$

which joins the fixed point 0, or a point 0′ of the surface S, with the fixed point 1. For the case of the transversal surface S, let $k_{00'}$ be any curve which lies entirely on S and in the field, and which joins 0 with 0′. This curve is transversal to the extremals of the field. Then from the properties of the integral J^*,

$$\begin{aligned} J(\bar{C}_{0'1}) - J(C_{01}) &= J(\bar{C}_{0'1}) - J^*(C_{01}) \\ &= J(\bar{C}_{0'1}) - J^*(k_{00'}) - J^*(\bar{C}_{0'1}) \\ &= J(\bar{C}_{0'1}) - J^*(\bar{C}_{0'1}). \end{aligned}$$

Hence by making use of the E-function defined in equation (37), it follows that

$$(45) \qquad J(\bar{C}_{0'1}) - J(C_{01}) = \int_{\bar{C}_{0'1}} E(x, y, z, \phi', \psi', \chi', \bar{x}', \bar{y}', \bar{z}') \, dt,$$

where ϕ', ψ', χ' are thought of as functions of x, y, z as explained above. Evidently if $E \geqq 0$ along the curve $\bar{C}$, then

$$J(\bar{C}_{0'1}) \geqq J(C_{01}).$$

The same conclusion holds when C passes through two fixed points 0 and 1.

By a method similar to that used in § 5 applied to the expression (38), the E-function can be expressed in any one of three forms similar to those in (40). The first of these is

$$2F_{x'x'} \cdot E = [(x_u - x')F_{x'x'} + (y_u - y')F_{x'y'} + (z_u - z')F_{x'z'}]^2 + (y'z_u - z'y_u)^2 F_1.$$

Now the Legendre condition of § 5 is supposed to be satisfied, i. e., $F_1 > 0$, while $F_{x'x'}$, $F_{y'y'}$, $F_{z'z'}$ are not negative and vanish only in the coördinate directions respectively.

It follows for a regular problem, in which Legendre's necessary condition for a minimum is satisfied, that the Weierstrass E-function (37) *is never negative, and is equal to zero only when* $x_u : y_u : z_u = x' : y' : z'$.*

Hence in the field of extremals under consideration the E-function vanishes only when the direction of the comparison curve $\bar{C}$ at the point x, y, z coincides with the direction of the extremal of the field which passes through that point. Therefore from (45)

$$J(\bar{C}_{0'1}) > J(C_{01}),$$

unless the direction of $\bar{C}_{0'1}$ coincides at every point with the direction of the extremal of the field through that point. But this can occur only when $\bar{C}_{0'1}$ coincides throughout its entire extent with the extremal C_{01}. For along the curve $\bar{C}_{0'1}$ the parameters u and v are functions of t whose derivatives, from equations (42), are

$$\frac{du}{dt} = u_x \bar{x}' + u_y \bar{y}' + u_z \bar{z}',$$

$$\frac{dv}{dt} = v_x \bar{x}' + v_y \bar{y}' + v_z \bar{z}'.$$

Since u and v are constant along any one of the extremals of the field, the values of these derivatives are zero when the direction $(\bar{x}', \bar{y}', \bar{z}')$ coincides with that of one of the extremals. Hence if the direction of $\bar{C}_{0'1}$ always coincides with the direction of some extremal, the second members of the above equations vanish identically, and along $\bar{C}_{0'1}$ the values of u and v must be constant. In other words, $\bar{C}_{0'1}$ must be an extremal, and since it passes through the point 1 it must be the extremal C_{01} itself. These results may be summarized in the following theorem.

---

* The expression (37) shows that E vanishes when the directions $x_u : y_u : z_u$ and $x' : y' : z'$ have the same sense, but not necessarily when they are opposite.

An extremal arc C_{01} joins two fixed points 0 *and* 1, *or joins* 1 *to a surface* S *which cuts* C_{01} *transversally. For a regular problem* ($F_1 \neq 0$) *the integral* J *will be minimized by this arc in either case provided that Legendre's necessary condition for a minimum is satisfied along* C_{01}, *and that* C_{01} *has upon it no point conjugate to the fixed end-point* 0 *in the former case, or to the surface* S *in the latter.*

§ 7. *Necessary conditions when one end point is variable on a curve.*

The results of the preceding sections apply with some modifications to the case where the end point 0 may vary on a given curve L,

$$(L) \qquad x = x(u), \qquad y = y(u), \qquad z = z(u),$$

instead of on a surface.

