# EXISTENCE AND OSCILLATION THEOREM FOR A CERTAIN BOUNDARY VALUE PROBLEM*

BY

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It is the purpose of this paper to consider the existence and oscillation of the real solutions of a linear differential equation of the second order

$$\frac{d^2u}{dx^2} + q(x, \lambda)u = 0 \qquad (a \leqq x \leqq b), \tag{1}$$

subject to the self-adjoint boundary conditions

$$\begin{aligned} \alpha_0 u'(a) + \beta_0 u(a) &= \gamma_0 u'(b) + \delta_0 u(b), \\ \alpha_1 u'(a) + \beta_1 u(a) &= \gamma_1 u'(b) + \delta_1 u(b), \\ \alpha_0\beta_1 - \beta_0\alpha_1 &= \gamma_0\delta_1 - \delta_0\gamma_1, \end{aligned} \tag{2}$$

where the real coefficients $\alpha_0$, $\beta_0$, $\gamma_0$, $\delta_0$ and $\alpha_1$, $\beta_1$, $\gamma_1$, $\delta_1$ are not proportional.

The function $q(x, \lambda)$ is assumed to be continuous in $(x, \lambda)$ for all real values of $\lambda$ when $x$ lies in $(a, b)$ and to increase steadily with $\lambda$ in such a way that

$$\lim_{\lambda=-\infty} q(x, \lambda) = -\infty, \qquad \lim_{\lambda=+\infty} q(x, \lambda) = +\infty. \tag{3}$$

We lay aside the trivial solution $u \equiv 0$.

STURM considered the problem under the special boundary conditions

$$\alpha_0 u'(a) + \beta_0 u(a) = 0, \qquad \gamma_1 u'(b) + \delta_1 u(b) = 0.\dagger \tag{4}$$

MASON ‡ has proved the existence of an infinite number of values of $\lambda$ furnishing a solution, when $q(x, \lambda)$ has the form $\lambda A(x) - g(x)$, and has given also an oscillation theorem for the special conditions

$$u(a) = u(b), \qquad u'(a) = u'(b). \tag{5}$$

BÔCHER § employing other methods has removed MASON's restriction on

* Presented to the Society (Chicago), January 1, 1908.

† Liouville's Journal, ser. 1, vol. 1 (1836), pp. 106–186. His linear differential equation appears in a more general form.

‡ These Transactions, vol. 7 (1906), pp. 337–360.

§ Comptes Rendus, vol. 140 (1905), p. 928.

$q(x, \lambda)$ for the conditions (5) except that a certain uniqueness of the values $\lambda$ is not proved. The method of this paper is like BÔCHER's, being based on certain theorems due to STURM.

## § 1. *Condition for a solution.*

In order to have a convenient notation let us write

$$(6)\qquad \begin{aligned} L_0[u(x)] &\equiv \alpha_0 u'(x) + \beta_0 u(x), & M_0[u(x)] &\equiv \gamma_0 u'(x) + \delta_0 u(x), \\ L_1[u(x)] &\equiv \alpha_1 u'(x) + \beta_1 u(x), & M_1[u(x)] &\equiv \gamma_1 u'(x) + \delta_1 u(x). \end{aligned}$$

Furthermore we assume

$$(7)\qquad \alpha_0\beta_1 - \beta_0\alpha_1 = \gamma_0\delta_1 - \delta_0\gamma_1 = 1,$$

as we may, since we can change either $\alpha_0, \beta_0, \gamma_0, \delta_0$, or $\alpha_1, \beta_1, \gamma_1, \delta_1$ by a constant factor.*

We now obtain an explicit condition that $l$ shall be a value of $\lambda$ for which there is a solution of (1), (2). Determine $u_0(x, \lambda)$, $u_1(x, \lambda)$ as the solutions of (1) for which

$$(8)\qquad \begin{aligned} L_0[u_0(a, \lambda)] &= 0, & L_1[u_0(a, \lambda)] &= 1, \\ L_0[u_1(a, \lambda)] &= 1, & L_1[u_1(a, \lambda)] &= 0. \end{aligned}$$

These equations fix $u_0, u_0', u_1, u_1'$ at $x = a$, and hence determine $u_0$ and $u_1$. We readily find that

$$(9)\qquad L_0[u_0(x, \lambda)]L_1[u_1(x, \lambda)] - L_0[u_1(x, \lambda)]L_1[u_0(x, \lambda)] = -1,$$

for the left-hand member reduces to $u_0'u_1 - u_1'u_0 =$ constant, and this constant must be $-1$ by (8). Also we have

$$(10)\qquad M_0[u_0(x, \lambda)]M_1[u_1(x, \lambda)] - M_0[u_1(x, \lambda)]M_1[u_0(x, \lambda)] = -1,$$

since here also the left-hand member reduces to $u_0'u_1 - u_1'u_0$.

