DIFFERENTIAL EQUATIONS AND IMPLICIT FUNCTIONS IN INFINITELY MANY VARIABLES*

BY

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The main purpose of the present paper is threefold. In the first place, there will be developed certain theorems concerning a type of real valued functions of infinitely many real variables. There will then be considered the problem of infinite systems of ordinary differential equations,

$$(1) \qquad \frac{dx_i}{dt} = f_i(x_1, x_2, \cdots; t) \qquad (i = 1, 2, \cdots),$$

in which the f_i are functions of the type treated in the initial theorems. Then, in the third part of the paper, the fundamental problem of implicit function theory in this field for a system of equations

$$(2) \qquad f_i(x_1, x_2, \cdots; y_1, y_2, \cdots) = 0 \qquad (i = 1, 2, \cdots)$$

will be discussed. The results of all three sections of the paper include as special cases the corresponding theorems on functions of a finite number of variables.

The notion of a function f of infinitely many variables $(x_1, x_2, \cdots)$ was brought into mathematics in the wake of the infinite determinant, the consideration of infinite systems of linear equations and the researches of Hilbert, Hellinger, Toeplitz, Westfall, and others on bilinear and linear forms in infinitely many variables in connection with the study of integral equations. In the present paper, the region of points $\xi = (x_1, x_2, \cdots)$, in which the functions considered will be supposed defined, will be similar to the generalized parallelepipedon

$$(3) \qquad \mathbf{R} : |x_i - a_i| \leqq r_i \qquad (i = 1, 2, \cdots).$$

Such a region is not necessarily a part of a Hilbert space, that is, a space in which $\sum_{i=1}^{\infty} |x_i^2|$ converges, such as has been used in many investigations in this field.

In Part I of the present discussion there is defined the notion of *complete continuity* for a function $f(\xi)$ at a point ξ_0 of a region $\mathbf{R}$. Complete continuity

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reduces to ordinary continuity in case the f depends on only a finite number of variables, and, as will be noted later, is similar to current definitions of continuity for functions of an infinite number of variables. The theorems on completely continuous functions, derived in Part I, include as special cases the analogous classical theorems on continuous functions of a finite number of variables.

A general study of functions of infinitely many variables has been made by M. J. Le Roux.* In addition to certain theorems on what he terms *convergent* functions of infinitely many variables, Le Roux obtained an analog of the Taylor series expansion for functions $f(\xi)$ representable in the special form

$$f(\xi) = \sum_{m=1}^{\infty} f_m(x_1, x_2, \cdots, x_m) \qquad (|x_i - a_i| \leqq r_i), \tag{4}$$

where the sum is uniformly convergent and where f_m is analytic in the variables $(x_1, \cdots, x_m)$, regular at the point $x_i = a_i$. In addition to the fact that Le Roux deals with complex function theory whereas the present paper deals with reals, there remains a difference between the two investigations due to the fact that, as Le Roux noted, *convergent* and *continuous* functions are not necessarily identical; a continuous function may or may not be a convergent function, and conversely. Functions of the type (4) are completely continuous in their arguments $(x_1, x_2, \cdots)$.

The system of differential equations considered in Part II is of the form (1). The problem of infinite systems of differential equations was first treated by H. von Koch.† He considered a system of the analytic type, and, under certain dominance hypotheses, established the existence of a solution $[x_1(t), x_2(t), \cdots]$ where all the components $x_i(t)$ were analytic functions of the variable t. A very similar problem has been treated by F. R. Moulton.‡ The problem of an infinite system of linear differential equations was considered by E. H. Moore§ as a special case of a more general investigation made in the sense of Moore's General Analysis.||

In Part II the f_i of (1) are assumed to be completely continuous functions satisfying certain supplementary hypotheses which are, for the most part,

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* Nouvelles Annales de Mathématiques, vol. 4, ser. 4 (1900), pp. 448–458.

† Ofversigt af Kongliga Vetenskaps Akademiens Förhandlingar, vol. 56 (1899), pp. 395–411.

‡ Proceedings of the National Academy of Sciences, vol. I, pp. 350–354.

§ *Atti dei IV Congresso Internazionale dei Matematici* (Roma, 6–11 Aprile, 1908), vol. II, page 98.

|| Compare E. H. Moore, *Introduction to a Form of General Analysis*, New Haven Mathematical Colloquium. In the future, this paper will be referred to as Moore's Paper I.

of a functional character. The unique existence of a continuous solution is established by a method analogous to the Picard scheme of successive approximation for the solution of a finite number of differential equations. The system treated by E. H. Moore does not come under the type considered in Part II. It will be proved, however, that the results of F. R. Moulton, when restricted to reals, are a special case of the theorems of this paper. It follows from this fact that the *unique analytic* solution of Moulton, when restricted to real values of the argument t, is the *only continuous* solution of the system which he treats.

In Part III, a system of equations of form (2) is treated in which the f_i are completely continuous functions satisfying certain additional hypotheses. The analog of the fundamental existence theorem of classical implicit function theory is obtained by a method of successive approximations. The form of this method is, in certain respects, similar to that used by Goursat* in the classical case of a finite number of equations.

Infinite systems of implicit functions have been considered by H. von Koch† and R. d'Adhemar.‡ Von Koch considered a system of the analytic type, defined in the field of complex numbers, and established the existence of an analytic solution. His work, however, is valid only if the sum of the numbers r_i of the region (3) in which his system is defined converges in a special manner. This limitation is not mentioned in von Koch's work. R. d'Adhemar treated a special system which arose in a problem he considered in integral equation theory. The results of von Koch, when restricted to reals, are a particular case of the theorems of the present paper.

In the sequel, it will be convenient to use the following notations, some of which are due to E. H. Moore (cf. Moore, I).

Suppose that p is a variable belonging to a given range $\mathbf{P}$ and that $\mu(p)$ is a real valued function of the argument p on the range $\mathbf{P}$, for short, on $\mathbf{P}$. Then, the statement that a certain numerical condition (i. e., a condition defined for numbers) is satisfied by μ will mean that, for every value of the argument p, $\mu(p)$ satisfies this condition. For example, $\mu < 1$ means that for every p of $\mathbf{P}$, there is the inequality $\mu(p) < 1$.

Let η_m $(m = 1, 2, \cdots)$ be a sequence of functions on $\mathbf{P}$ for which $\lim_{m=\infty} \eta_m(p) = \eta(p)$, uniformly for all values p of $\mathbf{P}$. Such a convergence will be indicated by writing (cf. Moore, I, p. 5)

$$\lim_{m=\infty} \eta_m = \eta \qquad (\mathbf{P}).$$

*Bulletin de la Société Mathématique de France, vol. 31 (1903), p. 184.

†Bulletin de la Société Mathématique de France, vol. 27 (1899), pp. 215–227.

‡Bulletin de la Société Mathématique de France, vol. 36 (1908), pp. 195–204.

Suppose, for example, that the range $\mathbf{P}$ is the set of integers $\mathbf{I} = (1, 2, \cdots)$. In the future a point ξ in a space of infinitely many dimensions will be thought of as a function on $\mathbf{I}$; that is, $\xi(i) = x_i$. Then, in agreement with the conventions explained above, the equation

$$(5) \qquad \lim_{m=\infty} \xi_m = \xi \qquad (\mathbf{I}),$$

means that the convergence of the coördinates x_{im} of ξ_m to the coördinates x_i of ξ is uniform with respect to i. If the notation $(\mathbf{I})$ were omitted in (5), this equation would merely state the convergence, for every i, of the coördinates x_{im} to the value x_i.

In the succeeding pages, if a point ξ belongs to a region $\mathbf{R}_0$ in a space of infinitely many dimensions, it will be indicated by writing* $\xi^{\mathbf{R}_0}$. Moreover, in this paper, Greek letters will always represent functions, one of whose arguments has the range $\mathbf{I}$. A Greek functional symbol with an added notation will represent the function whose ith component is the ith component, derived from the original symbol, with the given notation added. Thus, for example,

$$\xi' = (x_1', x_2', \cdots),$$

$$\xi_k = (x_{1k}, x_{2k}, \cdots),$$

$$\xi(t) = [x_1(t), x_2(t), \cdots].$$

It will be said that an infinite series converges *absolutely-uniformly* if the series formed by replacing each term by its absolute value converges uniformly.

PART I

General theorems on completely continuous functions

1. **Introduction.** In the present part of the paper, it is proposed to study functions of the type $f(\xi)$, where ξ lies in the region $\mathbf{R}$ defined by (3) and where f is *completely continuous* according to a definition to be given in § 3. Generalizations to infinitely many variables are obtained of some of the fundamental properties possessed by continuous functions of a finite number of variables. Among the more important theorems derived mention may be made of extensions of Weierstrass's theorem on uniformly convergent sequences of continuous functions, and of Taylor's Theorem with an integral form for the remainder term.

* This notation is a special case of a general convention adopted by Moore (cf. I, p. 16). If P represents a property of functions, then in case a function μ has the property P, it is denoted by writing μ^P; and, if $\mathfrak{M}$ is a class of functions, $\mathfrak{M}^P$ means that every function of $\mathfrak{M}$ has the property P.

2. **The fundamental lemma for points of R.** In the current theory of functions of a finite number of variables the Weierstrass condensation theorem, or some theorem of like content, is of fundamental importance. There is need of a similar result for the region $\mathbf{R}$ and it is met by the following

LEMMA 1. *Let* $S = (\xi_n; n = 1, 2, \cdots)$ *be an infinite sequence of points of* $\mathbf{R}$. *Then there exists a point* ξ' *of* $\mathbf{R}$ *and an infinite sub-sequence* $S' = (\xi'_n; n = 1, 2, \cdots)$ *of* S *such that*

(6) $$\lim_{n=\infty} \xi'_n = \xi'.$$

The lemma is exactly equivalent to a theorem proved [cf. Bolza, *Vorlesungen über Variationsrechnung*, p. 423, part (*b*)] as a lemma for the Hilbert proof of the existence of an absolute minimum in certain types of problems in the Calculus of Variations. The lemma is due originally to Hilbert. There is a theorem of the same nature* when points ξ are considered which satisfy $\sum_{i=1}^{\infty} |x_i^2| \leqq M$.

In the future, such a point as ξ' of the lemma will be called a limit point for the sequence S.

3. **Definition of complete continuity and fundamental properties of completely continuous functions.** DEFINITION 1. *A function* f, *defined in the region* $\mathbf{R}$, *is completely continuous at the point* ξ_0 *of* $\mathbf{R}$—*in notation,* $C_1(\xi_0)$—*if, whenever*

$$\lim_{n=\infty} \xi_n = \xi_0 \qquad (\xi_n^{\mathbf{R}}),$$

it follows that

$$\lim_{n=\infty} f(\xi_n) = f(\xi_0).$$

This concept of complete continuity is the same as that of *Vollstetigkeit* which has been much used by Hilbert and his followers. However, in the present paper, the functions satisfying Definition 1 are supposed defined in a region of points very different from that used by Hilbert. Moreover, entirely different applications of the concept are made in this paper.

If f is $C_1(\xi_0)$ for every $\xi_0^{\mathbf{R}}$, it will be said that f is completely continuous in $\mathbf{R}$—in notation,

(7) $$f \text{ is } C_1(\mathbf{R}).$$

Suppose the given function f depends on a finite or a denumerably infinite number of variables lying in the same or different spaces of infinitely many dimensions, as

$$f(\xi^{(1)}, \xi^{(2)}, \cdots, \xi^{(k)}) \qquad (k \text{ finite or infinite}),$$

where $\xi^{(i)}$ belongs to a certain region $\mathbf{R}^{(i)}$ of the same type as $\mathbf{R}$. Then it will be said that f is completely continuous simultaneously in its arguments

* Cf. Riesz, *Équations linéaires*, p. 57.

if the conditions

$$\lim_{n=\infty} \xi_n^{(i)} = \xi^{(i)} \qquad (i = 1, 2, \cdots, k),$$

imply the convergence

$$\lim_{n=\infty} f(\xi_n^{(1)}, \cdots, \xi_n^{(k)}) = f(\xi^{(1)}, \cdots, \xi^{(k)}).$$

Such a continuity property is seen to be equivalent to complete continuity in the single variable η in the space $\mathbf{Q}$, where the coördinates are those represented in the array (doubly or one way infinite according as k is infinite or finite),

$$\begin{array}{cccccc} x_1^{(1)}, & x_2^{(1)}, & \cdots, & x_n^{(1)}, & \cdots, \\ \cdots & \cdots & & \cdots & \\ x_1^{(k)}, & x_2^{(k)}, & \cdots, & x_n^{(k)}, & \cdots. \end{array}$$

The following obvious proposition relates this complete continuity to another generalization of the ordinary definition of continuity for a function of a finite number of variables.