The curve C_{01} which minimizes the integral J with respect to other curves joining the curve L with the fixed point 1 must in the first place be an extremal which is cut by L transversally, i. e., at the point of intersection 0 of C_{01} with L, the equation

$$l_0 F_{x'} + m_0 F_{y'} + n_0 F_{z'} = 0$$

must be satisfied, where l_0, m_0, n_0 are the values of the direction cosines l, m, n of L at the point 0, and the arguments of $F_{x'}, F_{y'}, F_{z'}$ are the values of x, y, z, x', y', z' on the extremal C_{01} at this point.

The extremal C_{01}, cut transversally by L at 0, may be imbedded in a two-parameter family of extremals, all cut transversally by L. The equations

$$(46) \qquad \begin{aligned} x'^2 + y'^2 + z'^2 &= 1, \\ lF_{x'} + mF_{y'} + nF_{z'} &= 0, \end{aligned}$$

have by hypothesis the solution x_0', y_0', z_0' which is given by the values of x', y', z' on C_{01} at the point 0. Let (α, β, γ), (λ, μ, ν) denote the direction cosines of the principal normal and binormal respectively along L. Let v be a variable such that

$$\alpha \cos v + \lambda \sin v = a(u, v), \quad \beta \cos v + \mu \sin v = b(u, v), \quad \gamma \cos v + \nu \sin v = c(u, v)$$

are the direction cosines of the perpendicular to (x', y', z') and to (l, m, n). Then

$$(47) \qquad x'a + y'b + z'c = 0.$$

The three equations (46) and (47) can now be solved for x', y', z' as functions of v and of the parameter u of the curve L, in the neighborhood of the values u_0, v_0 which correspond to the point 0 and the direction of the extremal C_{01}. The functional determinant of the left members of the equations with respect to x', y', z' is

$$\begin{vmatrix} 2x' & 2y' & 2z' \\ lF_{x'x'} + mF_{x'y'} + nF_{x'z'} & lF_{y'x'} + mF_{y'y'} + nF_{y'z'} & lF_{z'x'} + mF_{z'y'} + nF_{z'z'} \\ a & b & c \end{vmatrix}.$$

Let θ be the angle between (x', y', z') and (l, m, n). Since these directions are perpendicular to the direction (a, b, c) it follows that

$$(48)\quad ny' - mz' = a\sin\theta, \qquad lz' - nx' = b\sin\theta, \qquad mx' - ly' = c\sin\theta.$$

The value of $\sin\theta$ cannot be zero, for, if it were, F would vanish on account of the second equation of (46) and (4), and F has been assumed different from

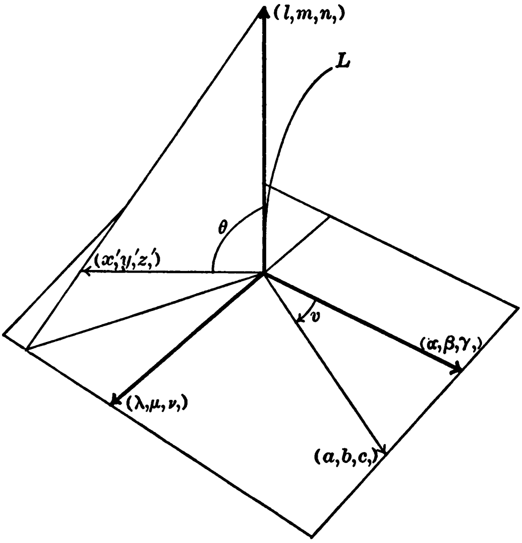

FIG. 4.

zero. The value of the above functional determinant, when calculated by the aid of (48) is twice the quadratic form

$$Q(l, m, n) = l^2F_{x'x'} + m^2F_{y'y'} + n^2F_{z'z'} + 2mnF_{y'z'} + 2nlF_{z'x'} + 2lmF_{x'y'}$$

divided by $\sin\theta$. This form vanishes only when the directions (l, m, n) and (x', y', z') coincide, or are opposite, i. e., only when $\sin\theta = 0$, and this cannot happen, as was seen above.

If the values $x(u)$, $y(u)$, $z(u)$ along the curve L, and the functions $x'(u, v)$, $y'(u, v)$, $z'(u, v)$ determined as above from equations (46), (47) *are substituted for x_0, y_0, z_0, x_0', y_0', z_0' in the equations of the extremals* (14), *a family of extremals*

$$x = \phi(s, u, v), \qquad y = \psi(s, u, v), \qquad z = \chi(s, u, v)$$

is determined, each of which is cut transversally by the curve L. The extremal C_{01} is represented for the particular parameter values u_0, v_0.