*The necessary and sufficient condition that there exists a solution $u \not\equiv 0$ when $\lambda = l$ is that $\phi(l) = 0$, where*

$$(11)\qquad \phi(\lambda) \equiv M_0[u_1(b, \lambda)] + M_1[u_0(b, \lambda)] - 2.$$

In fact the solutions $u_0$ and $u_1$ of (1) are linearly independent by (8), and therefore we may write any solution $u$ of (1) in the form

$$c_0 u_0 + c_1 u_1.$$

If this expression be substituted for $u$ in the conditions (2), we obtain, after sim-

* If, however,

$$\alpha_0\beta_1 - \beta_0\alpha_1 = \gamma_0\delta_1 - \delta_0\gamma_1 = 0,$$

this is not possible. In this case the conditions are of the Sturmian type (4) and we do not need to treat this case.

plifying by means of (8),

$$
(12)\qquad \begin{aligned} c_1 &= c_0 M_0[u_0(b, \lambda)] + c_1 M_0[u_1(b, \lambda)], \\ c_0 &= c_0 M_1[u_0(b, \lambda)] + c_1 M_1[u_1(b, \lambda)]. \end{aligned}
$$

These two equations are linear in $c_0$ and $c_1$; a solution $u \not\equiv 0$ which satisfies the conditions (1) and (2) will therefore exist if and only if the determinant

$$
(13)\qquad \phi(\lambda) \equiv \begin{vmatrix} -M_0[u_0(b, \lambda)], & 1 - M_0[u_1(b, \lambda)] \\ 1 - M_1[u_0(b, \lambda)], & -M_1[u_1(b, \lambda)] \end{vmatrix}
$$

vanishes. By means of (10) for $x = b$, this identity reduces to (11).

## § 2. *Simple and double solutions.*

A value $l$ of $\lambda$ for which a solution $u \not\equiv 0$ of (1), (2) exists is said to be *simple* if all the solutions are linearly dependent: if, however, there exist two linearly independent solutions of (1), (2) for $\lambda = l$, the value $l$ is said to be *double*. *The necessary and sufficient condition that $\lambda = l$ is a double value is that*

$$
(14)\qquad M_0[u_0(b, l)] = M_1[u_1(b, l)] = 0, \qquad M_0[u_1(b, l)] = M_1[u_0(b, l)] = 1.
$$

In fact two linearly independent solutions of (1), (2) exist if and only if the elements of the determinant $\phi(\lambda)$ given by (13) all vanish.

The question now arises: How does $\phi(\lambda)$ behave at simple and double values? It is the object of this section to show that $\phi(\lambda)$ changes sign at a simple value of $\lambda$ and preserves its sign at a double value.

It is essential first to derive some formulas. Let us change $\lambda$ to $\bar{\lambda} = \lambda + \delta\lambda$; all functions $f$ of $\lambda$ will then change to $\bar{f} = f + \delta f$. The function $\delta u_0(x, \lambda)$ fulfills the conditions

$$
(15)\qquad \frac{d^2}{dx^2}[\delta u_0(x, \lambda)] + q(x, \lambda)\delta u_0(x, \lambda) = -\delta q(x, \lambda) u_0(x, \lambda),
$$

$$
(16)\qquad \delta u_0(a, \lambda) = \frac{d}{dx}[\delta u_0(a, \lambda)] = 0;
$$

the equation (15) being obtained by subtracting from one another the equations satisfied by $u_0$ and $\bar{u}_0$, while the equations (16) are a consequence of the fact that $u_0(a, \lambda)$ and $u_0'(a, \lambda)$, as determined from (8), are independent of $\lambda$. A like set of equations holds for $\delta u_1(x, \lambda)$.

From the non-homogeneous linear differential equation (15) and the conditions (16), the function $\delta u_0(x, \lambda)$ can be explicitly obtained; we have

$$
(17)\qquad \delta u_0(x, \lambda) = \int_a^x [u_0(x, \lambda) u_1(\xi, \lambda) - u_1(x, \lambda) u_0(\xi, \lambda)] \delta q(\xi, \lambda) \bar{u}_0(\xi, \lambda)\, d\xi,
$$

as may be proved by a direct substitution. If we differentiate (17) as to $x$ we obtain

$$(18)\quad \frac{d}{dx}[\delta u_0(x,\lambda)] = \int_a^x [u_0'(x,\lambda)u_1(\xi,\lambda) - u_1'(x,\lambda)u_0(\xi,\lambda)]\,\delta q(\xi,\lambda)\bar{u}_0(\xi,\lambda)d\xi.$$

From (17) and (18) we get

$$(19)\quad \delta M_0[u_0(b,\lambda)] = \int_a^b \{M_0[u_0(b,\lambda)]u_1(\xi,\lambda) - M_0[u_1(b,\lambda)]u_0(\xi,\lambda)\}\,\delta q(\xi,\lambda) \times \bar{u}_0(\xi,\lambda)d\xi,$$

$$(20)\quad \delta M_1[u_0(b,\lambda)] = \int_a^b \{M_1[u_0(b,\lambda)]u_1(\xi,\lambda) - M_1[u_1(b,\lambda)]u_0(\xi,\lambda)\}\,\delta q(\xi,\lambda) \times \bar{u}_0(\xi,\lambda)d\xi,$$

and in like manner one can prove

$$(21)\quad \delta M_0[u_1(b,\lambda)] = \int_a^b \{M_0[u_0(b,\lambda)]u_1(\xi,\lambda) - M_0[u_1(b,\lambda)]u_0(\xi,\lambda)\}\,\delta q(\xi,\lambda) \times \bar{u}_1(\xi,\lambda)d\xi,$$

$$(22)\quad \delta M_1[u_1(b,\lambda)] = \int_a^b \{M_1[u_0(b,\lambda)]u_1(\xi,\lambda) - M_1[u_1(b,\lambda)]u_0(\xi,\lambda)\}\,\delta q(\xi,\lambda) \times \bar{u}_1(\xi,\lambda)d\xi.$$