Proposition 1. *If f is $C_1(\xi_0)$, then, whenever**

(8) $$\lim_{n=\infty} \xi_n = \xi_0 \qquad (\mathbf{I}),$$

where $\xi_n^{\mathbf{R}}$, it follows that

(9) $$\lim_{n=\infty} f(\xi_n) = f(\xi_0).$$

In general, if a function f is such that (8) implies (9), it will be said that f is $C_0(\xi_0)$, and a notation similar to (7) will indicate that this property holds for all points of $\mathbf{R}$.

Proposition 2. *If f is $C_1(\xi_0)$, it follows that for every function β on* $\mathbf{I}$ *and for every $e > 0$, there exists a number $d_e > 0$ such that, for all $\xi^{\mathbf{R}}$ satisfying*

(10) $$|\xi - \xi_0| \leqq d_e|\beta|$$

there is the relation

(11) $$|f(\xi) - f(\xi_0)| \leqq e.$$

And, moreover, if for $\beta = 1$ such a condition exists for every $e > 0$, it follows that f is $C_0(\xi_0)$.

The two statements of Proposition 2 are proved by the same method as is used in establishing the equivalence of the *limit* and the (ϵ, δ) definitions of continuity for functions of a finite number of variables. In the future, "*f is $C_0(\xi_0)$*" will refer usually to the (e, d_e), form of statement in Proposition 2 with $\beta = 1$.

The Definition 1 has no characteristics permitting of a useful definition of *uniform* complete continuity, but uniformity is obtained for the C_0 property in the following

* Cf. (5).

PROPOSITION 3. *If f is $C_1(\mathbf{R})$, then f is $C_0(\xi)$, uniformly for all $\xi^{\mathbf{R}}$; that is, to every $e > 0$ there corresponds a number $d_e > 0$, such that if*

$$|\xi_1 - \xi_2| \leqq d_e \qquad (\xi_1^{\mathbf{R}}, \xi_2^{\mathbf{R}}),$$

then it follows that

$$|f(\xi_1) - f(\xi_2)| \leqq e.$$

The proof of this proposition is made by an indirect argument. Suppose that the statement as to the uniformity is not true. Then there exists a certain number $e_0 > 0$ which has the property that, corresponding to the sequence (d_n; $n = 1, 2, \cdots$) for which $\lim_{n=\infty} d_n = 0$, there exists a sequenec S of pairs (η_n', η_n'') of points of $\mathbf{R}$ such that

$$(12) \qquad |\eta_n' - \eta_n''| \leqq d_n,$$

$$(13) \qquad |f(\eta_n') - f(\eta_n'')| > e_0.$$

The fundamental lemma for the sequence (η_n'), in connection with (12) shows that there exists a point γ of $\mathbf{R}$ and an infinite sub-sequence of pairs of points (ζ_n', ζ_n''; $n = 1, 2, \cdots$) belonging to S, such that

$$(14) \qquad \lim_{n=\infty} \zeta_n' = \lim_{n=\infty} \zeta_n'' = \gamma.$$

From (14) it follows that

$$(15) \qquad \lim_{n=\infty} f(\zeta_n') = \lim_{n=\infty} f(\zeta_n'') = f(\gamma).$$

But, since (15) contradicts (13), it is seen that Proposition 3 is true.

Certain useful theorems regarding completely continuous functions will now be proved which are analogous to the theorem that a continuous function of a continuous function is a continuous function.

THEOREM I. *Suppose f is $C_1(\mathbf{R})$. Let η be a point of a region $\mathbf{R}'$ of type* (3). *Then, if for every $\eta^{\mathbf{R}'}$ the function $\gamma(\eta)$ belongs to $\mathbf{R}$ while, for a certain η_0 of $\mathbf{R}'$, all coördinates of $\gamma(\eta)$ are $C_1(\eta_0)$, it follows that $f[\gamma(\eta)]$ is $C_1(\eta_0)$.*

This theorem is established by virtue of the fact that, if $\lim_{n=\infty} \eta_n = \eta_0$, then

$$(16) \qquad \lim_{n=\infty} \gamma(\eta_n) = \gamma(\eta_0);$$

as a consequence of (16), it follows that

$$(17) \qquad \lim_{n=\infty} f[\gamma(\eta_n)] = f[\gamma(\eta_0)].$$

It is obvious that (17) states the complete continuity at η_0 of $f[\gamma(\eta)]$.

THEOREM II. *Suppose f is $C_1(\mathbf{R})$. Let $\zeta = (x_2, x_3, \cdots)$ and suppose that* $\lim_{n=\infty} \zeta_n = \zeta_0$, *where the coördinates of ζ_n and of ζ_0 satisfy* (3). *Then*

$$\lim_{n=\infty} f(x_1, x_{2n}, x_{3n}, \cdots) = f(x_1, x_{20}, x_{30}, \cdots)$$

uniformly for all values of x_1 *on the interval*

$$|x_1 - a_1| \leqq r_1. \tag{18}$$

The proof of this theorem is accomplished by a direct method. Because of Proposition 3, it is seen that, for every $e > 0$ a number $d_e > 0$ can be found so that, if $|\xi' - \xi''| \leqq d_e$, then

$$|f(\xi') - f(\xi'')| \leqq \tfrac{1}{3}e. \tag{19}$$

Now divide the interval (18) into a finite number of parts of length at most d_e by the division points $(x_{1j};\ j = 1, 2, \cdots, k_e)$. Since f is $C_1(\mathbf{R})$, it follows that an integer n_e can be chosen so that for $(n \geqq n_e;\ j = 1, 2, \cdots, k_e)$

$$|f(x_{1j}, x_{2n}, x_{3n}, \cdots) - f(x_{1j}, x_{20}, x_{30}, \cdots)| \leqq \tfrac{1}{3}e. \tag{20}$$

Consider a value x_1 such that $x_{1j} \leqq x_1 \leqq x_{1\,(j+1)}$. Then from (19) there is obtained

$$\begin{cases} |f(x_{1j}, x_{2n}, \cdots) - f(x_1, x_{2n}, \cdots)| \leqq \tfrac{1}{3}e & (n = 1, 2, \cdots), \\ |f(x_{1j}, x_{20}, x_{30}, \cdots) - f(x_1, x_{20}, x_{30}, \cdots)| \leqq \tfrac{1}{3}e. \end{cases} \tag{21}$$

From (20) and (21) it is seen that

$$|f(x_1, x_{2n}, x_{3n}, \cdots) - f(x_1, x_{20}, x_{30}, \cdots)| \leqq e \qquad (n \geqq n_e). \tag{22}$$

Since the n_e in (22) was chosen independently of the value of j, it follows that the theorem is completely established.

In the following theorem there is given a result which includes as a particular case the Weierstrass theorem on uniformly convergent sequences of continuous functions.

THEOREM III. *If* f_n $(n = 1, 2, \cdots)$ *are* $C_1(\xi_0)$ *while*

$$\lim_{n=\infty} f_n(\xi) = f(\xi)$$

uniformly for $\xi^{\mathbf{R}}$, *it follows that* f *is* $C_1(\xi_0)$.

Let it be supposed that $\lim_{m=\infty} \xi_m = \xi_0$. Then for every $e > 0$ there can be found integers n_e and m_e such that if $\xi^{\mathbf{R}}$

$$|f_{n_e}(\xi) - f(\xi)| \leqq \tfrac{1}{3}e, \tag{23}$$

$$|f_{n_e}(\xi_m) - f_{n_e}(\xi_0)| \leqq \tfrac{1}{3}e \qquad (m \geqq m_e). \tag{24}$$

On taking (23) for $\xi = \xi_0$ and $\xi = \xi_m$ $(m \geqq m_e)$ in connection with (24), it follows that

$$|f(\xi_0) - f(\xi_m)| \leqq e \qquad (m \geqq m_e),$$

which, in accordance with Definition 1, shows that f is $C_1(\xi_0)$.

In the succeeding theorem it will be shown that completely continuous functions have upper and lower bounds which they attain.

THEOREM IV. *If f is $C_1(\mathbf{R})$, then there exist finite numbers m and M, and two points $(\xi_1^{\mathbf{R}})$, $(\xi_2^{\mathbf{R}})$, such that*

$$m \leqq f(\xi) \leqq M \qquad (\xi^{\mathbf{R}}), \tag{25}$$

$$m = f(\xi_1), \tag{26}$$

$$M = f(\xi_2). \tag{27}$$

Let the proof of the second inequality of (25) be considered. Suppose there exists no quantity M satisfying (25). Then to the sequence of integers $(n = 1, 2, \cdots)$ there corresponds a sequence of points $\eta_n^{\mathbf{R}}$ such that

$$f(\eta_n) > n. \tag{28}$$

Let η' be a limit point of this sequence; then $f(\eta') = l$. But then, since f is $C_1(R)$, $\lim_{n=\infty} f(\eta_n) = l$, which contradicts (28) for n sufficiently large. Hence there exists an M satisfying (25) and, in a similar fashion, it can be shown that an m exists.

Let M be the least upper bound of f for points in $\mathbf{R}$. Then it follows that there is a sequence of points γ_n $(n = 1, 2, \cdots)$ of $\mathbf{R}$ for which

$$\lim_{n=\infty} f(\gamma_n) = M.$$

Thus it is easily seen that every limit point γ' of this sequence is effective as the ξ_2 in (27). On taking m as the greatest lower bound of f it can be established in the same way that a point ξ_1 exists satisfying (26).

4. **The mean-value theorem for $f(\xi)$.** The object of the present section is to derive a mean-value theorem for functions $f(\xi)$ of a certain type. This result will then be used in § 5 in deducing for a corresponding type of functions f an expansion consisting of n terms, each of which is an infinite sum, with a remainder term which is the sum of an infinite number of integrals. In view of the fact that, in case $f(\xi)$ depends on only a finite number of variables, this expansion reduces to Taylor's Formula with n terms, the theorem of § 5 will be called Taylor's Theorem for the function $f(\xi)$.

THEOREM V. *Suppose that the function f and the partial derivatives $\partial f/\partial x_i$ $(i = 1, 2, \cdots)$ are $C_1(\mathbf{R})$, and that*

$$\sum_{j=1}^{\infty} r_j \left| \frac{\partial f(\eta)}{\partial x_j} \right| \tag{29}$$

converges, uniformly for all $\eta^{\mathbf{R}}$. Then, for every pair of points $(\xi^{\mathbf{R}}, \xi'^{\mathbf{R}})$,

$$f(\xi') - f(\xi) = \sum_{j=1}^{\infty} (x_j' - x_j) \int_0^1 \frac{\partial f[\xi + u(\xi' - \xi)]}{\partial x_j} du. \tag{30}$$

For convenience, let the following notations be adopted:

$$\zeta_0 = \xi', \qquad \zeta_n = (x_1', x_2', \cdots, x_n', x_{n+1}, \cdots) \quad (n = 1, 2, \cdots), \tag{31}$$

$$\eta(n, u) = \xi + u(\zeta_n - \xi) \qquad (n = 0, 1, 2, \cdots), \tag{32}$$

$$S_n = \sum_{j=1}^{n} (x_j' - x_j) \int_0^1 \frac{\partial f[\eta(0, u)]}{\partial x_j} du, \qquad \lim_{n=\infty} S_n = S, \tag{33}$$

$$s_n = \sum_{j=1}^{n} (x_j' - x_j) \int_0^1 \frac{\partial f[\eta(n, u)]}{\partial x_j} du \qquad (n = 1, 2, \cdots). \tag{34}$$

(35) *Let P be the function of $\gamma^{\mathbf{R}}$ defined by the expression*

$$P(\gamma) = \sum_{j=1}^{\infty} (x_j' - x_j) \frac{\partial f(\gamma)}{\partial x_j}.$$

In the sequel u will always have the range $0 \leqq u \leqq 1$, and, moreover, for the purpose of proving the theorem, it is supposed that a definite pair (ξ', ξ) is under consideration.

In reference to the preceding definitions, certain obvious propositions can be stated.

PROPOSITION 4. *For every n and for every value of u on the interval $0 \leqq u \leqq 1$ it follows that $\eta(n, u)$ is in $\mathbf{R}$.*

PROPOSITION 5. *The sequence (ζ_n) satisfies*

$$\lim_{n=\infty} \zeta_n = \xi'.$$

PROPOSITION 6. *The function $P(\gamma)$ is $C_1(\mathbf{R})$.*

The last result is obtained as a consequence of hypothesis (29) and of Theorem III.

