ON THE REGIONS OF CONVERGENCE OF POWER-SERIES WHICH REPRESENT TWO-DIMENSIONAL HARMONIC FUNCTIONS*

BY

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The function $u(x, y)$ is said to be harmonic throughout a two-dimensional continuum T in the (x, y)-plane† if it is continuous, has continuous first and second partial derivatives, and satisfies Laplace's equation

$$(1) \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

at every point of T. In this case it is well known that u is analytic throughout T, that is, it can be developed about every point of T into a double power-series. The main object of the present note is to determine the region of convergence of this double series.‡ For the sake of comparison I have also considered the simple series obtained from it by grouping together the terms of the same degree. In order to avoid a possible misunderstanding, I make the explicit statement that a double series shall be regarded as convergent when and only when every simple series into which its terms can be rearranged converges. According to this convention, every convergent double series is absolutely convergent.

The main results of this paper are recapitulated in § 3. Attention may also be called to the result stated at the close of § 1.

1. The double power-series. We approach our problem from the side of the Cauchy-Kowalewski existence theorem. Let $f_0(x)$ and $f_1(x)$ be two functions analytic when $x = 0$, and denote by K_0 and K_1 the radii of convergence

* Presented to the Society October 25, 1902, and subsequently further developed. Cf. Bulletin of the American Mathematical Society, series 2, vol. 9 (1903), p. 186–7, and OSGOOD, *Funktionentheorie*, vol. 1, p. 575.

† By a two-dimensional continuum I understand a connected region of the plane every point of which is an internal point. Throughout this note only real quantities are considered except where the contrary is explicitly stated.

‡ The only investigation of this question with which I am acquainted is contained in PICARD's *Traité d'Analyse* (1st or 2d edition), vol. 2, Chapter 1, § 14. It is there shown that if R is the distance from any point (x_0, y_0) of T to the nearest point of the boundary, the development about (x_0, y_0) converges throughout the square

$$|x - x_0| < \tfrac{1}{2}R, \qquad |y - y_0| < \tfrac{1}{2}R.$$

The present note shows that this is in reality only a portion of the region of convergence.

of the developments of these functions about the point $x=0$. If we denote by K the smaller of these two quantities, we may write

$$\left.\begin{aligned} f_0(x) &= a_0 + a_1 x + a_2 x^2 + \cdots \\ f_1(x) &= b_0 + b_1 x + b_2 x^2 + \cdots \end{aligned}\right\} \qquad (|x|<K).$$

Let us now try to find in the neighborhood of the origin a solution of (1) which satisfies the auxiliary conditions

$$(2) \qquad u\Big]_{y=0} = f_0(x), \qquad \frac{\partial u}{\partial y}\Big]_{y=0} = f_1(x).$$

Suppose that u is written as a double power-series in (x, y) with undetermined coefficients, and let us try to determine these coefficients in such a way that (1) and (2) are satisfied. If we arrange the double power-series according to powers of y, the first two terms will be uniquely determined by conditions (2), and we may write

$$(3) \qquad u(x, y) = f_0(x) + f_1(x)y + f_2(x)y^2 + \cdots$$

where $f_2(x)$, $f_3(x)$, $\cdots$ are power-series in x with undetermined coefficients. Substituting this series in (1), we readily find the formulæ

$$\left.\begin{aligned} f_{2n}(x) &= \frac{(-1)^n}{(2n)!} f_0^{[2n]}(x) \\ f_{2n+1}(x) &= \frac{(-1)^n}{(2n+1)!} f_1^{[2n]}(x) \end{aligned}\right\} \qquad (n=1, 2, 3, \cdots).$$

We thus see that there is only one double power-series in (x, y) which even formally satisfies (1) and (2). This series we will call S. When rearranged according to powers of y, the series S becomes*

$$(4) \quad f_0(x) + f_1(x)y - \frac{f_0''(x)}{2!}y^2 - \frac{f_1''(x)}{3!}y^3 + \frac{f_0^{\mathrm{IV}}(x)}{4!}y^4 + \frac{f_1^{\mathrm{IV}}(x)}{5!}y^5 - \frac{f_0^{\mathrm{VI}}(x)}{6!}y^6 - \cdots.$$

If we write

$$\left.\begin{aligned} F_0(x) &= |a_0| + |a_1|x + |a_2|x^2 + \cdots \\ F_1(x) &= |b_0| + |b_1|x + |b_2|x^2 + \cdots \end{aligned}\right\} \qquad (|x|<K),$$

we readily see that a necessary and sufficient condition for the convergence of S at a point (x, y) neither of whose coördinates is negative is the convergence of the two series

$$(5) \qquad F_0(x) + \frac{F_0''(x)}{2!}y^2 + \frac{F_0^{\mathrm{IV}}(x)}{4!}y^4 + \cdots,$$

$$(6) \qquad F_1(x)y + \frac{F_1''(x)}{3!}y^3 + \frac{F_1^{\mathrm{IV}}(x)}{5!}y^5 + \cdots.$$

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* The regions of convergence of S and (4) are, in general, different. Wherever S converges, (4) converges, but the converse is not necessarily true.