We are now in a position to consider the sign of $\phi$ near a value $l$ of $\lambda$ for which $\phi = 0$. We first take the case when $l$ is a simple value.

By means of (11) we have

$$\delta\phi = \delta M_0[u_1(b,\lambda)] + \delta M_1[u_0(b,\lambda)].$$

Therefore by (20) and (21) we obtain

$$(23)\quad \begin{aligned}\delta\phi = \int_a^b \{&M_0[u_0(b,\lambda)]u_1(\xi,\lambda)\bar{u}_1(\xi,\lambda) - M_0[u_1(b,\lambda)]u_0(\xi,\lambda)\bar{u}_1(\xi,\lambda)\\ &+ M_1[u_0(b,\lambda)]u_1(\xi,\lambda)\bar{u}_0(\xi,\lambda) - M_1[u_1(b,\lambda)]u_0(\xi,\lambda)\bar{u}_0(\xi,\lambda)\}\,\delta q(\xi,\lambda)d\xi.\end{aligned}$$

It is our purpose to investigate the sign of $\delta\phi$ for small $\delta\lambda$ by the aid of this formula.

The integrand on the right is the product of two factors, the second of which, $\delta q(\xi,\lambda)$, has the sign of $\delta\lambda$, since $q(\xi,\lambda)$ by hypothesis increases with $\lambda$. As the quantity $\delta\lambda$ becomes smaller, $\bar{u}_0$ and $\bar{u}_1$ approach $u_0$ and $u_1$, and the first factor of the integrand approaches a homogeneous quadratic form in $u_0(\xi,\lambda)$ and $u_1(\xi,\lambda)$ which is obtained when we replace $\bar{u}_0(\xi,\lambda)$ and $\bar{u}_1(\xi,\lambda)$ by $u_0(\xi,\lambda)$ and $u_1(\xi,\lambda)$ respectively. The discriminant of this quadratic form is

$$\{-M_0[u_1(b,\lambda)] + M_1[u_0(b,\lambda)]\}^2 + 4M_0[u_0(b,\lambda)]M_1[u_1(b,\lambda)].$$

If here we substitute the value

$$M_0[u_0(b,\lambda)]M_1[u_1(b,\lambda)] = -1 + M_0[u_1(b,\lambda)]M_1[u_0(b,\lambda)]$$

obtained from (10), the discriminant becomes

$$\{M_0[u_1(b,\lambda)] + M_1[u_0(b,\lambda)]\}^2 - 4,$$

which vanishes at $\lambda = l$ by (11) since $\phi(l) = 0$.

Now it is clear that the three coefficients of the form are not zero except at a double value; otherwise with the aid of (10) we derive equations (14). Also the functions $u_0$ and $u_1$ are linearly independent. Thus as $\bar{\lambda}$ tends to a value $l$, the limit of the first factor of the integrand is an expression of definite sign for $a \leqq \xi \leqq b$, the same as that of $M_0[u_0(b,l)]$ and $M_1[u_1(b,l)]$, save at isolated points at which it vanishes.

We conclude that *at a simple value $l$ of $\lambda$, $\phi(\lambda)$ changes sign in such a way that $\delta\phi/\delta\lambda$ is of the same sign as $M_0[u_0(b,\lambda)]$ and $M_1[u_1(b,\lambda)]$.**

Let us now pass to the case of a double value $l$. If we recall conditions (14), we find that for $\lambda = l$

$$M_0[u_0(b,\bar{\lambda})] = \delta M_0[u_0(b,\lambda)], \qquad M_0[u_1(b,\bar{\lambda})] = \delta M_0[u_1(b,\lambda)],$$

$$M_0[u_1(b,\bar{\lambda})] = 1 + \delta M_0[u_1(b,\lambda)], \quad M_1[u_1(b,\bar{\lambda})] = 1 + \delta M_1[u_1(b,\lambda)].$$

Substitute these values in (10), taking $\lambda = \bar{\lambda}$, and we find

$$\delta M_0[u_0(b,\lambda)]\,\delta M_1[u_1(b,\lambda)] - \{\delta M_0[u_1(b,\lambda)] + \delta M_1[u_0(b,\lambda)] + \delta M_0[u_1(b,\lambda)]\,\delta M_1[u_0(b,\lambda)]\} = 0$$

so that

$$\delta\phi = \delta M_0[u_0(b,\lambda)]\,\delta M_1[u_1(b,\lambda)] - \delta M_0[u_1(b,\lambda)]\,\delta M_1[u_0(b,\lambda)].$$

* This argument is satisfactory when the $\lambda$-derivative of $q(x,\lambda)$ exists on either side of $l$ and is positive for some $x$ of $(a,b)$.