In the proof of the theorem, it will be useful to have the following

LEMMA 1. *The sequence $P[\eta(n, u)]$, $(n = 1, 2, \cdots)$, is such that*

$$\lim_{n=\infty} P[\eta(n, u)] = P[\eta(0, u)] \text{ uniformly for } 0 \leqq u \leqq 1.$$

It has just been shown that P is $C_1(\mathbf{R})$. Hence, in view of Theorem I, it follows that $P[\theta(\gamma, u)]$, where

$$\theta(\gamma, u) = \xi + u(\gamma - \xi) \qquad [\gamma^{\mathbf{R}} = (y_1, y_2, \cdots)]$$

is completely continuous in the space R', where the coördinates are $(u, y_1, y_2, \cdots)$, in which y_i satisfies (3). The lemma, therefore, is an immediate consequence of Theorem II, when the x_1 of that discussion is identified with u.

COROLLARY 1. *For every $e > 0$ an integer n_e can be determined such that, for $n \geqq n_e$,*

$$|P_n[\eta(n, u)] - P_n[\eta(0, u)]| \leqq e \qquad (0 \leqq u \leqq 1),$$

where P_n is defined as the sum of the first n terms of P.

In view of hypothesis (29), it follows that, for every $e > 0$, an integer n_e can be found such that, for $0 \leqq u \leqq 1$; $n \geqq n_e$; $m = 1, 2, \cdots$,

$$|P_n[\eta(m, u)] - P[\eta(m, u)]| \leqq \tfrac{1}{3}e. \tag{36}$$

As a result of the Lemma 1, this n_e can be supposed chosen so that also

$$|P[\eta(0, u)] - P[\eta(n, u)]| \leqq \tfrac{1}{3}e \quad (0 \leqq u \leqq 1;\ n \geqq n_e). \tag{37}$$

From (36) for $m = n$ and $m = 0$, and from (37), there is obtained

$$|P_n[\eta(0, u)] - P_n[\eta(n, u)]| \leqq e \quad (n \geqq n_e,\ 0 \leqq u \leqq 1).$$

Now consider the proof of the theorem. It is seen that on account of Proposition 5

$$\lim_{n=\infty} f(\zeta_n) = f(\xi'). \tag{38}$$

From the law of the mean for the finite case, it follows that

$$f(\zeta_n) - f(\xi) = s_n.$$

In view of (38) there is obtained

$$\lim_{n=\infty} s_n = f(\xi') - f(\xi).$$

It is desired to show that $\lim_{n=\infty} s_n = S$.

From the definitions (31) to (35) it is seen that

$$|S_n - s_n| \leqq \int_0^1 |P_n[\eta(0, u)] - P_n[\eta(n, u)]| du.$$

Because of Corollary 1 to Lemma 1, it follows that, for every $e > 0$ an integer n_e can be found such that

$$|S_n - s_n| \leqq e \qquad (n \geqq n_e).$$

Consequently, since $\lim_{n=\infty} s_n$ exists, it follows that S exists and that $\lim_{n=\infty} s_n = S$, which completes the proof of the theorem.

It should be remarked that, as a result of (29), the series (30) converges absolutely-uniformly with respect to the pair (ξ^R, ξ'^R).

In deriving the Taylor Theorem in the next article, it will be convenient to have the following theorem which gives a means for computing the derivative of a function f of a differentiable function $\xi(v)$.

THEOREM VI. *Let $f(\xi)$ and the partial derivatives $\partial f/\partial x_i$ be as assumed in Theorem V. Let $\xi(v)$ be a function of $v\,(0 \leqq v \leqq 1)$ for which the derivative $\partial\xi/\partial v$ is continuous in v and for which*

$$\sum_{i=1}^{\infty} \left| \frac{dx_i(v_i)}{dv} \frac{\partial f(\eta)}{\partial x_i} \right| \tag{39}$$

converges uniformly for all $0 \leqq v_i \leqq 1$, *and* $\eta^{\mathbf{R}}$, *while, for every* v, $\xi(v)$ *is in* $\mathbf{R}$. *Then the function* $G(v) = f[\xi(v)]$ *is continuous and*

$$\frac{dG(v)}{dv} = \sum_{i=1}^{\infty} \frac{\partial f[\xi(v)]}{\partial x_i} \frac{dx_i(v)}{dv} = H(v). \tag{40}$$

The continuity of $G(v)$ follows from Theorem I. Moreover, $H(v)$ exists as a result of hypothesis (39); it remains to show the existence of dG/dv and the equality stated in (40).

As a consequence of Theorem V, it is seen that, for every pair of points $(\xi'^{\mathbf{R}}, \xi^{\mathbf{R}})$,

$$f(\xi') - f(\xi) = \sum_{j=1}^{\infty} (x_j' - x_j) \int_0^1 \frac{\partial f[\xi + u(\xi' - \xi)]}{\partial x_j} du, \tag{41}$$

where the infinite sum converges absolutely-uniformly for all points $(\xi'^{\mathbf{R}}, \xi^{\mathbf{R}})$. Suppose $\xi = \xi(v)$, $\xi' = \xi(v')$, in (41). Let $\Delta v = (v' - v)$ and

$$\eta(v', u) = \xi(v) + u[\xi(v') - \xi(v)].$$

Then

$$\frac{g(v') - g(v)}{\Delta v} = \sum_{j=1}^{\infty} \frac{x_j(v') - x_j(v)}{\Delta v} \int_0^1 \frac{\partial f[\eta(v', u)]}{\partial x_j} du. \tag{42}$$

The equation (42) is to be considered as v' approaches v. From the mean-value theorem for the function $x_j(v)$, there is obtained

$$x_j(v') - x_j(v) = \Delta v \frac{dx_j(v_{v',j})}{dv}, \tag{43}$$

where $v_{v',j}$ is suitably chosen between v' and v. In view of (43), equation (42) becomes

$$\frac{G(v') - G(v)}{\Delta v} = \sum_{j=1}^{\infty} \frac{dx_j(v_{v',j})}{dv} \int_0^1 \frac{\partial f[\eta(v', u)]}{\partial x_j} du.$$

Hence there results

$$\left| H(v) - \frac{G(v') - G(v)}{\Delta v} \right| = \left| \sum_{j=1}^{\infty} \frac{dx_j(v)}{dv} \frac{\partial f[\xi(v)]}{\partial x_j} - \frac{dx_j(v_{v',j})}{dv} \int_0^1 \frac{\partial f[\eta(v', u)]}{\partial x_j} du \right|, \tag{44}$$

$$\leqq \left| \sum_{j=1}^{\infty} \left[\frac{dx_j(v_{v',j})}{dv} - \frac{dx_j(v)}{dv} \right] \int_0^1 \frac{\partial f[\eta(v', u)]}{\partial x_j} du \right| + \left| \sum_{j=1}^{\infty} \frac{dx_j(v)}{dv} \left(\int_0^1 \frac{\partial f[\eta(v', u)]}{\partial x_j} du - \frac{\partial f[\xi(v)]}{\partial x_j} du \right) \right|. \tag{45}$$

Let $T_1(v')$ and $T_2(v')$, respectively, denote the first and second terms in (45). Consider $T_1(v')$. Because of the uniform convergence in (39), it is seen that, for every $e > 0$, an integer n_e can be determined for which

$$\sum_{j=n_e+1}^{\infty} \left| \left[\frac{dx_j(v_{v',j})}{dv} - \frac{dx_j(v)}{dv} \right] \int_0^1 \frac{\partial f[\eta(v', u)]}{\partial x_j} du \right| \leqq \tfrac{1}{3}e \quad (0 \leqq v' \leqq 1).$$

Moreover, from the continuity of $d\xi/dv$, it follows that a constant $b_e > 0$ can be chosen such that, if $|v' - v| \leqq b_e$, then the sum for $j = 1$ to $j = n_e$ of the same expression is at most $\frac{1}{3}e$. Hence it has been established that

$$T_1(v') \leqq \tfrac{2}{3}e \qquad (|v' - v| \leqq b_e).$$

Before considering $T_2(v')$, it is convenient to note, under the hypotheses of the present theorem,

Lemma 2. *Suppose that* ξ, ξ', ζ_m $(m = 1, 2, \cdots)$ *are points of* **R** *such that* $\lim_{m=\infty} \zeta_m = \xi'$. *Then it follows that*

$$\lim_{m=\infty} \frac{\partial f[\xi + u(\zeta_m - \xi)]}{\partial x_j} = \frac{\partial f[\xi + u(\xi' - \xi)]}{\partial x_j}$$

uniformly for all $0 \leqq u \leqq 1$.

This result is a consequence of Theorem II with the x_1 and $(x_2, x_3, \cdots)$ of that discussion identified with the u and ξ' of the present statement respectively.

In view of the lemma, it is seen that the absolutely-uniformly converging series

$$(46) \quad S(\xi_0) = \sum_{j=1}^{\infty} \frac{dx_j(v)}{dv} \left(\int_0^1 \frac{\partial f[\xi(v) + u(\xi_0 - \xi(v))]}{\partial x_j} du - \frac{\partial f[\xi(v)]}{\partial x_j} du \right)$$

is $C_1(\xi_0)$ for all $\xi_0^{\mathbf{R}}$.

On considering (45) it is seen that, because of (46) and Theorem I, the expression $T_2(v')$ is a continuous function of v'. Thus, a number d_e, where $0 < d_e \leqq b_e$, can be chosen such that

$$T_2(v') \leqq \tfrac{1}{3}e \qquad (|v' - v| \leqq d_e),$$

and, hence, $T_1(v') + T_2(v') \leqq e$ for the same values of v'. It has therefore been established that the quotient (42) has a unique limit as v' approaches v, which limit satisfies the equation (40).

It is easily proved that the derivative dG/dv is a continuous function of v because, in view of Theorem I, each term of $H(v)$ is continuous.

5. **Taylor's Theorem for $f(\xi)$.** The subject for discussion in the present article is the

THEOREM VII. *Suppose that the function $f(\xi)$ and, for a certain k, all partial derivatives $\partial^h f/\partial x_{i_1} \cdots \partial x_{i_h}$ $(h = 1, 2, \cdots, k)$ are $C_1(\mathbf{R})$, while*

$$\sum_{j_1, \ldots, j_h=1}^{\infty} r_{j_1} r_{j_2} \cdots r_{j_h} \left| \frac{\partial^h f(\eta)}{\partial x_{j_1} \cdots \partial x_{j_h}} \right| \qquad (h = 1, 2, \cdots, k), \tag{47}$$

converge uniformly for all η of $\mathbf{R}$. Then it follows that, for every pair of points $(\xi^{\mathbf{R}}, \xi'^{\mathbf{R}})$,

$$f(\xi') - f(\xi) = \sum_{j=1}^{\infty} (x_j' - x_j) \frac{\partial f(\xi)}{\partial x_j} + \frac{1}{2} \sum_{j_1, j_2=1}^{\infty} (x_{j_1}' - x_{j_1})(x_{j_2}' - x_{j_2}) \frac{\partial^2 f(\xi)}{\partial x_{j_1} \partial x_{j_2}}$$

$$+ \cdots + \frac{1}{(k-1)!} \sum_{j_1, \ldots, j_{k-1}=1}^{\infty} (x_{j_1}' - x_{j_1}) \cdots (x_{j_{k-1}}' - x_{j_{k-1}}) \frac{\partial^{k-1} f(\xi)}{\partial x_{j_1} \cdots \partial x_{j_{k-1}}}$$

$$+ \frac{1}{(k-1)!} \sum_{j_1, \ldots, j_k=1}^{\infty} (x_{j_1}' - x_{j_1}) \cdots (x_{j_k}' - x_{j_k}) \int_0^1 \frac{\partial^k f[\xi + u(\xi' - \xi)]}{\partial x_{j_1} \cdots \partial x_{j_k}} (1-u)^{k-1} du.$$

The method of proof consists in reducing the problem to a question of the expansion by Taylor's Formula of a function of a single variable; this reduction is accomplished by means of Theorem VI. In order to avoid complicated notation the proof will be given only for the case $k = 2$. The details in the general case are precisely similar.