Let us also write

$$\Phi(x) = \int_0^x F_1(x)\,dx.$$

Since $F_0(x)$ and $\Phi(x)$ are analytic in the complex x-plane throughout the circle $|x| < K$, we see by the Cauchy-Taylor theorem that the developments

$$F_0(x+y) = F_0(x) + F_0'(x)y + \frac{F_0''(x)}{2!}y^2 + \cdots,$$

$$\Phi(x+y) = \Phi(x) + F_1(x)y + \frac{F_1'(x)}{2!}y^2 + \cdots,$$

are valid when x and y are real and

$$0 \leqq x < K, \qquad 0 \leqq y < K - x.$$

Consequently, for these same values of (x, y) the series (5) and (6), and accordingly also the double series S, converge. From the well known property that a double power-series has a region of convergence which, except possibly for boundary points, is symmetrical with regard to the coördinate axes, we now infer that the series S converges throughout the square

$$|x| + |y| < K.$$

In order to see whether we have thus determined the complete region of convergence of S, let us first confine our attention to positive (not zero) values of x and y. Let (x_0, y_0) be any point whose coördinates are both positive and such that $x_0 + y_0 > K$. We wish to prove that S does not converge at (x_0, y_0). For this purpose we consider two cases:

(*a*) $x_0 < K$. We make use here of a theorem given by HADAMARD,* which says:

If a power-series in x with real non-negative coefficients has a radius of convergence $R > 0$, the analytic function represented by this series has a singular point at $x = R$.

Applying this theorem to the developments of $F_0(x)$ and $\Phi(x)$ about the point $x = 0$, we see that these functions have singular points at $x = K_0$ and $x = K_1$ respectively. Considering first the function F_0, we see that $F_0(x_0 + y)$ has a singular point at the point $y = K_0 - x_0$. Since $F_0(x_0 - y)$ is analytic at this point, the function

$$\tfrac{1}{2}[F_0(x_0 + y) + F_0(x_0 - y)]$$

has a singular point at $y = K_0 - x_0$. The development of this function according to positive integral powers of y cannot then converge when $y > K_0 - x_0$.

* *La Série de Taylor*, p. 20–21. The restriction there made to *positive* coefficients is not necessary as an examination of the proof shows.

This development is precisely what the series (5) becomes when $x = x_0$. Consequently (5), and hence also S, does not converge when $x = x_0$, $y > K_0 - x_0$.

Similar reasoning applied to the function

$$\tfrac{1}{2}[\Phi(x_0 + y) - \Phi(x_0 - y)]$$

shows that (6), and consequently S, does not converge when $x=x_0$, $y > K_1 - x_0$. Since y_0 satisfies at least one of the inequalities

$$y_0 > K_0 - x_0, \qquad y_0 > K_1 - x_0,$$

we thus see that S does not converge at (x_0, y_0).

(*b*) $x_0 \geqq K$. If S converged at (x_0, y_0), it would converge throughout the rectangle

$$0 < x < x_0, \qquad 0 < y < y_0.$$

Within this rectangle we could obviously choose a point (x_0', y_0') such that

$$0 < x_0' < K, \qquad 0 < y_0', \qquad x_0' + y_0' > K.$$

The convergence of S at this point is in contradiction with what we have proved under (*a*).

We have thus proved that if

$$x_0 > 0, \qquad y_0 > 0, \qquad x_0 + y_0 > K,$$

the series S does not converge at (x_0, y_0). From this we infer, on account of the symmetry of the region of convergence of a double power-series, that if neither x_0 nor y_0 is zero and

$$|x_0| + |y_0| > K,$$

the series S does not converge at (x_0, y_0). Thus we see that the complete region of convergence of S is the interior of the square

$$|x| + |y| < K$$

with possibly some (or all) points on its boundary, and possibly some (or all) points outside of it on the two coördinate axes. Since on the coördinate axes the double power-series reduces to a simple power-series, it is clear that the interval of convergence on each coördinate axis extends out the same distance from the origin in the two opposite directions, and that beyond the points thus reached we have divergence. At each one of the four points which terminate the intervals of convergence on the coördinate axes we may of course have convergence or divergence.