Assume now that the above statement is not true for $\phi$ on one side of $l$, say for $\lambda > l$. Replace $q$ in the interval $(l, l+d)$ by

$$q^*(x,\lambda) = q(x,l) + \frac{\lambda - l}{d}\,q(x, l+d),$$

which satisfies the above condition. Then we have

$$\phi^*(l) = \phi(l) = 0, \qquad \phi^*(l+d) = \phi(l+d).$$

But the above statement does hold for $\phi^*$. From this it follows that for a proper choice of $d$, as small as we please, we shall have $\phi^*(l') = 0$ where $l < l' \leqq l + d$. But for two successive simple values for which $\phi^*(\lambda) = 0$, $M_0[u_0^*(b,\lambda)]$ and $M_1[u_1^*(b,\lambda)]$ must change sign (or vanish) since the sign of $\delta\phi^*/\delta\lambda$ does, and at a double value both vanish. We conclude that

$$M_0[u_0^*(b,\lambda_1)] = 0, \qquad M_1[u_1^*(b,\lambda_2)] = 0, \qquad (l < \lambda_1,\ \lambda_2 \leqq l + d).$$

Let $d$ now tend to zero. Then we infer

$$M_0[u_0(b,l)] = M_1[u_1(b,l)] = 0.$$

Since $\phi(l) = 0$ these relations, combined with (10) when $x = l$, show that $l$ would then be a double value, contrary to hypothesis.

If now the expressions (19), (20), (21), (22) be simplified by the use of (14) and substituted in this expression for $\delta\phi$, and if we write for $\bar{u}_0(\xi, \lambda)$, $\bar{u}_1(\xi, \lambda)$ their limiting values $u_0(\xi, \lambda)$, $u_1(\xi, \lambda)$, we obtain

$$
(24) \qquad \begin{aligned} -\int_a^b u_0^2(\xi, \lambda)\delta q(\xi, \lambda)d\xi \int_a^b u_1^2(\xi, \lambda)\delta q(\xi, \lambda)d\xi \\ + \left[\int_a^b u_0(\xi, \lambda)u_1(\xi, \lambda)\delta q(\xi, \lambda)d\xi\right]^2. \end{aligned}
$$

But by a familiar inequality

$$\int_a^b f^2 d\xi \int_a^b g^2 d\xi - \left[\int_a^b fg d\xi\right]^2 > 0,$$

provided $f$ and $g$ are linearly independent real continuous functions. Hence, if we write

$$f = u_0(\xi, \lambda)\sqrt{\pm \delta q(\xi, \lambda)}, \qquad g = u_1(\xi, \lambda)\sqrt{\pm \delta q(\xi, \lambda)},$$

using $+$ or $-$ according as $\delta\lambda$ is positive or negative, it is apparent that the quantity (24) is negative. Therefore $\phi(\lambda)$ *preserves a negative sign at a double value of* $\lambda$.*

## § 3. *The existence theorem.*

Now let the infinite set of values of $\lambda$ furnishing a solution $u$ of (1) for the Sturmian conditions

$$(25) \qquad L_0[u(a)] = 0, \qquad M_0[u(b)] = 0,$$

be denoted by

$$\lambda_1, \lambda_2, \cdots \qquad (\lambda_1 < \lambda_2 < \cdots).$$

These values separate the $\lambda$-axis into the intervals

$$(26) \qquad (-\infty, \lambda_1), \qquad (\lambda_1, \lambda_2), \qquad (\lambda_2, \lambda_3), \cdots.$$

This division into intervals is not uniquely determined, for the conditions (2) can be replaced by any two linearly independent conditions which arise from (2) by linear combination.

THE EXISTENCE THEOREM. *There exists an infinite set of values* $l_1, l_2, \cdots$ *of* $\lambda$ *furnishing solutions of* (1), (2). *If we take these quantities in order of increasing magnitude counting each double value twice, there are the following*

* This argument is not satisfactory unless the $\lambda$-derivative of $q(x, \lambda)$ exists on either side of $l$, and is positive for $a \leqq x \leqq b$.

If the derivative does not exist, we proceed as in the previous footnote and prove that $M_0[u_0^*(b, \lambda)]$ and $M_1[u_1^*(b, \lambda)]$ vanish in the vicinity of $l$ as well as at $l$. This is impossible since the zeros of these functions are separated by finite intervals according to the results of STURM.