Assume, now, that a definite pair of points $(\xi^{\mathbf{R}}, \xi'^{\mathbf{R}})$ is given. Define $\eta(v) = \xi + v(\xi' - \xi)$. It is clear that, for $(0 \leqq v \leqq 1)$, the point $\eta(v)$ belongs to $\mathbf{R}$. Moreover, every component of η is continuous in v and

$$\frac{d\eta(v)}{dv} = (\xi' - \xi).$$

On placing $\eta(v) = [y_1(v), y_2(v), \cdots]$, it follows that

$$\sum_{j=1}^{\infty} \frac{dy_j(v_j)}{dv} \frac{\partial f(\zeta)}{\partial x_j} = \sum_{j=1}^{\infty} (x_j' - x_j) \frac{\partial f(\zeta)}{\partial x_j} \qquad (\zeta^{\mathbf{R}} = z_1, z_2, \cdots). \tag{48}$$

Since the right side of (48) is independent of the v_j, it follows from the hypothesis (47), for $k = 1$, that (48) converges absolutely-uniformly for all $\zeta^{\mathbf{R}}$, and $(0 \leqq v_j \leqq 1)$. Hence in view of Theorem VI, it is seen that, if $F(v)$ denotes $f[\eta(v)]$ then

$$\frac{dF(v)}{dv} = \sum_{j=1}^{\infty} (x_j' - x_j) \frac{\partial f[\eta(v)]}{\partial x_j},$$

where $dF(v)/dv$ is continuous in v. On applying Theorem VI a second time, it is similarly established that

$$\frac{d^2 F(v)}{dv^2} = \sum_{j_1, j_2=1}^{\infty} (x_{j_1}' - x_{j_1})(x_{j_2}' - x_{j_2}) \frac{\partial^2 f[\eta(v)]}{\partial x_{j_1} \partial x_{j_2}},$$

and that $d^2 F(v)/dv^2$ is continuous in v.

The Taylor Formula with two terms, applied to the function $F(v)$, gives

$$(49)\qquad F(v) - F(0) = v\frac{dF(0)}{dv} + v^2\int_0^1 (1-u)\frac{d^2F(uv)\,du}{dv^2}.$$

On placing $v = 1$ in (49), there is obtained

$$F(1) - F(0) = f(\xi') - f(\xi) = \sum_{j=1}^{\infty}(x_j' - x_j)\frac{\partial f(\xi)}{\partial x_j}$$

$$+ \sum_{j_1, j_2=1}^{\infty}(x_{j_1}' - x_{j_1})(x_{j_2}' - x_{j_2})\int_0^1 \frac{\partial^2 f[\eta(u)]}{\partial x_{j_1}\partial x_{j_2}}(1-u)\,du,$$

which establishes the desired result.

6. **Discussion of the results.** The special form of the hypotheses (29) of Theorem V, (39) of Theorem VI, and (47) of Theorem VII have as their purpose the distribution of the force of the assumptions both on to the region **R** through the introduction of the r_i, and on to the derivatives which are involved. For example, if the region **R** is such that $\sum_{i=1}^{i=\infty} r_i$ converges, then (29) is satisfied if it is merely assumed that the derivatives $\partial f/\partial x_j$ have a common upper bound. On the other hand, if there is a number $a > 0$ such that $r_i > a$, then it would be necessary, in order to satisfy (29), that

$$\sum_{i=1}^{\infty}\left|\frac{\partial f(\xi)}{\partial x_i}\right|$$

should converge uniformly for $\xi^{\mathbf{R}}$. Similar alternative hypotheses are possible in Theorems VI and VII. The general notion of such a distribution of the force of the hypotheses as is used here was first brought out by F. R. Moulton* in a paper on differential equations.

PART II

Ordinary differential equations in infinitely many variables

1. **Introduction.** The main purpose of the following discussion is the presentation of an existence proof for a solution of a certain infinite system of differential equations of the form (1). The existence theorem is divided into two sections. Under suitable conditions the unique existence of a continuous solution of (1) in a restricted neighborhood is first established in § 2 and then, in § 3, under further hypotheses the solution is shown to extend to the boundary of the region of points **R** in which (1) is defined. In this final theorem of § 3, **R** is taken as a general region possessing interior points, where the notion *interior* is given a definition reducing, for the finite case, to that used in a space of a finite number of dimensions.

* Loc. cit.

In carrying out the proof for a restricted neighborhood in § 2 by a method analogous to the Picard scheme for a finite system, it is found convenient to assume that the $f_i(\xi, t)$ in (1) satisfy what might be termed a generalized Lipschitz condition with respect to the variables $(x_1, x_2, \cdots)$. In § 4 it is shown that this hypothesis, which is of a formal character, can be replaced by a second assumption which, though stronger, is functional in nature.

In § 5 the results of F. R. Moulton are considered. It is established that the main results of his paper, so far as they apply to reals, can be obtained as special cases of the results of §§ 2, 3, and 4. It is proved that the system (1), which Moulton treats, satisfies the assumed conditions of the theorems of the present paper.

2. **Definition of the approximations and proof of the theorem in a restricted neighborhood.** Let the system (1) be represented by the notation

$$\frac{d\xi}{dt} = \Phi(\eta) \qquad [\Phi = (f_1, f_2, \cdots; \ \eta = (\xi, t) = (t, x_1, x_2, \cdots)], \tag{50}$$

where the initial conditions are

$$\xi(t) = \xi_0 = (a_1, a_2, \cdots) \qquad (t = t_0).$$

For the system (50) let the approximations ξ_k $(k = 0, 1, 2, \cdots)$ to a solution satisfying the initial conditions, be defined formally by the equations

$$\begin{aligned} &\xi_0(t) = \xi_0, \\ &\xi_k(t) = \xi_0 + \int_{t_0}^{t} \Phi[\eta_{k-1}(t)]\,dt \quad \{\eta_{k-1}(t) = [\xi_{k-1}(t), t];\ k = 1, 2, \cdots\}. \end{aligned} \tag{51}$$

The following theorem presents hypotheses under which the approximations (51) exist and converge to a solution of (50) for t restricted to a sufficiently small neighborhood of t_0. A simple form will be assumed for the region **R** in which (50) will be supposed defined. This procedure has as its purpose the formation of a basis for the proof in § 3 of the existence of a solution of (50) in an extended region when **R** is of a more general type.

Let **R** designate the region of points η for which t lies in the region

$$T : |t - t_0| \leqq r_0 \qquad (r_0 > 0), \tag{52}$$

and for which ξ lies in the region

$$S : |x_i - a_i| \leqq r_i \qquad (i = 1, 2, \cdots). \tag{53}$$

THEOREM VIII. *Suppose that the system* (50) *satisfies the following hypotheses*:

(H_1) *The function* Φ *is* $C_1(\mathbf{R})$—*in other words,* $f_1, f_2, \cdots$ *are each* $C_1(\mathbf{R})$.

(H_2) *There exist positive-valued functions* $A_{ij}(t, \xi, \xi')$, *where* (t^T, ξ^S, ξ'^S), *which are completely continuous in their arguments, and are such that, for every* (t^T, ξ^S, ξ'^S),

$$|f_i(\xi, t) - f_i(\xi', t)| \leqq \sum_{j=1}^{\infty} A_{ij}(t, \xi, \xi')|x'_j - x_j|,$$

where, moreover, it is assumed that, for all (t^T, ξ^S, ξ'^S),

$$\sum_{j=1}^{\infty} r_j A_{ij}(t, \xi, \xi') = V_i(t, \xi, \xi') \qquad (i = 1, 2, \cdots) \tag{54}$$

converges uniformly.

(H_3) *There exist finite numbers* M *and* K *such that the maxima* M_i *of the* $|f_i(\xi, t)|$, *and the maxima* K_i *of the* $V_i(t, \xi, \xi')$ *for points* (t^T, ξ^S, ξ'^S) *satisfy*

$$M_i \leqq r_i M, \tag{55}$$

$$K_i \leqq r_i K. \tag{56}$$

Then the approximations (51), *formed for the system* (50), *exist and converge to a function* $\xi(t)$ *for* $|t - t_0|$ *sufficiently small. Moreover,* $\xi(t)$ *is continuous in* t *and is a solution of* (50) *for which* $\xi(t_0) = \xi_0$. *Furthermore, there is no other continuous solution of* (50) *reducing to* ξ_0 *for* $t = t_0$.

It is interesting to note the following obvious relation which exists in connection with (H_3) when R is of a special form.

PROPOSITION 4. *Suppose there is a number* $a > 0$ *such that* $r_i > a$ $(i = 1, 2, \cdots)$. *Then if the constants* (K_i, M_i) *have a common finite bound* H, *it follows that there are numbers* M *and* K *satisfying* (H_3).

To prove Theorem VIII, it will first be shown that there is a number $d > 0$ such that, for $|t - t_0| \leqq d$, the approximations (51) are defined and satisfy

$$|\xi_k(t) - \xi_0| \leqq \rho \qquad [\rho = (r_1, r_2, \cdots)]. \tag{57}$$

It will next be established that, for $|t - t_0| \leqq d$, the sequence (51) converges uniformly, and then, finally, it will be proved that the limit of the sequence is the unique continuous solution of (50) reducing to ξ_0 for $t = t_0$.

To accomplish the first step of the proof, consider $|\xi_k(t) - \xi_0|$. It follows from (51) that

$$(58) \qquad \begin{aligned} |\xi_k(t) - \xi_0| &= \left|\int_{t_0}^{t} \Phi(\xi_0, t)\, dt\right| && (k = 1), \\ |\xi_k(t) - \xi_0| &\leqq \left|\int_{t_0}^{t} \rho M dt\right| \leqq M\rho|t - t_0| && (k = 1). \end{aligned}$$

Let d be a number such that $d \leqq 1/M$. Then, in view of (58),

$$|\xi_1(t) - \xi_0| \leqq \rho \qquad (|t - t_0| \leqq d).$$

It is similarly established that (58) holds true for $(k = 2, 3, \cdots)$ if

$$|t - t_0| \leqq d.$$

Now let $\gamma_k(t) = \xi_k(t) - \xi_{k-1}(t)$, and consider the problem of showing the convergence of the series $\sum_{k=1}^{\infty} \gamma_k(t)$ which is equivalent to the question of the convergence of the sequence (51). On placing

$$\gamma_k(t) = [y_{1k}(t), y_{2k}(t), \cdots],$$

it follows that

$$|y_{i1}(t)| = \left| \int_{t_0}^{t} f_i(\xi_0, t)\, dt \right| \leqq r_i M |t - t_0| \qquad (i = 1, 2, \cdots).$$

From the definition (51) there is obtained

$$|y_{i2}(t)| = \left| \int_{t_0}^{t} \{f_i[\eta_1(t)] - f_i[\eta_0(t)]\} dt \right|,$$

which, in view of (H_2), gives

$$\begin{aligned} |y_{i2}(t)| &\leqq \left| \int_{t_0}^{t} \sum_{j=1}^{\infty} A_{ij}(t, \xi_1(t), \xi_0) |x_{j1}(t) - a_j| dt \right|, \\ &\leqq M \left| \int_{t_0}^{t} \left(\sum_{j=1}^{\infty} A_{ij}(t, \xi_1(t), \xi_0) r_j \right) |t - t_0| dt \right|, \\ &\leqq \tfrac{1}{2} r_i MK |t - t_0|^2. \end{aligned}$$

On proceeding in a similar fashion, it follows by a simple induction that

$$|y_{ik}(t)| \leqq \frac{1}{k!} MK^{k-1} r_i |t - t_0|^k.$$

Therefore it has been established that*

$$\sum_{k=1}^{\infty} y_{ik}(t) \ll r_i M \sum_{j=1}^{\infty} \frac{K^{k-1}}{k!} |t - t_0|^k. \tag{59}$$

Since the series on the right converges uniformly for all $|t - t_0| \leqq d$, it follows that the series on the left converges absolutely-uniformly for all $|t - t_0| \leqq d$. Moreover, since each term of the series is a continuous function of t, it is seen that the sum represents a continuous function of t. Hence

* The sign $\ll$ is a notation introduced by Poincaré which means that the series on the left is term by term dominated by the series on the right.

it has been shown that there exists a continuous function $\xi(t)$ such that

$$\lim_{k=\infty} \xi_k(t) = \xi(t), \tag{60}$$

uniformly for all $|t - t_0| \leqq d$. From the form of the dominating series in (59) it is seen that the convergence in (60) is characterized by the statement that *for every number* $e > 0$ *there exists an integer* k_e *such that, for* $k \geqq k_e$,

$$|\xi_k(t) - \xi(t)| \leqq e\rho \qquad (|t - t_0| \leqq d). \tag{61}$$

This manner of convergence is a special case of uniform convergence relative to a scale function, a notion first made use of by E. H. Moore. To use Moore's terminology, it would be stated that, in (61), the sequence ξ_k $(k = 1, 2, \cdots)$ converges to ξ uniformly over the range **I** relatively to the scale function ρ. The general definition* of this type of convergence is as follows:

Let $\mu_n(p)$ $(n = 1, 2, \cdots)$, $\mu(p)$, and $\sigma(p)$ be complex valued functions on the range **P**. Then the sequence μ_n converges to μ uniformly over the range **P**, relatively to the scale function σ, if, for every $e > 0$, an integer n_e can be found such that, for $n \geqq n_e$,

$$|\mu_n - \mu| \leqq e|\sigma|.$$

In order to complete the proof of Theorem VIII it remains to show that $\xi(t)$ is a solution of equation (50). From the definition (51) it is seen that $\xi(t_0) = \xi_0$; moreover,

$$\lim_{k=\infty} x_{ik}(t) = a_i + \lim_{k=\infty} \int_{t_0}^{t} f_i[\eta_{k-1}(t)]\,dt. \tag{62}$$

Because of (61), it follows from Proposition 2 that, for every $e > 0$, there can be found an integer k_e such that

$$|f_i[\eta_{k-1}(t)] - f_i[\xi(t), t]| \leqq e \quad (k \geqq k_e;\ |t - t_0| \leqq d).$$

Therefore it is found that

$$\lim_{k=\infty} \int_{t_0}^{t} f_i[\eta_{k-1}(t)]\,dt = \int_{t_0}^{t} f_i[\xi(t), t]\,dt. \tag{63}$$

As a result of (62) and of (63), it follows that

$$\xi(t) = \xi_0 + \int_{t_0}^{t} \Phi(\xi(t), t)\,dt, \tag{64}$$

* I, p. 30.

and, therefore, there is obtained

$$\frac{d\xi(t)}{dt} = \Phi[\xi(t), t].$$

Thus $\xi(t)$ is a continuous solution of (50) which satisfies the initial conditions.