The value of the determinant

$$\Delta = \begin{vmatrix} \phi_s & \phi_u & \phi_v \\ \psi_s & \psi_u & \psi_v \\ \chi_s & \chi_u & \chi_v \end{vmatrix}$$

is in this case equal to zero at the point of intersection of L and C_{01}, since for $s = 0$ the functions ϕ, ψ, χ are identically equal to $x(u)$, $y(u)$, $z(u)$, and the derivatives ϕ_v, ψ_v, χ_v all vanish identically. But the value $s = 0$ is an isolated zero of Δ, whatever the values of u and v, for it can be shown that the derivative Δ_s does not vanish when $s = 0$. In fact the value of Δ_s for $s = 0$ is

$$\Delta_s = \begin{vmatrix} \phi_s & \phi_u & \phi_{sv} \\ \psi_s & \psi_u & \psi_{sv} \\ \chi_s & \chi_u & \chi_{sv} \end{vmatrix} = \begin{vmatrix} x'(u, v) & l(u) & x'_v(u, v) \\ y'(u, v) & m(u) & y'_v(u, v) \\ z'(u, v) & n(u) & z'_v(u, v) \end{vmatrix} \sqrt{\phi_u^2 + \psi_u^2 + \chi_u^2},$$

or, on account of (48),

$$\Delta_s = \sin\theta \sqrt{\phi_u^2 + \psi_u^2 + \chi_u^2}(x'_v a + y'_v b + z'_v c).$$

Now equation (47) holds identically in v, so that on differentiating (47) with respect to v the third factor in the above equation is seen from the expressions for $a(u, v)$, $b(u, v)$, $c(u, v)$ to be equal to

$$-(x'a_v + y'b_v + z'c_v) = -\left[x'a\left(u, v + \frac{\pi}{2}\right) + y'b\left(u, v + \frac{\pi}{2}\right) + z'c\left(u, v + \frac{\pi}{2}\right)\right],$$

and from the figure it follows that this is equal to $-\sin\theta$. The value of Δ_s for $s = 0$ is therefore

$$\Delta_s = -\sin^2\theta \sqrt{\phi_u^2 + \psi_u^2 + \chi_u^2}.$$

Since (x', y', z') can never coincide with (l, m, n), as was seen before, it follows that Δ_s *is different from zero for $s = 0$, and consequently the value $s = 0$ is an isolated zero of $\Delta(s, u, v)$.*

Consider now the extremal C_{01} of the family, corresponding to $u = u_0$, $v = v_0$, which intersects L at 0 and passes through the point 1. There will be an arc of C_{01} in the neighborhood of the point 0 along which Δ does not vanish. If Δ vanishes while one at least of the determinants of the matrix (24) is different from zero, at some point other than 0 on C_{01}, then the equation

$$\Delta(s, u, v) = 0$$

defines an enveloping surface D for the extremals of the family. Exactly as in § 3 a one-parameter family can be selected which has an enveloping curve d on

D, and an arc joining L to the point 1 can be found which gives to the integral J a smaller value than is given by C_{01}.

The following are therefore necessary conditions for a minimum with respect to curves joining a fixed curve L with a fixed point 1 *when the problem at hand is a regular one. The minimizing arc must be an extremal C_{01} which is cut transversally by the curve L at the intersection point* 0 *of L with C_{01}. The function F_1 must be positive, and the functions $F_{x'x'}$, $F_{y'y'}$, $F_{z'z'}$ must not be negative along C_{01}. Finally the envelope of the family of extremals to which L is transversal must not touch the arc C_{01}.**

§ 8. *Determination of a field of extremals to which a curve is transversal.*

The construction of a field for the case of an endpoint variable on a curve is somewhat more complicated than for the case of an endpoint variable on a surface. It has been shown that an extremal C_{01} to which L is transversal can be imbedded in a two-parameter family of neighboring extremals all cut transversally by L. These extremals were obtained by applying theorems on implicit functions to equations (46), (47), and consequently not only cut L at points in the neighborhood of the point 0, but have initial directions at their intersections with L which must be near to the direction of C_{01} at the point 0. In order to construct a field entirely surrounding L in the neighborhood of the point 0 it will be shown that the family can be extended to include extremals whose directions are not thus restricted, that is, that equations (46), (47) have unique solutions for all values of u near u_0 and all values of v between 0 and 2π, and furthermore that these solutions are periodic in v with period 2π.