Before going on to other matters, we will note that the results so far obtained hold without change for the differential equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0,$$

practically no modification being required in the proofs. In this case the region of convergence of the double power-series is bounded by characteristics of the differential equation.

2. The series of homogeneous polynomials. Let us now rearrange the double series S into a simple series P whose terms are homogeneous polynomials in (x, y) of ascending degrees. It is clear that the terms of P are themselves solutions of Laplace's equation, and since it is well known, and easily proved, that every homogeneous polynomial of the nth degree in (x, y) which satisfies Laplace's equation may be written in the form

$$(A \cos n\theta + B \sin n\theta) r^n,$$

where

$$r = \sqrt{x^2 + y^2}, \qquad \tan \theta = \frac{y}{x},$$

it follows that the series P may be written in the form *

$$P \qquad u(x, y) = \sum_{n=0}^{\infty} (A_n \cos n\theta + B_n \sin n\theta) r^n.$$

Since this series has been obtained from S by a rearrangement of the terms and an insertion of parentheses, we see that P converges wherever S converges. We shall presently find that it also converges elsewhere.

For this purpose we first establish the

Lemma. If when $r = r_0$ the series P converges for a continuous range of values of θ, however short, it converges absolutely for all values of θ when $r < r_0$, and if r_0' is any positive constant less than r_0, it converges uniformly in (x, y) when $r < r_0'$.

Proof. From the convergence of the trigonometric series P when $r = r_0$ for a continuous range of values of θ, we infer by a well known theorem of G. CANTOR † that the coefficients

$$A_n r_0^n, \qquad B_n r_0^n$$

approach zero as n becomes infinite. Let us denote by M the maximum of the absolute values of these coefficients. The terms of P are then readily seen to be in absolute value not greater than the corresponding terms of the series

$$\sum_{n=0}^{\infty} 2M \left(\frac{r}{r_0}\right)^n,$$

and since, when $r < r_0$, this series converges, it follows that P converges absolutely when $r < r_0$.

* The coefficients here may readily be determined to have the values

$$A_n = a_n, \qquad B_n = \frac{b_{n-1}}{n}.$$

† Mathematische Annalen, vol. 4 (1871), p. 139.

An obvious modification of this proof shows that P converges uniformly when $r < r_0'$.

From the lemma just proved we infer at once that throughout the circle $r < K$ the series P converges absolutely, and that throughout any smaller circle which has the origin as center it converges uniformly; * for if r_0 is any positive constant less than K, we can always find on the circle $r = r_0$ an arc lying within the square $|x| + |y| < K$, and on this arc the series S, and therefore also P, converge.

If the series P converged throughout a continuum lying outside of the circle $r < K$, we see, by an application of our lemma, that it would converge uniformly throughout a circle of radius $K' > K$. By a theorem of HARNACK according to which a series of harmonic functions which is uniformly convergent throughout a two dimensional continuum can be differentiated term by term, $\partial u / \partial y$ could then be obtained when $r < K'$ by differentiating P term by term. If in this series, and also in P itself, we let $y = 0$, we should get two power-series in x which are convergent when $|x| < K'$, and which, when $|x| < K$, have the values $f_1(x)$ and $f_0(x)$ respectively. They have therefore by hypothesis the radii of convergence K_1 and K_0 respectively, at least one of which is less than K'. Thus the assumption that P converges throughout a continuum outside of the circle $r = K$ leads to a contradiction.

If on each ray which radiates from the origin O we lay off a distance OQ, either finite or infinite, the totality of all the points on the segments OQ, excluding the points Q themselves, may be called a *star* with O as center. The points Q may be called the vertices of this star. It is then readily seen that the complete region of convergence of any series whose terms are homogeneous polynomials in (x, y) is a star with O as center together with perhaps some or all of its vertices. For on any particular ray (except on the rays $x = 0$) we may write $y = \lambda x$, where λ is a constant, and when this value of y is substituted, the series becomes a power-series in x. This reasoning in fact shows that the star in question is symmetrical with regard to the origin. Except possibly at the vertices of the star, the convergence will be absolute.

Turning now to the series P above considered, we see that all the rays of its star will reach out at least as far as the circumference of the circle $r = K$, so that this circle forms, so to speak, a solid nucleus of the star. Outside of it the rays do not completely fill any sector however narrow and short. Simple examples † show that there may be even an infinite number of these external rays

* These facts can readily be deduced without making use of CANTOR's Theorem by reference to the values of A_n and B_n given in the first footnote in this section.