The uniqueness of the continuous solution $\xi(t)$ is established by an indirect argument. Suppose that there exists a second continuous solution $\gamma(t)$ differing from $\xi(t)$, lying in S for $|t - t_0| \leqq d$, and satisfying $\gamma(t_0) = \xi_0$. Then

$$\gamma(t) = \xi_0 + \int_{t_0}^{t} \Phi[\gamma(t), t]\, dt.$$

It follows from the fact that γ is in S, that

$$|\gamma(t) - \xi_0| \leqq \rho.$$

Therefore there is obtained, on placing $\gamma(t) = [y_1(t), y_2(t), \cdots]$,

$$\begin{aligned} |y_i(t) - x_{i1}(t)| &= \left| \int_{t_0}^{t} \{f_i[\gamma(t), t] - f_i[\xi_0, t]\}\, dt \right| \\ &\leqq \left| \int_{t_0}^{t} \sum_{j=1}^{\infty} A_{ij}(t, \gamma(t), \xi_0) |y_j(t) - a_j|\, dt \right| \\ &\leqq r_i K |t - t_0|. \end{aligned}$$

On proceeding as in the discussion leading to (59), there results

$$|y_i(t) - x_{ik}(t)| \leqq \frac{r_i K^k |t - t_0|^k}{k!} \qquad (i, k = 1, 2, \cdots).$$

Thus it follows that the sequence $\xi_k(t)$ converges to $\gamma(t)$ and therefore $\gamma = \xi$. Hence, the assumption as to the existence of the second distinct solution has been contradicted.

It is easily seen that Theorem VIII holds in its entirety if in its statement the approximations ξ_k of (51) are replaced by those defined as follows:

$$(65)\qquad \begin{aligned} \xi_0(t) &= \zeta(t) & [\zeta(t_0) = \xi_0], \\ \xi_k(t) &= \xi_0 + \int_{t_0}^{t} \Phi[\eta_{k-1}(t)]\, dt & (k = 1, 2, \cdots), \end{aligned}$$

in which $\eta_{k-1}(t) = [\xi_{k-1}(t), t]$ and where, for $|t - t_0| \leqq r_0$, the function $\zeta(t)$ is continuous and satisfies

$$|\zeta(t) - \xi_0| \leqq b\rho, \qquad b < 1.$$

However, the neighborhood of t_0 in which the ξ_k of (65) converge to a solution is perhaps different from that found for the ξ_k of (51).

The question naturally arises as to whether the solution $\xi(t)$ possesses a second derivative under further hypotheses similar to those added in the finite case in deriving the corresponding result. This query is answered affirmatively in

COROLLARY 1. *In addition to the hypotheses of Theorem VIII, assume that the functions* $\partial f_i/\partial x_j$ *and* $\partial f_i/\partial t$ $(i, j = 1, 2, \cdots)$ *are* $C_1(\mathbf{R})$ *and that each of the sums*

$$\sum_{j=1}^{\infty}\left|\frac{\partial f_i(\gamma, t)}{\partial x_j} r_j\right| \qquad (i = 1, 2, \cdots) \tag{66}$$

converges uniformly for (t^T, γ^S). *Then it follows that, for* $|t - t_0| \leqq d$, $\xi(t)$ *has a second derivative given by*

$$\frac{d^2 x_i}{dt^2} = \sum_{j=1}^{\infty}\frac{\partial f_i[\xi(t), t]}{\partial x_j} f_j[\xi(t), t] + \frac{\partial f_i[\xi(t), t]}{\partial t} \quad (i = 1, 2, \cdots).$$

To establish this corollary, recall Theorem VI. Let the function $f_i(\xi, t)$ and the point $\eta(t) = [\xi(t), t]$ replace respectively the function f and the point $\xi(v)$ of Theorem VI. To show that all hypotheses in the former case are satisfied by f_i it remains to prove that

$$W = \frac{\partial f_i[\eta(t')]}{\partial t} + \sum_{j=1}^{\infty}\left|\frac{\partial f_i(\gamma, t)}{\partial x_j} f_j[\eta(t_j)]\right|$$

converges uniformly for all (t^T, γ^S) and for all (t_j, t') on the interval

$$|t - t_0| \leqq d.$$

From (H_3) it follows that

$$W \ll \sum_{j=1}^{\infty} r_j M\left|\frac{\partial f_i(\gamma, t)}{\partial x_j}\right| + \left|\frac{\partial f_i[\eta(t')]}{\partial t}\right|,$$

which, in view of hypothesis (66), is seen to converge uniformly for all (t^T, γ^S). Therefore by a direct application of Theorem VI the corollary is fully established.

In a similar fashion, further hypotheses can be added which insure the existence of third, fourth, and higher derivatives of $\xi(t)$. The procedure would be to impose sufficient conditions to permit the successive application of Theorem VI.

3. **The existence of a solution in an extended region.** Let $\mathbf{R}$ denote a region of points $\eta = (\xi, t)$. Then, η_0 will be termed an interior point of $\mathbf{R}$ if *there exist positive numbers* $(d_0, d_1, \cdots)$ *such that all points in the neighbor-*

hood

$$|t - t_0| < d_0, \qquad |\xi - \xi_0| < \delta \qquad [\delta = (d_1, d_2, \cdots)]$$

are in the region R. All other points of $\mathbf{R}$ will be called boundary points.

The theorem to be presented in this article gives extension properties of the solution of (50) in a region possessing interior points which, if the number of variables and equations in (50) is finite, reduces to the usual theorem on the solution of a system of differential equations of finite order.

THEOREM IX. *Let* $\Phi(\eta)$ *be* C_1 *for all* η *in a region* $\mathbf{R}$ *possessing interior points. Suppose that, for every interior point* η_0, *there exists a neighborhood* $\mathbf{R}_0$ *of interior points of* $\mathbf{R}$, *of type* (52) *and* (53), *and functions* $A_{ij}^{(0)}(t, \xi, \xi')$, *which are* C_1 *for all points* $(t^{T_0}, \xi^{S_0}, \xi'^{S_0})$, *simultaneously in all their arguments. At every* η_0 *let the hypotheses of Theorem VIII be supposed satisfied with the* $\mathbf{R}$ *and the* A_{ij} *of that theorem replaced by the* $\mathbf{R}_0$ *and* $A_{ij}^{(0)}$ *of the present statement. Then, if* (ξ_0, t_0) *is interior to* $\mathbf{R}$, *it follows that there is a unique continuous solution* $\xi(t)$ *of* (50) $(\xi(t_0) = \xi_0)$ *which is defined on an interval* $t^{(1)} < t < t^{(2)}$ *including* $t_0 (t^{(1)} < t_0 < t^{(2)})$, *on which the point* $(\xi(t), t)$ *is interior to* $\mathbf{R}$. *Furthermore,* $t^{(i)}$ $(i = 1, 2)$ *are such that, as* t *approaches* $t^{(i)}$, *the only limit points of* $\eta(t) = [\xi(t), t]$ *are on the boundary of* $\mathbf{R}$ *or else are interior points such as* $(\gamma, t^{(i)})$, *which are not approached by any sequence of points* $[\xi(t_k), t_k]$ $(k = 1, 2, \cdots)$, *where the* $\xi(t_k)$ *converge to* γ *uniformly over the range* $\mathbf{I}$ *relatively to the scale function* $\rho' = (r_1', r_2', \cdots)$, *where the* r_i' *are the numbers corresponding by hypothesis to the interior point* $(\gamma, t^{(i)})$.

In order to establish this result,* first apply Theorem VIII at the point η_0. It follows that there is a continuous solution $\xi(t)$, defined over a certain interval $|t - t_0| \leqq e_1$. It is seen that, since the $\mathbf{R}_0$ corresponding to η_0 consists entirely of interior points, $\eta_1 = \eta(t_0 + e_1)$ is interior to $\mathbf{R}$. From the uniqueness of the continuous solution $\xi(t)$ through ξ_0, it follows that the solution $\xi'(t)$ through $\xi(t_0 + e_1)$, given by a second application of Theorem VIII, is a continuation of $\xi(t)$. Hence $\xi(t)$ is defined, continuous, and interior to R for

$$t_0 \leqq t \leqq t_0 + e_1 + e_2, \qquad (e_2 > 0).$$

Let $t^{(2)}$ be the least upper bound of the intervals on which $\xi(t)$ is defined, continuous and such that the point $\eta(t)$ is interior to $\mathbf{R}$. Suppose that, as t approaches $t^{(2)}$, the point $\eta(t)$ has a limit point $(\gamma, t^{(2)})$ interior to $\mathbf{R}$ and that $(r_0', r_1', r_2', \cdots)$ are the numbers of (52) and (53) corresponding to it by hypothesis. Suppose that t_k $(k = 1, 2, \cdots)$ are a sequence of values of t such that the corresponding sequence $\xi(t_k)$ converges to γ, uniformly over $\mathbf{I}$, relatively to the scale function ρ'. Then the theorem will be completely

* The existence of $t^{(2)}$ will be considered. A similar argument would suffice for the case of $t^{(1)}$.

proved if, under these assumptions, it is shown that $t^{(2)}$ cannot be the upper bound as stated above.

Let M_i', M', K_i', K', and $A_{ij}'(t, \xi, \xi')$ be the elements of the hypothesis of Theorem VIII corresponding to the point $(\gamma, t^{(2)})$. Take a value g such that

$$|t^{(2)} - t_g| < \tfrac{1}{2}r_0', \ \frac{1}{2M'}; \quad |\xi(t_g) - \gamma| < \tfrac{1}{2}\rho'$$

At the point $[t = t_g, \xi = \xi(t_g)]$ the hypotheses of Theorem VIII are satisfied with the **R** of that discussion replaced by the region

$$|\xi - \xi(t_g)| \leqq \tfrac{1}{2}\rho'; \qquad |t - t_g| \leqq \tfrac{1}{2}r_0',$$

and with $\rho = \rho'/2$, $M = 2M'$, $K = K'$, and $A_{ij} = A_{ij}'$. Hence the work following (58) shows that there passes through $\eta(t_g)$ a solution which, from its uniqueness, coincides with $\xi(t)$ for $t \leqq t_g$ and which is defined, continuous, and interior to **R** for $|t - t_g| \leqq 1/2M'$. But this last interval includes $t^{(2)}$ as an interior point. Therefore $t^{(2)}$ is not the least upper bound as assumed above and thus Theorem IX is proved.