*possible cases:*

$\mathrm{I}_a$ $\lambda_1 < l_1 \leqq \lambda_2 \leqq l_2 < \lambda_3 < l_3 \leqq \lambda_4 \leqq l_4 < \lambda_5 \cdots,$

$\mathrm{I}_b$ $l_1 < \lambda_1 < l_2 \leqq \lambda_2 \leqq l_3 < \lambda_3 < l_4 \leqq \lambda_4 \leqq l_5 < \lambda_5 \cdots,$

$\mathrm{II}_a$ $\lambda_1 \leqq l_1 < \lambda_2 < l_2 \leqq \lambda_3 \leqq l_3 < \lambda_4 < l_4 \leqq \lambda_5 \cdots,$

$\mathrm{II}_b$ $l_1 \leqq \lambda_1 \leqq l_2 < \lambda_2 < l_3 \leqq \lambda_3 \leqq l_4 < \lambda_4 < l_5 \leqq \lambda_5 \cdots.$

*Proof.* Clearly $u_0$ is the only solution of (1) (save for a constant factor) which satisfies the first condition (25), and if the second of these conditions is also to be fulfilled, we must have $M_0[u_0(b, \lambda)] = 0$. Hence we see that

$$M_0[u_0(b, \lambda)] = 0 \qquad \text{for } \lambda = \lambda_1, \lambda_2, \cdots, \tag{27}$$

but for no other $\lambda$. It follows from (10) that

$$M_0[u_1(b, \lambda_i)] M_1[u_0(b, \lambda_i)] = 1 \qquad (i = 1, 2, \cdots). \tag{28}$$

Substituting the value for $M_0[u_1(b, \lambda_i)]$ obtained from this last equation in (11), we find

$$\phi(\lambda_i) = \frac{1}{M_1[u_0(b, \lambda_i)]}\{1 - M_1[u_0(b, \lambda_i)]\}^2 \qquad (i = 1, 2, \cdots). \tag{29}$$

We infer that $\phi$ *has the same sign as* $M_1[u_0(b, \lambda)]$ *at the values* $\lambda = \lambda_1$, $\lambda_2, \cdots$, *unless* $M_1[u_0(b, \lambda)] = 1$, *when* $\phi$ *vanishes.*

But since $u_0(a, \lambda)$, $u_0'(a, \lambda)$ are independent of $\lambda$, it is a consequence of familiar theorems due to STURM that the roots of $M_0[u_0(b, \lambda)] = 0$ and $M_1[u_0(b, \lambda)] = 0$ separate each other, and that $M_0[u_0(b, \lambda)]$ and $M_1[u_0(b, \lambda)]$ change sign when they vanish.*

Accordingly by (27) it is clear that $M_1[u_0(b, \lambda)]$ *alternates in sign at the values* $\lambda_1, \lambda_2 \cdots$. Either $M_1[u_0(b, \lambda)]$ is positive at $\lambda_1, \lambda_3, \cdots$ and negative at $\lambda_2, \lambda_4, \cdots$, or *vice versa.* Thus two cases arise:

$$\text{CASE I} \begin{cases} \phi \geqq 0 \text{ at } \lambda_1, \lambda_3, \cdots, \\ \phi < 0 \text{ at } \lambda_2, \lambda_4, \cdots, \end{cases}$$
$$\text{when } M_1[u_0(b, \lambda_1)] > 0. \tag{30}$$
$$\text{CASE II} \begin{cases} \phi < 0 \text{ at } \lambda_1, \lambda_3, \cdots, \\ \phi \geqq 0 \text{ at } \lambda_2, \lambda_4, \cdots, \end{cases}$$
$$\text{when } M_1[u_0(b, \lambda_1)] < 0.$$

We see that there must exist values $l$ of $\lambda$ as follows. In Case I there exist at least two values $l$, say $l_1$, $l_2$, in each double interval $(\lambda_{2p}, \lambda_{2p+2})$,

* Loc. cit., p. 139, 142.

$p = 1, 2, \cdots$, such that

$$\lambda_{2p} < l_1 \leqq \lambda_{2p+1} \leqq l_2 < \lambda_{2p+2},$$

and at least one $l_2$ such that

$$\lambda_1 \leqq l_2 < \lambda_2.$$

In Case II there exist at least two values $l_1$, $l_2$ in each double interval $(\lambda_{2p-1}, \lambda_{2p+1})$, $p = 1, 2 \cdots$ such that

$$\lambda_{2p-1} < l_1 \leqq \lambda_{2p} \leqq l_2 < \lambda_{2p+1}.$$

For this deduction it is necessary to know (see § 2) that $\phi$ preserves its sign only at a double value of $l$ which we count as two.

It remains to prove first that there exist only the two values $l_1$, $l_2$ in the double intervals; secondly that there exist in Case I either two values $l_1$, $l_2$ in $(-\infty, \lambda_2)$ such that

$$l_1 \leqq \lambda_1 \leqq l_2 < \lambda_2 \qquad \text{(Subcase I}_b\text{)},$$

or one value $l_2$ such that

$$\lambda_1 \leqq l_2 < \lambda_2 \qquad \text{(Subcase I}_a\text{)};$$

and lastly that there exists in Case II only one value $l_2$ in $(-\infty, \lambda_1)$ such that

$$l_2 \leqq \lambda_1. \qquad \text{(Subcase II}_b\text{)},$$

or none (Subcase II$_a$).