In the first place, in any plane through a point x, y, z and containing the direction (l, m, n) there are at most two directions (x', y', z') to which (l, m, n) is transversal. For let (x_0', y_0', z_0') be one such direction. Then any other direction in the plane has the direction cosines

$$x_0' \cos\theta + l \sin\theta, \qquad y_0' \cos\theta + m \sin\theta, \qquad z_0' \cos\theta + n \sin\theta.$$

The expression

$$\Phi = lF_{x'} + mF_{y'} + nF_{z'}$$

with these arguments in $F_{x'}$, $F_{y'}$, $F_{z'}$ has the derivative

$$\frac{d\Phi}{d\theta} = l(x_\theta' F_{x'x'} + y_\theta' F_{x'y'} + z_\theta' F_{x'z'}) + m(x_\theta' F_{y'x'} + y_\theta' F_{y'y'} + z_\theta' F_{y'z'}) + n(x_\theta' F_{z'x'} + y_\theta' F_{z'y'} + z_\theta' F_{z'z'}).$$

This derivative vanishes when $\theta = \pi/2$ or $3\pi/2$ on account of equations (5).

* The exceptional case where the envelope has a singular point needs, as before, further consideration.

For other values of θ the expression takes the following form, obtained by substituting the values of x'_θ, y'_θ, z'_θ and using equations (5):

$$\frac{d\Phi}{d\theta} = \sec\theta(l^2F_{x'x'} + m^2F_{y'y'} + n^2F_{z'z'} + 2mnF_{y'z'} + 2nlF_{z'x'} + 2lmF_{x'y'}).$$

On account of the properties of this quadratic form, as shown in § 5, it is seen that the function $\Phi(\theta)$ is zero for $\theta = 0, 2\pi$ and has a derivative which changes sign only when $\theta = \pi/2, 3\pi/2$. Consequently, since Φ is periodic, it vanishes once between $\theta = \pi/2$ and $\theta = 3\pi/2$ and has no other zeros except $\theta = 0, 2\pi$. Hence in the plane through the point (x, y, z) and containing the directions (l, m, n) and (x'_0, y'_0, z'_0) there is but one other direction besides (x'_0, y'_0, z'_0) to which (l, m, n) is transversal, and that direction lies on the opposite side of the line (l, m, n) from (x'_0, y'_0, z'_0).

With these results in mind it is evident from the theory of implicit functions that the solutions x', y', z' of the equations (46) and (47), for a constant value of u, can be extended step by step over the whole interval from v_0 to $v_0 + 2\pi$. For if this were not the case, and an upper bound $V < v_0 + 2\pi$ for the extension existed, then the solutions x', y', z' would have at least one set of limiting values X', Y', Z' as v approached V. But any set of limiting values must be a solution of the equations, and since there can be but one solution in the half plane corresponding to $v = V$, it follows that X', Y', Z' form the only set. The solutions could therefore be extended beyond the value $v = V$, which is contrary to assumption concerning V. It is furthermore evident that the values of x', y', z' for $v = v_0 + 2\pi$ coincide with the values x'_0, y'_0, z'_0. These results may be stated in the following theorem:

*If a fixed line through the point (x, y, z) in the direction (l, m, n) is transversal to a single line (x'_0, y'_0, z'_0) through this point, then each half plane through the line (l, m, n) contains one and only one line (x', y', z') to which (l, m, n) is transversal at (x, y, z). The direction cosines of these lines are given by functions $x'(v), y'(v), z'(v)$ of class C', where v is the angle which determines the position of the half plane.**

Since for any particular value of u in the neighborhood of u_0 the equations (46), (47) have a unique solution for each value of v in the interval $(v_0, v_0 + 2\pi)$, it follows immediately from the theory of implicit functions that the totality of solutions $x'(u, v)$, $y'(u, v)$, $z'(u, v)$ so defined are functions of u and v of class C'.

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* It follows readily that *any surface element through a point (x, y, z) is transversal to at most one direction on each side of the element.* Suppose there were two directions on the same side of a surface element E to which E is transversal at the point P. Then every line in E through P would be transversal to the two directions, in particular the line L common to E and to the plane determined by the two directions. But this would contradict the theorem proved above, since the two directions would lie in the same half plane through L.

There exists, therefore, a two-parameter family of extremals

$$x = \phi(s, u, v), \qquad y = \psi(s, u, v), \qquad z = \chi(s, u, v),$$

each cut transversally for $s = 0$ *by the curve* L, *where the functions* ϕ, ψ, χ *are defined in a region*

$$0 \leqq s \leqq S, \qquad |u - u_0| \leqq \delta, \qquad 0 \leqq v \leqq 2\pi,$$

and are periodic in v *with the period* 2π.