4. **Substitution of a functional hypothesis in place of (H_2).** In the case of a function f of a finite number of variables $(x_1, x_2, \cdots, x_n)$ it is known that the existence of a Lipschitz inequality

$$|f(x_1, \cdots, x_n) - f(x_1', \cdots, x_n')| \leqq K\sum_{i=1}^{n} |x_i' - x_i| \qquad (|x_i - a_i| \leqq r_i)$$

is implied by the assumption that f has continuous derivatives $\partial f/\partial x_i$ for points in $|x_i - a_i| \leqq r_i$. A somewhat similar result in regard to the hypothesis (H_2), which is of the nature of a Lipschitz condition, is stated in the following

THEOREM X. *Let f be defined in the region* **R** *given by* (52) *and* (53), *and such that $f(\eta)$ and $\partial f(\eta)/\partial x_j$ $(j = 1, 2, \cdots)$ are $C_1(\mathbf{R})$. Suppose, also, that*

$$\sum_{j=1}^{\infty}\left|\frac{\partial f(\eta)}{\partial x_j}\right| r_j \tag{67}$$

converges uniformly for $\eta^{\mathbf{R}}$. Then it follows that there exist positive-valued functions $A_j(t, \xi, \xi')$ $(j = 1, 2, \cdots)$ which are C_1 simultaneously in the arguments (t^T, ξ^S, ξ'^S) and are such that

$$\sum_{j=1}^{\infty} A_j(t, \xi, \xi')r_j \tag{68}$$

converges uniformly for all (t^T, ξ^S, ξ'^S); and, moreover,

$$|f(\xi, t) - f(\xi', t)| \leqq \sum_{j=1}^{\infty} A_j(t, \xi, \xi')|x_j - x_j'|. \tag{69}$$

In order to prove this theorem, consider, first, an application of the mean-value theorem to $f(\xi, t)$. From the present hypotheses and from Theorem V it follows that

$$(70)\quad f(\xi', t) - f(\xi, t) = \sum_{j=1}^{\infty} (x_j' - x_j) \int_0^1 \frac{\partial f[\xi + u(\xi' - \xi); t]}{\partial x_j} du.$$

By the aid of (67) it is easily established that

$$(71)\qquad \sum_{j=1}^{\infty} r_j \left| \int_0^1 \frac{\partial f[\xi + u(\xi' - \xi); t]}{\partial x_j} du \right|$$

converges uniformly in the arguments (t^T, ξ^S, ξ'^S).

Now define the function $A_j(t, \xi, \xi')$ by the equation

$$(72)\qquad A_j(t, \xi, \xi') = \left| \int_0^1 \frac{\partial f[\xi + u(\xi' - \xi); t]}{\partial x_j} du \right|.$$

By virtue of (70) and (71) it is seen that (68) and (69) hold for this definition of A_j. The complete continuity of the A_j of (72) in its arguments is a consequence of Theorem II, when u is identified with the x_1 of that discussion and the totality $(u; t; x_1, x_2, \cdots; x_1', x_2', \cdots)$ is considered as a single point in a space of infinitely many dimensions.

Corollary 1. *Hypothesis* (H_2) *of Theorem VIII is implied by the following assumption*:

(H_2'). *There exist derivatives* $\partial f_i/\partial x_j$ $(i, j = 1, 2, \cdots)$ *which are* $C_1(\mathbf{R})$ *and such that each of the sums*

$$(73)\qquad \sum_{j=1}^{\infty} r_j \left| \frac{\partial f_i(\eta)}{\partial x_j} \right| \qquad (i = 1, 2, \cdots)$$

converges uniformly for all $\eta^{\mathbf{R}}$.

To establish the corollary, define

$$(74)\qquad A_{ij}(t, \xi, \xi') = \left| \int_0^1 \frac{\partial f_i[\xi + u(\xi' - \xi); t]}{\partial x_j} du \right|.$$

As an immediate consequence of Theorem X it is seen that hypothesis (H_2) is true for this interpretation of A_{ij}.

Corollary 2. *Expression* (56) *of hypothesis* (H_3) *of Theorem VIII is implied by* (H_2') *and the further assumption that the maxima* K_i' *of the sums of* (73) *for points* $\eta^{\mathbf{R}}$ *satisfy*

$$K_i' \leqq r_i K'.$$

To see this result, suppose that in (H_2) the A_{ij} have the definition (74).

Then it follows that

$$\sum_{j=1}^{\infty} r_j A_{ij}(t, \xi, \xi') \leqq \int_0^1 \sum_{j=1}^{\infty} r_j \left| \frac{\partial f_i[\xi + u(\xi' - \xi); t]}{\partial x_j} \right| du,$$
$$\leqq K'_i.$$

Hence (56) of hypothesis (H_3) is true with $K = K'$.

5. **Comparison with Moulton's results.** In order to make a definite comparison with Moulton's paper, it will be convenient to have his main theorem stated explicitly in the following form:

THEOREM M. *Let there be given the system of differential equations*

$$\frac{dx_i}{dt} = f_i(t, \xi) = a_i + \sum_{j=1}^{\infty} f_i^{(j)}(t, \xi) \qquad (i = 1, 2, \cdots) \tag{75}$$

with the initial conditions $\xi(0) = 0$, *where "* a_i *" is a constant and* $f_i^{(j)}$ *is the totality of terms of* f_i *which are homogeneous in* $(t, x_1, x_2, \cdots)$ *of degree* j. *Suppose there exist real positive numbers* $(c_0, c_1, \cdots)$, $(r_0, r_1, \cdots)$ *and* (A, a) *such that* $c_0 t + c_1 x_1 + \cdots = s(t, \xi)$ *converges for points* $\eta = (t, \xi)$ *in the region of complex values*

$$|t| \leqq r_0, \qquad |\xi| \leqq \rho \qquad [\rho = (r_1, r_2, \cdots)],$$

and such that $A r_i s^j$ *dominates* $f_i^{(j)}$ *term by term, and* $|a_i| \leqq A r_i a$. *Then there exists for* (75) *a unique analytic solution* $\xi = \xi(t)$, $[\xi(0) = 0]$, *for* $|t|$ *sufficiently small.*

In Moulton's paper the discussion was for complex values of the quantities entering. The work of the present paper is concerned only with reals, and hence, let Theorem M be considered for real values of the symbols involved. Then we have

THEOREM XI. *The Theorem M when restricted to reals is a corollary of Theorem VIII; and, in addition, it follows that the analytic solution of* (75) *is the unique continuous solution satisfying the initial conditions.*

To establish these facts, it will first be shown that, under Moulton's hypotheses, all those of Theorem VIII are satisfied. On choosing, in common with Moulton, a number G so that $s(t, \xi) \leqq w < 1$ for all η in the region **R** defined by

$$T: |t| \leqq \frac{r_0}{g}; \qquad S: |x_i| \leqq \frac{r_i}{g} \qquad (i = 1, 2, \cdots),$$

it follows from the dominance assumption of Theorem M that

$$f_i(t, \xi) = a_i + \sum_{j=1}^{\infty} f_i^{(j)} \ll r_i M \quad \left[M = A\left(a + \sum_{j=1}^{\infty} w^j\right) \right]. \tag{76}$$

It will be proved that, in the region $\mathbf{R}$, all the hypotheses of Theorem VIII hold true for (75).

In view of Theorem III it is seen from (76) that $f_i(\eta)$ is $C_1(\mathbf{R})$, which is the hypothesis (H_1) of Theorem VIII. Moreover,

$$\frac{\partial f_i}{\partial x_k} \ll r_i\, Ac_k \left[\sum_{j=1}^{\infty} j w^{j-1} \right] = r_i\, Ac_k\, H \qquad \left(H = \sum_{j=1}^{\infty} j w^{j-1} \right),$$

and hence $\partial f_i / \partial x_k$ $(i, k = 1, 2, \cdots)$ are $C_1(\mathbf{R})$. From the fact that

$$\sum_{k=1}^{\infty} r_k \left| \frac{\partial f_i(\eta)}{\partial x_k} \right| \ll r_i \left(AH \sum_{k=1}^{\infty} c_k r_k \right) = r_i K \quad \left(K = AH \sum_{k=1}^{\infty} c_k r_k \right)$$

for all $\eta^{\mathbf{R}}$, and from the Corollaries 1 and 2 of Theorem X, it is seen that hypotheses (H_2) and (56) of (H_3) are satisfied. Moreover, because of (76), (55) of (H_3) is true. Thus, all conditions of Theorem VIII are satisfied and, therefore, for $|t| \leqq d$, (d sufficiently small), there is an unique solution $\xi(t)$ of (75) satisfying $\xi(0) = 0$. On examining the form of the approximations (51) for the system (75), it is seen that all $\xi_k(t)$ are analytic in t and hence, the γ_k of Theorem VIII are analytic. From the equality $\xi(t) = \sum_{k=1}^{\infty} \gamma_k(t)$ and from the absolute-uniform convergence of this series for $|t| \leqq d$ [cf. (59)], it follows that $\xi(t)$ is analytic; therefore the proof of Theorem XI is complete.

The equation (59) of this paper also establishes another result obtained in Moulton's paper which can be summarized in the statement that *there exists a continuous function* $p(t)$, $(p(0) = 0)$, *such that* $|\xi(t)| \leqq \rho p(t)$.

PART III

Implicit functions in infinitely many variables

1. **Introduction.** Let the infinite system of equations (2) defining $(y_1, y_2, \cdots)$ as functions of $(x_1, x_2, \cdots)$ be represented by the notation

$$\Phi(\gamma) = 0 \qquad [\Phi = (f_1, f_2, \cdots)] \tag{77}$$

where γ is in the space of infinitely many dimensions with the coördinates $(x_1, x_2, \cdots;\ y_1, y_2, \cdots)$ or, $\gamma = (\xi, \eta)$, where $\xi = (x_1, x_2, \cdots)$ and $\eta = (y_1, y_2, \cdots)$.

Let $(\xi = \alpha, \eta = \beta)$ be a solution of (77). Then the problem of the present chapter is to obtain for the system (77) with this given initial solution an analog of the fundamental theorem on implicit functions for the finite case. In carrying over the theory from the finite to the infinite case, alternative methods of procedure are open, due to the different possible choices for the region of definition of (2) and for the postulated character of the function $\Phi(\gamma)$. In the sequel, the classical theory is generalized in a direction parallel to that taken in the theorems of Part I.

In dealing with the classical case, certain finite systems of linear equations play an important part. Similarly in the present transcendental case, certain infinite systems of linear equations enter in a fundamental fashion. The hypotheses which will be imposed on the system (77) are arranged so that these infinite systems come under the theory of infinite systems of linear equations with normal determinants. In § 2 there is given a brief summary of the auxiliary results on infinite determinants and infinite systems of linear equations which will be used in the subsequent parts of the paper.

In § 3 there is presented, under suitable hypotheses, the analog for (77) of the classical theorem on implicit functions. Then, in § 4, additional hypotheses are imposed under which it is shown that the solution of (77) possesses first partial derivatives with respect to the variables x_i.

2. **Normal infinite determinants and infinite systems of linear equations.** In the sequel it will be convenient to have the theorems outlined below. References are made to the pages in Kowalewski's book on determinants,* where the proofs of the results are to be found. In the future, the notation "$(K$, p. —$)$" refers to a page in this book.

Let the notation for an infinite determinant be

$$A = |a_{ij}|_{(i,\,j=1,\,2,\,\cdots)} \quad (i \textit{ designates the row, } j \textit{ the column}).$$

Then A is defined as a *normal determinant* if

$$S = \sum_{i,\,j=1}^{\infty} |a_{ij} - d_{ij}|$$

converges, where d_{ij}, the Kronecker symbol, is zero or unity according as $i \neq j$ or $i = j$. Certain properties of normal determinants will be stated.

Property 1. (K, p. 372) *If A_n represents the determinant*

$$A_n = |a_{ij}|_{(i,\,j=1,\,2,\,\cdots,\,n)},$$

then the determinant A exists in the sense

$$A = \lim_{n=\infty} A_n.$$

Property 2. (K, p. 374) *The determinant A can be expressed as a series $A = 1 + \sum_{i=1}^{\infty} T_i$ which satisfies*

$$1 + \sum_{i=1}^{\infty} T_i \ll 1 + \sum_{i=1}^{\infty} \frac{S^i}{i!}.$$

Property 3. (K, p. 385) *For every k, the series $\sum_{j=1}^{\infty} D_{kj}$ where D_{kj} is the co-factor of a_{kj} in A, can be arranged in a series which is term by term*

* *Einführung in die Determinanten-Theorie.*

dominated by

$$2\left(1+\frac{S}{1!}+\frac{S^2}{2!}+\cdots\right),$$

and, thus, these sums ($k = 1, 2, \cdots$) *have a common upper bound* B.