Let us take up the first point. If there exist additional values $l$ in some double interval $(\lambda_q, \lambda_{q+2})$, there will be at least four, since $\phi(\lambda_q)$ and $\phi(\lambda_{q+2})$ have the same sign. If there is no double value, there must at least two simple values fall within one of the intervals $(\lambda_q, \lambda_{q+1})$ or $(\lambda_{q+1}, \lambda_{q+2})$, at which $\delta\phi/\delta\lambda$ is of opposite signs. But this is impossible since by the last section $\delta\phi/\delta\lambda$ has at all such values within one interval the sign of $M_0[u_0(b, \lambda)]$, which is invariable within the interval. On the other hand if a double value exists, it must fall at $\lambda_{q+1}$ by (14); then $\phi$ in the neighborhood of $\lambda_{q+1}$ is negative by § 2. Other values $l$ besides the double value would then imply at least two roots $l$ existing in $(\lambda_q, \lambda_{q+1})$ or $(\lambda_{q+1}, \lambda_{q+2})$, inasmuch as $\phi(\lambda_q)$ and $\phi(\lambda_{q+2})$ are negative. This is impossible, as we have just seen.

Thus the first point is proved, and the second and third may be treated in the same manner.

## § 4. *Discrimination between the four sub-cases.*

We have Case I or Case II according as $M_1[u_0(b, \lambda_1)] > 0$ or $M_1[u_0(b, \lambda_1)] < 0$, and under each of these subcase $a$ or $b$ according as $\phi$ is of the same sign as $M_1[u_0(b, \lambda_1)]$ or of opposite sign, for large negative $\lambda$. This is evident from (30).

It remains now to determine the sign of these quantities.

*Sign of* $M_1[u_0(b, \lambda_1)]$.

We have $M_0[u_0(b, \lambda)] = 0$ for $\lambda = \lambda_1$ but not for $\lambda < \lambda_1$. By theorems of STURM * $u_0(x, \lambda)$ does not vanish for $a < x < b$ and $\lambda \leqq \lambda_1$. There are four sub-cases:

(1) $\alpha_0 \neq 0, \ \gamma_0 \neq 0$.

Since $\gamma_0 u_0'(b, \lambda_1) + \delta_0 u_0(b, \lambda_1) = 0$, we have by (7)

$$M_1[u_0(b, \lambda_1)] = \gamma_1\left(-\frac{\delta_0}{\gamma_0}u_0(b, \lambda_1)\right) + \delta_1 u_0(b, \lambda_1) = \frac{u_0(b, \lambda_1)}{\gamma_0}.$$

But $u_0(b, \lambda_1)$ has the same sign as $u_0(a, \lambda_1) = \alpha_0$. Hence the *sign of* $M_1[u_0(b, \lambda_1)]$ *is that of* $\alpha_0/\gamma_0$.

(2) $\alpha_0 \neq 0, \ \gamma_0 = 0$.

Here we have $u_0(b, \lambda_1) = 0$, and $u_0'(b, \lambda_1)$ will have the opposite sign to $u_0(a, \lambda_1) = \alpha_0$. Therefore the *sign of* $M_1[u_0(b, \lambda_1)]$ *is that of* $-\alpha_0\gamma_1$.

(3) $\alpha_0 = 0, \ \gamma_0 \neq 0$.

Here, as in case (1),

$$M_1[u_0(b, \lambda_1)] = \frac{u_0(b, \lambda_1)}{\gamma_0}.$$

But $u_0(b, \lambda_1)$ has the same sign as $u_0'(a, \lambda_1) = -\beta_0$ since $u_0(a, \lambda_1) = \alpha_0 = 0$. *The sign of* $M_1[u_0(b, \lambda_1)]$ *is that of* $-\beta_0/\gamma_0$,

(4) $\alpha_0 = 0, \ \gamma_0 = 0$.

Here, as in case (2),

$$M_1[u_0(b, \lambda_1)] = \gamma_1 u_0'(b, \lambda_1),$$

where $u_0'(b, \lambda_1)$ has the sign opposite to $u_0'(a, \lambda_1) = -\beta_0$ since $u_0(a, \lambda_1) = 0$ and $u_0(b, \lambda_1) = 0$. *The sign of* $M_1[u_0(b, \lambda_1)]$ *is that of* $-\beta_0\delta_0$.

*Sign of* $\phi(\lambda)$ *for large negative* $\lambda$.

It remains to determine the sign of $\phi(\lambda)$ for large negative $\lambda$. Let us assume first that $\alpha_0 \neq 0$.