It is readily seen from the properties of the solutions of the Euler equations that the function ϕ, and similarly for ψ and χ, is continuous together with all its first derivatives and all its second derivatives that are formed by differentiating ϕ_s with respect to s, u, or v.

§ 9. *Sufficient conditions when one end point is variable on a curve.*

Consider a region in space containing in its interior the arc C_{01} to which the curve L is transversal at the point 0. This region may be restricted in the first place, so that through any point of it not on L there passes one and but one of the extremals determined in the preceding section, to which L is transversal at points in the neighboring of the point 0; and, in the second place, so that the determinant Δ is different from zero for values of s, u, v corresponding to points in the field.* The equations

$$(49) \qquad \phi(s, u, v) = x, \qquad \psi(s, u, v) = y, \qquad \chi(s, u, v) = z$$

have a unique solution

$$(50) \qquad s = s(x, y, z), \qquad u = u(x, y, z), \qquad v = v(x, y, z)$$

of class C', for all values of x, y, z in the field except those which define a point on L. The reasoning of § 6 now shows that the value of the integral

$$J^* = \int (\bar{x}' F_{x'} + \bar{y}' F_{y'} + \bar{z}' F_{z'}) dt$$

taken along a curve K of class D'

$$(51) \qquad x = \bar{x}(t), \qquad y = y(t), \qquad z = \bar{z}(t)$$

which lies entirely in the field and has no point in common with L, depends only on the end points, and not at all on the form of the curve K. The derivatives of F are considered as functions of x, y, and z by means of equations (49), (50), and of t by (51). The integrand of J^*, thought of as a function of x, y, z, $\bar{x}'$, $\bar{y}'$, $\bar{z}'$, is not defined for points on the curve L since the function v is itself not defined for these points. The integral J^* has however a defi-

* The proof of the existence of such a field has been made by the writers and will be submitted to the Transactions for publication in the near future.

nite value when taken along a curve which has a finite number of points in common with L, since the integrand is limited and the only possible discontinuities are at the intersections with L. Let the value of J^* be defined as zero when K coincides with a portion of L. Then it may readily be seen that the integral is independent of the path for all curves which lie in the field, including those which intersect L or have an arc in common with L.

For let K and K_1 be two curves joining the same two points, where K_1 has no point in common with L. Suppose in the first place that K intersects L at a point P but has no other point in common with L. Let $\bar{K}$ be a curve which has no point in common with L and which coincides with K except along an arc which includes the point P. Then $J^*(\bar{K}) = J^*(K_1)$, and since by taking $\bar{K}$ near enough to K the value of $J^*(\bar{K})$ may be made to differ from $J^*(K)$ by an arbitrarily small amount, it follows also that $J^*(K) = J^*(K_1)$. Suppose now that K has an arc K_{56} which coincides with L. Then by definition $J^*(K_{56}) = 0$. Let $\bar{K}_{56}$ be a curve in the field which joins 5 to 6 but has no other points in common with L. Recalling the meaning of J^* as derived in § 5, it is seen that $J^*(\bar{K}_{56}) = 0$. In fact $J^*(\bar{K}_{56})$ is equal to the difference in the values of the integral J when taken along extremals of the set from L to 5 and from L to 6, and both of these are zero, since 5 and 6 lie on L. Then the portion of K which coincides with L may be replaced by the curve $\bar{K}_{56}$, which has only its end points on L, without altering the value of $J^*(K)$. It follows from the case above considered that $J^*(K) = J^*(K_1)$. From the preceding argument it is seen that, *at least for curves in the field which have a finite number of intersections or arcs in common with L, the integral J^* is independent of the path.*

Let $\bar{C}_{0'1}$ be any curve of class D' joining a point $0'$ of the curve L to the fixed point 1 and lying entirely in the field. Then, exactly as in § 6, the equation

$$J(\bar{C}_{0'1}) - J(C_{01}) = \int_{\bar{C}_{0'1}} E(x, y, z, \phi', \psi', \chi', x', y', z')\,dt$$

is derived by means of the invariant integral J^*. As has been seen before, this integral is always positive unless the comparison curve $\bar{C}_{0'1}$ coincides throughout with C_{01}, and the following theorem may be stated:

Let C_{01} be an arc of an extremal which joins a fixed curve L to a fixed point 1, and which is cut transversally by the curve L at the point 0. Furthermore suppose that Legendre's condition for a minimum is satisfied along C_{01}. If then the determinant $\Delta(s, u, v)$, for the set of extremals to which L is transversal, vanishes only at the point 0 on C_{01}, this arc will minimize the integral J with respect to curves joining L with the point 1.