The fundamental theorem on the solution of certain types of infinite systems of linear equations is stated as

PROPERTY 4. (K, p. 383) *In the infinite system of linear equations*

$$\sum_{j=1}^{\infty} a_{ij}\, x_j = b_j \qquad (i = 1, 2, \cdots),$$

suppose that the determinant A *is normal and distinct from zero and that* $|b_j| \leqq b$, (b *finite*; $j = 1, 2, \cdots$). *Then among all bounded sequences of numbers* ($x_1, x_2, \cdots$) *there exists uniquely a solution*

$$x_j = \sum_{k=1}^{\infty} \frac{b_k\, D_{kj}}{A} \qquad (j = 1, 2, \cdots).$$

3. **The theorem for a restricted neighborhood.** Let $\mathbf{R}$ represent the region of points $\gamma = (\xi, \eta)$ defined by

(78) $$S : |\xi - \alpha| \leqq \rho; \qquad T : |\eta - \beta| \leqq \rho' \qquad [\rho = (r_1, r_2, \cdots);\ \rho' = (r_1', r_2', \cdots)],$$

in which $r_i \leqq r$, $r_i' \leqq r$. Then the result for a restricted neighborhood is stated in the following

THEOREM XII. *Suppose in* (77) *that the function* Φ *and the derivatives* $\partial f_i/\partial y_j$ ($i, j = 1, 2, \cdots$) *are* $C_1(\mathbf{R})$, *and that the maxima* M_i *of the* $|f_i(\gamma)|$ *for points* $\gamma^{\mathbf{R}}$ *satisfy*

(79) $$M_i \leqq r_i\, M.$$

Assume that the double sum

(80) $$\sum_{i,j=1}^{\infty} \left| d_{ij} - \frac{\partial f_i(\gamma)}{\partial y_j} \right|$$

converges uniformly for all $\gamma^{\mathbf{R}}$, *and that the normal infinite determinant*

(81) $$F(\gamma) = \left| \frac{\partial f_i(\gamma)}{\partial y_j} \right|_{(i,\, j=1,\, 2,\, \cdots)}$$

is different from zero for ($\xi = \alpha$, $\eta = \beta$). *Suppose, also, that there exists a number* G *such that the convergent series* $\sum_{k=1}^{\infty} |D_{ki}|$, *where* D_{ki} *is the co-factor of the element* a_{ki} *in*

$$D = F(\alpha, \beta) = |a_{ij}|_{(i,\, j=1,\, 2,\, \cdots)} \qquad \left[a_{ij} = \frac{\partial f_i(\alpha, \beta)}{\partial y_j} \right],$$

satisfies the inequality

$$\sum_{k=1}^{\infty} |D_{ki}| \leqq DGr'_i \qquad (i = 1, 2, \cdots), \tag{82}$$

and let the function Φ have the property defined as follows:
(83) *For every $e > 0$ there can be found a number $d_e > 0$ such that, for*

$$|\xi - \alpha| \leqq d_e \rho, \qquad (\xi^S),$$

it follows that

$$|\Phi(\xi, \beta)| \leqq e.$$

Then, if $\Phi(\alpha, \beta) = 0$, it follows that there are positive constants (c, d), $(0 < d \leqq 1, 0 < c \leqq d)$, such that to every ξ in $|\xi - \alpha| \leqq c\rho$ there corresponds one and only one solution of (77) *in $|\eta - \beta| \leqq d\rho'$. Furthermore, the function $\eta(\xi)$, so determined, is completely continuous for all points ξ in the region $|\xi - \alpha| \leqq c\rho$.*

In the proof of Theorem XII it is convenient to have the auxiliary function $\Psi(\xi, \eta)$ whose components $h_i(\xi, \eta)$ $(i = 1, 2, \cdots)$ are defined by the system of linear equations

$$f_i(\xi, \eta) + \sum_{j=1}^{\infty} a_{ij}[h_j(\xi, \eta) - y_j] = 0. \tag{84}$$

On solving (84) it follows from Property 4 and from the present hypotheses that

$$h_i(\xi, \eta) = y_i - \frac{F_i(\xi, \eta)}{D}, \tag{85}$$

where F_i represents D with the ith column replaced by $(f_1, f_2, \cdots)$; i. e.,

$$h_i(\xi, \eta) = y_i - \sum_{j=1}^{\infty} f_j(\xi, \eta) a'_{ji} \qquad \left[a'_{ji} = \frac{D_{ji}}{D}\right]. \tag{86}$$

Since D is a normal determinant, it follows from § 2 that $\sum_{j=1}^{\infty} |a'_{ji}|$ converges and, therefore, as a consequence of (79) and of Theorem III, the h_i are $C_1(\mathbf{R})$.

In order to establish the theorem, first consider the proof of certain lemmas.

LEMMA 1. *Let w be a positive number and $0 < w < 1$. Then there exists a number $d \leqq 1$ $(d > 0)$ such that*

$$\sum_{j=1}^{\infty} \left| \frac{\partial \Psi(\gamma)}{\partial y_j} \right| \leqq \frac{w\rho'}{r} \quad (|\xi - \alpha| \leqq d\rho, \ |\eta - \beta| \leqq d\rho'). \tag{87}$$

It follows from (85) that

$$\frac{\partial h_i}{\partial y_j} = d_{ij} - \frac{\dfrac{\partial F_i}{\partial y_j}}{D}.$$

In view of assumption (80), it is seen that

$$\sum_{k=1}^{\infty}\left|\frac{\partial f_k(\gamma)}{\partial y_j}D_{ki}\right|$$

converges uniformly for $\gamma^{\mathbf{R}}$, and hence $\partial F_i/\partial y_j$ may be expanded, giving

$$\frac{\partial h_i}{\partial y_j} = d_{ij} - \sum_{k=1}^{\infty}\frac{\partial f_k}{\partial y_j}a'_{ki}. \tag{88}$$

On making the substitution

$$d_{ij} = \sum_{k=1}^{\infty} a_{kj}\, a'_{ki},$$

and summing (88) with respect to j after dividing by r'_i, there results

$$\frac{1}{r'_i}\sum_{j=1}^{\infty}\left|\frac{\partial h_i}{\partial y_j}\right| \leqq \frac{1}{r'_i}\sum_{j,k=1}^{\infty}\left|\left(a_{kj} - \frac{\partial f_k}{\partial y_j}\right)a'_{ki}\right|.$$

It is seen that

$$\frac{1}{r'_i}\sum_{k=1}^{\infty}\left|\left(a_{kj} - \frac{\partial f_k}{\partial y_j}\right)a'_{ki}\right| \leqq \frac{1}{r'_i}\left(\sum_{k=1}^{\infty}|a'_{ki}|\right)\left(\sum_{k=1}^{\infty}\left|a_{kj} - \frac{\partial f_k}{\partial y_j}\right|\right),$$

$$\leqq G\sum_{k=1}^{\infty}\left|a_{kj} - \frac{\partial f_k}{\partial y_j}\right|;$$

and, therefore, it follows that

$$\frac{1}{r'_i}\sum_{j=1}^{\infty}\left|\frac{\partial h_i(\gamma)}{\partial y_j}\right| \leqq GV(\gamma) \qquad \left(V(\gamma) = \sum_{j,k=1}^{\infty}\left|a_{kj} - \frac{\partial f_k(\gamma)}{\partial y_j}\right|\right)$$

In view of (80) it is verified that $V(\gamma)$ converges absolutely-uniformly for all $\gamma^{\mathbf{R}}$, and hence $V(\gamma)$ is $C_1(\mathbf{R})$.

From the fact that $V(\alpha,\beta) = 0$, it follows, in view of Proposition (2), Part I, that there is a number $d > 0$ such that, if

$$|\xi - \alpha| \leqq d\rho, \qquad |\eta - \beta| \leqq d\rho',$$

then

$$V(\gamma) \leqq \frac{w}{Gr},$$

and, consequently,

$$\sum_{j=1}^{\infty}\left|\frac{\partial h_i}{\partial y_j}\right| \leqq \frac{r'_i\, w}{r}.$$

Lemma 2. *There exists a number $c \leqq d$ such that*

$$|\Psi(\xi,\beta) - \beta| \leqq d\rho'(1 - w)$$

for all values of ξ in the region S' defined by $|\xi - \alpha| \leqq c\rho$.

To obtain this result, start from the inequality

$$\frac{1}{r_1'}|h_i(\xi,\beta)-b_i| \leqq \frac{1}{r_i'}\sum_{k=1}^{\infty}|a'_{ki}f_k(\xi,\beta)|, \tag{89}$$

which is derived from (86) by letting $\gamma=(\xi,\beta)$. Condition (83) shows that there exists a number $c \leqq d$ such that

$$|\Phi(\xi,\beta)| \leqq \frac{d(1-w)}{G} \qquad (|\xi-\alpha| \leqq c\rho),$$

and, hence, it follows from (89) and from the definition of G that

$$|\Psi(\xi,\beta)-\beta| \leqq \rho' d(1-w).$$

For convenience denote by T' the set of points satisfying $|\eta-\beta| \leqq d\rho'$.

Now consider the proof of Theorem XII. Define a sequence $\eta_k(\xi)$ of approximations to a solution of (77) through the initial point (α,β) by the equations

$$\eta_k(\xi) = \Psi[\xi, \eta_{k-1}(\xi)] \qquad [\eta_0(\xi)=\beta;\ k=1,2,\cdots]. \tag{90}$$

It will first be shown that, for all $\xi^{S'}$, the points $\eta_k(\xi)$ are in T'. Then it will be established that, for these values of ξ, the sequence (90) converges to a solution through (α,β). Finally, the uniqueness of this solution will be established by an indirect argument.

From Theorem I it follows that $\eta_1(\xi)$ is $C_1(S')$. Moreover, from the definition of the sequence (90) and from Lemma 2 it is seen that

$$|\eta_1(\xi)-\beta| = |\Psi(\xi,\beta)-\beta| \leqq d\rho'(1-w) < d\rho',$$

and thus it follows that $\eta_2(\xi)$ is $C_1(S')$. From (90) there results

$$|\eta_2(\xi)-\eta_1(\xi)| = |\Psi[\xi,\eta_1(\xi)]-\Psi[\xi,\eta_0(\xi)]|. \tag{91}$$

On applying Taylor's Theorem (cf. Part I) to the right-hand member in (91), there is obtained

$$\begin{aligned}|\eta_2(\xi)-\eta_1(\xi)| &\leqq \sum_{j=1}^{\infty}|y_{j1}(\xi)-y_{j0}(\xi)|\cdot\left|\int_0^1 \frac{\partial\Psi[\xi;\beta+u(\eta_1(\xi)-\beta)]}{\partial y_j}du\right| \\ &\leqq dr(1-w)\int_0^1\sum_{j=1}^{\infty}\left|\frac{\partial\Psi[\xi;\beta+u(\eta_1(\xi)-\beta)]}{\partial y_j}\right|du \\ &\leqq d\rho' w(1-w),\end{aligned}$$

and, therefore,

$$|\eta_2(\xi)-\beta| \leqq d\rho'(1-w^2) < d\rho'.$$

On proceeding as above, it is established by a simple induction that all $\eta_k(\xi)$

are $C_1(S')$ and satisfy

$$(92)\qquad \begin{aligned} |\eta_k(\xi) - \eta_{k-1}(\xi)| &\leqq d\rho' w^{k-1}(1-w), \\ |\eta_k(\xi) - \beta| &\leqq d\rho'(1-w^k) < d\rho'. \end{aligned}$$

To prove the convergence of the sequence $\eta_k(\xi)$, consider the equivalent problem of the convergence of the telescopic series

$$(93)\qquad \Theta(\xi) = \beta + \sum_{k=1}^{\infty}\big(\eta_k(\xi) - \eta_{k-1}(\xi)\big).$$

In view of (92) it is seen that $\Theta(\xi)$ satisfies the dominance equation

$$(94)\qquad \Theta(\xi) \ll |\beta| + d\rho'(1-w)\sum_{k=1}^{\infty} w^{k-1} \qquad (\xi^{S'}).$$

Since the right member of (94) converges, the same result holds for the sequence of approximations; and it can be easily established that the manner of convergence is characterized by the statement that *there exists a function*

$$\eta(\xi) = \Theta(\xi),$$

or $\xi^{S'}$, *such that, to every* $e > 0$ *there corresponds an integer* k_e *such that*

$$(95)\qquad |\eta_k(\xi) - \eta(\xi)| \leqq e\rho' \qquad (k \geqq k_e;\ \xi^{S'}).$$

That is, the convergence is uniform with respect to $\xi^{S'}$ and uniform over the range I relative to the scale function ρ'. Moreover it is seen from Theorem III that $\eta(\xi)$ is $C_1(S')$.