We have, in view of the relation (10) between the solutions of (1), the equation

$$u_1(x, \lambda) = \left(m + n\int_a^x \frac{dx}{[u_0(x, \lambda)]^2}\right) u_0(x, \lambda).$$

By means of (8) we determine the values of the constants $m$ and $n$ and

* Loc. cit., p. 140.

have

$$u_1(x, \lambda) = \left(-\frac{\alpha_1}{\alpha_0} + \int_a^x \frac{dx}{[u_0(x, \lambda)]^2}\right) u_0(x, \lambda).$$

If we substitute this expression for $u_1(x, \lambda)$ in (11), we find

$$(31) \quad \begin{aligned} \phi(\lambda) = {} & \gamma_1 u_0'(b, \lambda) + \delta_1 u_0(b, \lambda) \\ & + \gamma_0 \left[\left(-\frac{\alpha_1}{\alpha_0} + \int_a^b \frac{dx}{[u_0(x, \lambda)]^2}\right) u_0'(b, \lambda) + \frac{1}{u_0(b, \lambda)}\right] \\ & + \delta_0 \left(-\frac{\alpha_1}{\alpha_0} + \int_a^b \frac{dx}{[u_0(x \ \lambda)]^2}\right) u_0(b, \lambda) - 2. \end{aligned}$$

Now let us recall that for $x > a$

$$\lim_{\lambda=-\infty} u_0'(x, \lambda) = \lim_{\lambda=-\infty} u_0(x, \lambda) = \infty,$$

$$\lim_{\lambda=-\infty} \frac{u_0'(x, \lambda)}{u_0(x, \lambda)} = +\infty.$$

It is then apparent that $\phi(\lambda)$ has the sign of $(\alpha_0\gamma_1 - \gamma_0\alpha_1)u_0'(b, \lambda)/\alpha_0$ when $\lambda$ is large, if $\alpha_0\gamma_1 - \gamma_0\alpha_1 \neq 0$. But $u_0'(b, \lambda)$ has the sign of $u_0(a, \lambda) = \alpha_0$. Hence *for large negative* $\lambda$ *the sign of* $\phi(\lambda)$ *is that of* $(\alpha_0\gamma_1 - \gamma_0\alpha_1)$, *provided that* $\alpha_0\gamma_1 - \gamma_0\alpha_1 \neq 0$.

If $\alpha_0 = 0$, the above statement remains true. In this case we express $u_0(x, \lambda)$ in terms of $u_1(x, \lambda)$ since $\alpha_1 \neq 0$.

Let us next attend to the case where $\alpha_0\gamma_1 - \gamma_0\alpha_1 = 0$ but $\alpha_0 \neq 0$. In this case we transform the second of conditions (2) so that $\alpha_1 = \gamma_1 = 0$ by linear combination. Furthermore we choose a multiplier so that (7) also holds, whence $\beta_1 = 1/\alpha_0$, $\delta_1 = 1/\gamma_0$.

When these values of $\alpha_1$, $\beta_1$, $\gamma_1$, $\delta_1$ are substituted in (31) we get

$$(32) \quad \begin{aligned} \phi(\lambda) = {} & \gamma_0 \left(\frac{1}{\gamma_0^2} u_0(b, \lambda) + \int_a^b \frac{dx}{[u_0(x, \lambda)]^2} u_0'(b, \lambda) + \frac{1}{u_0(b, \lambda)}\right) \\ & + \delta_0 \int_a^b \frac{dx}{[u_0(x, \lambda)]^2} u_0(b, \lambda) - 2. \end{aligned}$$

Now $u_0(b, \lambda)$, $u_0'(b, \lambda)$ have the same sign as $u_0(a, \lambda) = \alpha_0$. Therefore by (32), *for large negative* $\lambda$, $\phi(\lambda)$ *has the sign of* $\alpha_0\gamma_0$ *in the case* $\alpha_0\gamma_1 - \gamma_0\alpha_1 = 0$, $\alpha_0 \neq 0$, *if the second condition of* (2) *is so chosen that*

$$\alpha_1 = 0, \qquad \beta_1 = \frac{1}{\alpha_0}, \qquad \gamma_1 = 0, \qquad \delta_1 = \frac{1}{\gamma_0}.$$

Finally let us consider the case where $\alpha_0\gamma_1 - \gamma_0\alpha_1 = 0$ and $\alpha_0 = 0$. In this case $\alpha_1$ is not zero since $\alpha_0\beta_1 - \beta_0\alpha_1 = 1$. By symmetry it appears that *for large negative* $\lambda$, $\phi(\lambda)$ *has the sign of* $-\alpha_1\gamma_1$ *if both* $\alpha_0\gamma_1 - \gamma_0\alpha_1 = 0$ *and* $\alpha_0 = 0$.

It should be noted that the above classification depends on the selection of the first condition $L_0[u(a)] = M_0[u(b)]$ out of the double infinity of conditions.

## § 5. *The oscillation theorem.*

THEOREM. *The solution $u_p(x)$ of* (1), (2) *which corresponds to $\lambda = l_p$ vanishes $p-1$, $p$, or $p+1$ times for $a < x \leqq b$ in accordance with the following table, in which*