† For instance, the series

$$\sum_{n=1}^{\infty} r^{n!} \sin (n! \theta)$$

obviously converges when $r < 1$. By CANTOR's Theorem it does not converge for any continuissuing from every arc, however short, of the circle; and that some or all of these rays may even extend to infinity.

A question of fundamental importance, at which we have not yet looked, is whether the series S and P represent harmonic functions throughout the whole continuum where they converge. This question is of course to be answered in the affirmative in the case of S, since a power series which converges throughout a continuum can always be differentiated there as often as we please, term by term. In the case of P it is also to be answered in the affirmative, since by HARNACK's Theorem a series of harmonic functions uniformly convergent throughout a two dimensional continuum represents a function harmonic throughout this continuum.

If S converges on rays which project from the extremities of the horizontal diagonal of the square $|x| + |y| < K$, the function represented by S need not be the analytic extension of the harmonic function within the square, since the function $f_1(x)$ may have a singular point at the point $x = +K$ or $x = -K$, and this function is equal to the value of $\partial u / \partial y$ on the horizontal diagonal of the square. Similar reasoning applies to the vertical diagonal. If, however, u admits of analytic extension along a two dimensional strip surrounding any one of these four rays, this analytic extension, so far as its values on the ray itself are concerned, will clearly be given by the series S if this series converges on this ray.

Since at points on the sides of the square, which are not vertices, u is harmonic and is represented by the series P, it is evident that if S converges at any such point it represents u there.

Precisely similar results hold with regard to the series P and the rays which project from the circle $r = K$, as we see by turning the coördinate axes.

Turning finally to another question, we note that if the harmonic function u is extended analytically so far as possible, and if R denotes the distance from the origin to the nearest singular point of u, it follows readily either by the use of Poisson's Integral (cf. PICARD, loc. cit.) or by an application of the Cauchy-Taylor development of the analytic function of which u is the real part, that u can be developed into a series of the form P which converges throughout the circle $r < R$. Since two developments of the form P for one and the same function are impossible,* it follows that P itself converges when $r < R$. It can

ous range of values of θ when $r = 1$. When θ is any rational multiple of π, all the terms in the series after a certain point are zero, and hence the series, for these values of θ everywhere dense, converges for all values of r.

* We use here the easily established theorem: If the series

$$\phi_0 + \phi_1 + \phi_2 + \cdots,$$

where ϕ_n is a homogeneous polynomial of degree n in (x, y), converges to the value zero at all points of a sector

$$\theta_1 \leqq \theta \leqq \theta_2, \qquad 0 \leqq r \leqq \rho,$$

obviously converge in no larger circle, as otherwise u would be harmonic throughout this larger circle. Consequently we have $K = R$.

3. Recapitulation of Results. We are now in a position to recapitulate our main results as follows:

If $u(x, y)$ is a function harmonic throughout the neighborhood of the point (x_0, y_0), and if, when this function is continued analytically,* the distance from (x_0, y_0) to the nearest singular point of u which lies in the same sheet of the Riemann's Surface generated by the analytic continuation in which (x_0, y_0) lies is K, then

(a) The Taylor's development S of $u(x, y)$ about (x_0, y_0) converges throughout the square

$$\Sigma \qquad |x - x_0| + |y - y_0| < K;$$

(b)† The development of $u(x, y)$ in a series P proceeding according to homogeneous polynomials of ascending degrees in $x - x_0$ and $y - y_0$ converges and represents u throughout the circle

$$\Pi \qquad \sqrt{(x - x_0)^2 + (y - y_0)^2} < K;$$

(c) The series S converges throughout no continuum which does not lie in Σ, and P converges throughout no continuum which does not lie in Π;

(d) The only points outside of Σ where S can converge are, first, points on the boundary of Σ, and at these it represents u (except at the vertices of Σ in case u is not defined there); and, secondly, points on the lines $x = x_0$ and $y = y_0$, at which it may or may not represent u.

(e) The complete region of convergence of P is a star having (x_0, y_0) as center, and perhaps some or all of its vertices. At the points of this region outside of Π the series P may or may not represent u. It will represent it at every point of the region to which a rectilinear analytic extension from (x_0, y_0) is possible.

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where $\theta_2 > \theta_1$, $\rho > 0$, then all the polynomials ϕ vanish identically. This is at once obvious if we write $y = \lambda x$ and thus express the series as a power-series in x whose coefficients are polynomials in λ.

* In the following statements u is supposed to be defined only at the points reached by this analytic continuation. Thus the region of definition of u is a continuum whose boundary points are the singular points of u.

† The fact here stated is, of course, well known, and is restated here only for the sake of completeness.