To show that $\eta(\xi)$ is a solution of (77), first note that, because of the complete continuity of $\Psi(\gamma)$, the convergence (95) shows that

$$\lim_{k=\infty} \Psi[\xi, \eta_k(\xi)] = \Psi[\xi, \eta(\xi)].$$

But from definition (90) it follows that

$$\lim_{k=\infty} \eta_k(\xi) = \lim_{k=\infty} \Psi(\xi, \eta_{k-1}(\xi)) = \Psi(\xi, \eta(\xi)),$$

and thus $\eta(\xi)$ satisfies

$$\eta(\xi) = \Psi(\xi, \eta(\xi)).$$

Also, since $D \neq 0$, it is clear that (77) is equivalent to the system $\eta = \Psi(\xi, \eta)$ and therefore $\eta(\xi)$ is a solution of (77).

In order to establish the uniqueness of the solution $\eta(\xi)$ suppose that, for a certain $\xi_1^{S'}$, there existed a second solution $\eta_1^T \neq \eta(\xi_1)$. Then

$$\Phi(\xi_1, \eta_1) = 0.$$

Also, since $\eta(\xi_1)$ is a solution of (77),

$$|\eta(\xi_1) - \eta_1| = |\Psi(\xi_1, \eta(\xi_1)) - \Psi(\xi_1, \eta_1)|,$$

(96)
$$\leqq \sum_{j=1}^{\infty} |y_j(\xi_1) - y_{j1}| \int_0^1 \left| \frac{\partial \Psi[\xi_1; \eta(\xi_1) + u(\eta_1 - \eta(\xi_1))]}{\partial y_j} \right| du,$$

where the last expression is obtained by an application of Taylor's Theorem (cf. Theorem V). Since $|y_{j1} - y_j(\xi_1)|$ is bounded for $(j = 1, 2, \cdots)$, and since $\eta_1 \neq \eta(\xi_1)$, it follows that there exists a least upper bound $B > 0$ for the values of

$$\left| \frac{y_{j1} - y_j(\xi_1)}{r'_j} \right| \qquad (j = 1, 2, \cdots).$$

Therefore an index i can be found for which

(97)
$$|y_{i1} - y_i(\xi_1)| > Bwr'_i.$$

Then, from (96) it is seen that $|y_i(\xi_1) - y_{i1}|$ is at most equal to

(98)
$$\sum_{j=1}^{\infty} |y_j(\xi_1) - y_{j1}| \cdot \left| \int_0^1 \frac{\partial h_i[\xi_1; \eta(\xi_1) + u(\eta_1 - \eta(\xi_1))]}{\partial y_j} du \right|.$$

That is

$$|y_i(\xi_1) - y_{i1}| \leqq \frac{Brwr'_i}{r} = Bwr'_i.$$

But, since (98) contradicts (97), it follows that the assumption as to the existence of a distinct η_1 was wrong and, thus, the uniqueness of the solution $\eta(\xi)$ is proved.

Corollary 1. *To every number $g > 0$ there corresponds a number $m > 0$ such that, for all ξ in $|\xi - \alpha| \leqq m\rho$,*

$$|\eta(\xi) - \beta| \leqq g\rho'.$$

The corollary is a result of the inequality (94). From this it is seen that

$$|\eta(\xi) - \beta| \leqq \rho' \sum_{k=1}^{\infty} w^{k-1} \qquad (|\xi - \alpha| \leqq c\rho),$$

where it is to be recalled that c depends on the choice of w in Lemma 1. Suppose, now, that w_0 is so small that $\sum_{k=1}^{\infty} w_0^{k-1} \leqq g$. Then, if m is chosen as the number c corresponding to the w_0 through Lemmas 1 and 2, it follows that the conclusion of the corollary is true.

4. **Differentiation of the solution $\eta(\xi)$ of § 3.** An hypothesis in addition to those of Theorem XII is necessary in order to insure the existence of partial derivatives of the solution $\eta(\xi)$ with respect to the components x_j of ξ. The

consideration of the existence of these derivatives for a representative component (e. g., x_1) is found in

THEOREM XIII. *Suppose that, in addition to the hypotheses of Theorem XII, there exist in* $\mathbf{R}$ *partial derivatives* $\partial f_i(\gamma)/\partial x_1$ $(i = 1, 2, \cdots)$ *which are* $C_1(\mathbf{R})$ *and are such that their maximum absolute values,* N_i', *for points* $\gamma^{\mathbf{R}}$ *satisfy* $N_i \leqq N$. *Then, for* $|\xi - \alpha| \leqq b\rho$, *(b sufficiently small), there exists a derivative function* $\partial\eta(\xi)/\partial x_1$ *whose components are given by the solution of the system*

$$\sum_{j=1}^{\infty} \frac{\partial f_i}{\partial y_j}\frac{\partial y_j}{\partial x_1} + \frac{\partial f_i}{\partial x_1} = 0 \qquad (i = 1, 2, \cdots)$$

for $\partial y_j/\partial x_1$. *Moreover,* $\partial\eta/\partial x_1$ *is* $C_1(\xi)$ *for* $|\xi - \alpha| \leqq b\rho$ *and there is a number* $Q > 0$ *such that* $|\partial\eta(\xi)/\partial x_1| \leqq Q$.

In establishing the theorem the components $(x_2, x_3, \cdots)$ of ξ are considered fixed and so, for simplicity, think of x_1 as the only independent variable in the problem.

If (x_1, η) and $(x_1' = x_1 + \Delta x_1,\ \eta' = \eta + \Delta\eta)$ are two solutions of (77) where x_1 and x_1' are in S', then

$$(99) \qquad f_i(x_1, \eta') - f_i(x_1, \eta) = f_i(x_1, \eta') - f_i(x_1', \eta') \quad (i = 1, 2, \cdots).$$

On expanding both sides by the mean-value theorem, there results, on dividing by Δx_1, for $(i = 1, 2, \cdots)$,

$$(100) \qquad \sum_{j=1}^{\infty} \frac{\Delta y_j}{\Delta x_1}\int_0^1 \frac{\partial f_i(x_1, \eta + u\Delta\eta)}{\partial y_j}du = \int_0^1 \frac{\partial f_i(x_1 + u\Delta x_1, \eta + \Delta\eta)}{\partial x_1}du.$$

It is desired to show that there exists $\lim_{\Delta x_1 = 0} \Delta y_j/\Delta x_1$.

LEMMA 1. *For every* (x_1^S, μ^T, μ'^T) *the expressions*

$$\int_0^1 \frac{\partial f_i(x_1, \mu + u\Delta\mu)}{\partial y_j}du \quad (i, j = 1, 2, \cdots;\ \Delta\mu = \mu' - \mu),$$

are $C_1(x_1, \mu, \mu')$.

The lemma is obtained as a consequence of Theorem II as was done in a similar case following (72), Part II.

LEMMA 2. *For every* (x_1^S, μ^T, μ'^T) *the infinite determinant*

$$A(x_1, \mu, \mu') = \left| \int_0^1 \frac{\partial f_i(x_1, \mu + u\Delta\mu)}{\partial y_j}du \right| \qquad (i, j = 1, 2, \cdots)$$

is normal and is $C_1(x_1, \mu, \mu')$.

In order to establish the normal character of A consider the inequality

$$Y(x_1, \mu, \mu') = \sum_{i,j=1}^{\infty} \left| d_{ij} - \int_0^1 \frac{\partial f_i(x_1, \mu + u\Delta\mu)}{\partial y_j}du \right|$$
$$\leqq \int_0^1 \sum^{\infty} \left| d_{ij} - \frac{\partial f_i(x_1, \mu + u\Delta\mu)}{\partial y_j} \right| du.$$

In view of (80) the right side of this equation is seen to converge uniformly for all (x_1^S, μ^T, μ'^T). Hence, because of Theorem III and of Lemma 1, Y is C_1 in all its arguments simultaneously; and, moreover, it is seen from definition that $A(x_1, \mu, \mu')$ is a normal determinant. From the Property 2 of § 2 it follows that

$$(101) \qquad |A(x_1, \mu, \mu')| \leqq 1 + \frac{Y(x_1, \mu, \mu')}{1!} + \frac{Y^2(x_1, \mu, \mu')}{2!} + \cdots .$$

Since Y has an upper bound for the values (x_1^S, μ^T, μ'^T) of its arguments, it is seen from (101) that $A(x_1, \mu, \mu')$, expressed as $\lim (n = \infty) A_n$ (the nth order determinant in the upper left-hand corner of A), converges uniformly for such values of its arguments and therefore is $C_1(x_1^S, \mu^T, \mu'^T)$.

We turn now to the consideration of (100). In view of the hypotheses as to the N_i, it follows from Property 4 of § 2 that the values $\Delta y_j / \Delta x_1$ are given by

$$(102) \qquad \begin{aligned} \frac{\Delta y_j}{\Delta x_1} &= \frac{1}{A(x_1, \eta, \eta')} z_j(x_1, x_1') \qquad (j = 1, 2, \cdots), \\ z_j(x_1, x_1') &= \sum_{k=1}^{\infty} A_{k,j}(x_1, \eta, \eta') \int_0^1 \frac{\partial f_k(x_1 + u\Delta x_1, \eta')}{\partial x_1} du, \end{aligned}$$

where $A_{k,j}$ represents the co-factor of the element of A in the row k and the column j. Now choose b', at most equal to c of Theorem XII, so that in the region

$$(103) \qquad |\xi - \alpha| \leqq b' \rho, \qquad |\mu - \beta| \leqq b' \rho', \qquad |\mu' - \beta| \leqq b' \rho',$$

the infinite determinant $A(\xi, \mu, \mu') \neq 0$. Select $b \leqq b'$ so that, for all ξ in $|\xi - \alpha| \leqq b\rho$,

$$|\eta(\xi) - \beta| \leqq b' \rho',$$

as is possible in view of the Corollary (1) of Theorem XII. Suppose that x_1 of (99) is in $|x_1 - a_1| \leqq b r_1$. Then, since A is $C_1(\mu')$ and since

$$A(x_1, \mu, \mu) = F(x_1, \mu),$$

it follows that

$$\lim_{x_1' = x_1} A(x_1, \eta, \eta') = F(x_1, \eta) \neq 0.$$

Furthermore, from the uniform convergence of A in its arguments, it is seen that $z_j(x_1, x_1')$ converges uniformly in the argument x_1'. Hence, since the expression

$$\int_0^1 \frac{\partial f_k(x_1 + u\Delta x_1, \mu')}{\partial x_1} du$$

is continuous in x_1', it follows that $z_j(x_1, x_1')$ is continuous in x_1'. Therefore

$$\lim_{x_1' = x_1} z_j(x_1, x_1') = \sum_{k=1}^{\infty} \frac{\partial f_k(x_1, \eta)}{\partial x_1} A_{k,j}(x_1, \eta, \eta),$$

and it has been established that there exists $\partial y_j/\partial x_1$ given by

$$(104) \qquad \frac{\partial y_j}{\partial x_1} = \lim_{x_1'=x_1} \frac{\Delta y_j}{\Delta x_1} = \frac{1}{F(x_1, \eta)} \sum_{k=1}^{\infty} \frac{\partial f_k(x_1, \eta)}{\partial y_1} D_{k,j}(x_1, \eta),$$

where $D_{k,j}(\xi, \eta)$ is defined as the co-factor of $\partial f_k/\partial x_j$ in the infinite determinant $F(\xi, \eta)$. From the Property 3 of § 2 and from the uniform convergence assumed in (80) it follows that the sum (104), as a function of $(x_1, x_2, \cdots)$, where $(x_2, x_3, \cdots)$ are now allowed to vary, is $C_1(\xi)$ for all $|\xi - \alpha| \leqq b\rho$, and

$$\left|\frac{\partial y_j}{\partial x_1}\right| \leqq \frac{Nq}{p} \qquad (j = 1, 2, \cdots),$$

where q is the maximum in $\mathbf{R}$ of $(\sum_{j=1}^{\infty} |D_{kj}(\gamma)|;\ k = 1, 2, \cdots)$. And where $p > 0$ is the minimum of $|F(\xi, \eta)|$ for points in

$$(|\xi - \alpha| \leqq b\rho; \qquad |\eta - \beta| \leqq b'\rho').$$

The problem of obtaining continuation properties of the solution $\eta(\xi)$ is not developed in the present paper because the form of the hypotheses used here does not readily lend itself to this generalization.

CHICAGO,
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