$\bar{\alpha}_0 = \gamma_0\alpha_1 - \alpha_0\gamma_1$, $\bar{\beta}_0 = \gamma_0\beta_1 - \beta_0\gamma_1$, *and* $K = -\bar{\beta}_0/\bar{\alpha}_0$.*

| | | | |
|---|---|---|---|
| I. $\bar{\alpha}_0 > 0$, | $p = 2m$, | $\frac{u_p'(a)}{u_p(a)} \leqq K$; | $p+1$ times, |
| | | $\frac{u_p'(a)}{u_p(a)} > K$; | $p$ times, |
| | $p = 2m+1$, | $\frac{u_p'(a)}{u_p(a)} \leqq K$; | $p$ times, |
| | | $\frac{u_p'(a)}{u_p(a)} > K$; | $p-1$ times, |
| II. $\bar{\alpha}_0 < 0$, | $p = 2m$, | $\frac{u_p'(a)}{u_p(a)} \leqq K$ | $p$ times, |
| | | $\frac{u_p'(a)}{u_p(a)} > K$; | $p-1$ times, |
| | $p = 2m+1$, | $\frac{u_p'(a)}{u_p(a)} \leqq K$; | $p+1$ times, |
| | | $\frac{u_p'(a)}{u_p(a)} > K$; | $p$ times, |
| III. $\bar{\alpha}_0 = 0$, $\bar{\beta}_0 > 0$, | $p = 2m$; | | $p$ times, |
| | $p = 2m+1$; | | $p+1$ times, |
| $\bar{\alpha}_0 = 0$, $\bar{\beta}_0 < 0$, | $p = 2m$; | | $p+1$ times, |
| | $p = 2m+1$; | | $p$ times. |

*Proof.* The conditions may always be written in the form†

$$\begin{aligned} &\bar{\alpha}_0 u'(a) + \bar{\beta}_0 u(a) = u(b), \\ &\bar{\alpha}_1 u'(a) + \bar{\beta}_1 u(a) = -u'(b) + \bar{\delta}_1 u(b). \end{aligned} \tag{33}$$

* At a double value we can take any solution $u_p(x)$ as corresponding to $\lambda_p$. Also if $u_p(a) = 0$, we make the convention that $u_p'(a)/u_p(a) = -\infty$.

† If the conditions be taken in this form at the start, no transformation is necessary. To obtain the first of these new conditions from (2), multiply the first by $-\gamma_0$, the second by $\gamma_1$, and add. The second of these conditions is then any condition so taken as to satisfy (7).

Then $\bar{\phi}(\lambda)$ has the sign of $-\bar{\alpha}_0\bar{\gamma}_1=\bar{\alpha}_0$ at $\lambda_1$ and of $(\bar{\alpha}_0\bar{\gamma}_1-\bar{\gamma}_0\bar{\alpha}_1)=\bar{\alpha}_0$ for large negative $\lambda$. This follows from the last section. With these new conditions we therefore have case $I_a$ or $II_a$ of the existence theorem. Accordingly by the existence theorem, the value $l_p$ lies on the interval $(\lambda_p . \lambda_{p+1})$.

By well-known theorems of STURM, $\bar{u}_0(x, \lambda)$ will vanish $p$ times on $a<x<b$ for $\lambda_p \leqq \lambda \leqq \lambda_{p+1}$. But the roots of $u_p(x)$ and $\bar{u}_0(x, l_p)$ separate each other, or else coincide. Hence $u_p(x)$ *vanishes* $p-1$, $p$, *or* $p+1$ *times on* $a<x\leqq b$.

Consider first the case $\bar{\alpha}_0>0$, and assume that $u_p(a)$ and $u_p(b)$ do not vanish, a retriction which is easily removed. If we have $p=2m$ and $u_p'(a)/u_p(a)\leqq K$, we proceed as follows: Since, by definition the equation $\bar{u}_0'(a, l_p)/\bar{u}_0(a, l_p)=K$ holds, the function $u_p(x)$ vanishes at least as often as $\bar{u}_0(x, l_p)$ and hence $p$ or $p+1$ times. But also from (33) we have $\bar{\alpha}_0 u_p(a)[u_p'(a)/u_p(a)-K]=u_p(b)$. Thus $u_p(a)$ and $u_p(b)$ have opposite signs, and $u_p(x)$ vanishes an odd number of times. This excludes the first possibility. Hence $u_p(x)$ vanishes $p+1$ times.

Likewise if $p=2m$ and $u_p'(a)/u_p(a)>K$, we see that $u_p(x)$ cannot vanish more often than $\bar{u}_0(x, l_p)$, and therefore vanishes $p-1$ or $p$ times. However, $u_p(a)$ and $u_p(b)$ have like signs, so that $u_p(x)$ must vanish $p$ times.

Also if $p=2m+1$, $u_p'(a)/u_p(a)\leqq K$, we find that $u_p(x)$ must vanish $p$ times.

Also if $p=2m+1$, $u_p'(a)/u_p(a)>K$, we find that $u_p(x)$ must vanish $p-1$ times.

A precisely similar discussion is possible when $\bar{\alpha}_0<0$.

When one has $\bar{\alpha}_0=0$, $\bar{\beta}_0>0$, the first condition (33) shows that $u_p(a)$ and $u_p(b)$ have like signs. Also $u_p(x)$ has at most $p$ or $p-1$ zeros, since $\bar{u}_0(x, l_p)$ vanishes for $x=a$. From this follows the table for this case.

In a like manner one may discuss the case $\bar{\alpha}_0=0$, $\bar{\beta}_0<0$.

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